GOOD $\ell_2$-SUBSPACES OF $L_p$, $p > 2$

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ABSTRACT. We give an alternate proof of the result due to Haydon, Odell and Schlumprecht that subspaces of $L_p$, $p > 2$, which are isomorphic to $\ell_2$ contain subspaces which are well isomorphic to $\ell_2$ and well complemented.

1. INTRODUCTION AND PRELIMINARIES

In a recent paper Haydon, Odell and Schlumprecht show that any subspace of $L_p$, $p > 2$, which is isomorphic to $\ell_2$ contains a subspace on which the $L_p$ norm behaves similarly to its behavior on the span of independent, mean 0, Gaussian variables. (See [2, Section 6].) Using this subspace they obtain a well complemented subspace $(1 + \epsilon)$-isomorphic to $\ell_2$. In order to find this subspace the authors use types [3] and random measures [1].

In this note we show that the same result can be produced without as much machinery by using a version of the Central Limit Theorem for martingales [4]. In the proof of Lemma 6.6 of [2] the Central Limit Theorem also plays an essential role. Below we assume that $(\Omega, P)$ is an atomless probability space and denote by $E(\cdot|\mathcal{F})$ the conditional expectation with respect to the $\sigma$-algebra $\mathcal{F}$. $N(\mu, \sigma^2)$ denotes, as usual, the normal distribution with center at $\mu$ and variance $\sigma^2$. Below $\mathcal{F}_{n,0} = \{\Omega, \emptyset\}$, the trivial $\sigma$-algebra. We follow the convention of suppressing the measure space variable, i.e., $\{f > r\} = \{\omega : f(\omega) > r\}$.

Theorem 1.1. [4, VII.8 Theorem 4] Suppose that for each $n$, $(f_{n,k})_{k=1}^n$ is a square integrable martingale difference sequence adapted to $(\mathcal{F}_{n,k})_{k=1}^n$ satisfying...
the Lindeberg condition: for every $\epsilon > 0$,

$$\lim_{n \to \infty} \sum_{k=1}^{n} E(f_{n,k}^2 1_{\{|f_{n,k}| > \epsilon\}} | \mathcal{F}_{n,k-1}) = 0,$$

in probability. If

$$\lim_{n \to \infty} \sum_{k=1}^{n} E(f_{n,k}^2 | \mathcal{F}_{n,k-1}) = \sigma^2$$

in probability then

$$\lim_{n \to \infty} \sum_{k=1}^{n} f_{n,k} = N(0, \sigma^2)$$

in distribution.

Our argument uses the limiting conditional variance as in [2] and we borrow a few facts from that paper. If $(x_n)$ is a weakly null sequence in $L_p$ such that $(x_n^2)$ converges weakly in $L_{p/2}$ to $v$, then $v$ is the limiting conditional variance of $(x_n)$. Recall that a subset $U$ of $L_p(\Omega, P)$ is said to be $p$-uniformly integrable if for every $\epsilon > 0$ there is a $\delta$ such that if $A$ is measurable and $P(A) < \delta$ then for all $f \in U$, $E(|f|^p 1_A) < \epsilon$. An equivalent definition is that there exists $K$ such that for all $f \in U$, $E(|f|^p 1_{\{|f| > K\}}) < \epsilon$. Also it is easy to see that the negation is equivalent to: there is an $\epsilon > 0$ and a sequence $(f_j)$ in $U$ and a sequence of disjoint measurable sets $(A_j)$ such that $E(|f_j|^p 1_{A_j}) \geq \epsilon$ for all $j$.

**Lemma 1.2.** [2, Lemma 5.3] Let $(x_n)$ be a martingale difference sequence which is $p$-uniformly integrable. Then set of linear combinations of $(x_n)$ with coefficients in the unit sphere of $\ell_2$ is $p$-uniformly integrable.

Our goal is to prove the following.

**Theorem 1.3.** Suppose that $X$ is a subspace of $L_p$ for some $p$, $2 < p < \infty$, which is isomorphic to $\ell_2$. Then for every $\epsilon > 0$ there is a sequence $(z_n)$ in $X$ such that for all $(c_n)$ in $\ell_2$,

$$\left(1 - \epsilon\right) \left(\sum_{n=1}^{\infty} c_n^2\right) \leq \left\|\sum_{n=1}^{\infty} c_n z_n\right\| \leq \left(1 + \epsilon\right) \left(\sum_{n=1}^{\infty} c_n^2\right)$$

and $[z_n]$ is complemented in $L_p$ with a projection of norm less than $(1 + \epsilon)\gamma_p$, where $\gamma_p$ is the norm of a symmetric Gaussian random variable. Moreover, if $(x_n)$ is a normalized sequence in $X$ equivalent to the unit vector basis of $\ell_2$ with limiting conditional variance $v$, then $(z_n)$ can be chosen with limiting conditional variance $v$.

This result is implicit in the proof of Theorem 6.8 of [2]. The proof given here can also be used to prove that result since it produces a stabilized $\ell_2$-sequence with the required limiting conditional variance.
2. Proof of the Theorem

The proof is mostly reductions to a simple situation where the Central Limit Theorem can be easily applied. Except for the results cited above the proof uses only basic measure theory and functional analysis.

Proof. Let \((x_n)\) be a normalized sequence in \(X\) equivalent to the unit vector basis of \(\ell_2\) with limiting conditional variance \(\nu\). Because \((x_n)\) is weakly null, we may assume by passing to a subsequence and perturbing that \((x_n)\) is a martingale difference with respect to some increasing family of \(\sigma\)-algebras, \((\mathcal{G}_n)\). We may assume that the simple functions with respect to \(\cup \mathcal{G}_n\) are dense in \(L_p(\Omega, P)\) for all \(p, 1 \leq p < \infty\).

Our first step is to replace \((x_n)\) by a sequence of blocks \((y_k)\) of \((x_n)\) so that \((y_k)\) is uniformly \(p\)-integrable. (See Lemma 5.4 of [HOS] for a similar argument.)

Let

\[
a = \sup \{ \epsilon : \text{there exists } (n_j) \text{ and disjoint measurable sets } (A_j) \text{ such that } \| x_{n_j|A_j} \| \geq \epsilon \}.
\]

Then choose a subsequence \((n_j)\) and a sequence of disjoint sets \((A_j)\) so that the supremum is achieved, i.e., \(\lim \| x_{n_j|A_j} \| = a\). Moreover by taking an appropriate subsequence of the \(\sigma\)-algebras and relabeling we may assume that \(A_j\) is in \(\mathcal{G}_{n_j}\) for each \(j\). It follows that \((x_{n_j} 1_{\Omega|A_j})\) is a martingale difference. Moreover by the choice of \((A_{n_j})\), \((x_{n_j} 1_{\Omega|A_j})\) must be \(p\)-uniformly integrable.

Let \((K_m)\) be finite subsets of \(\mathbb{N}\) such that \(\max K_m < \min K_{m+1}\) for all \(m\) and strictly increasing cardinality. Let \(y_m = |K_m|^{-1/2} \sum_{j \in K_m} x_{n_j}\) for all \(m\), where \(|F|\) denotes the cardinality of \(F\).

Observe that

\[
\| |K_m|^{-1/2} \sum_{j \in K_m} x_{n_j} 1_{A_j} \| = |K_m|^{-1/2} \left( \sum_{j \in K_m} \| x_{n_j|A_j} \|^p \right)^{1/p} \leq a |K_m|^{(2-p)/(2p)}.
\]

Because \(p > 2\), this goes to zero as \(m\) increases. A norm perturbation of a \(p\)-uniformly integrable sequence is also \(p\)-uniformly integrable and thus it follows from Lemma 1.2 that \((y_m)\) is \(p\)-uniformly integrable. Further notice that \((y_m^2)\) converges weakly to \(\nu\). Indeed

\[
y_m^2 = |K_m|^{-1} \sum_{j \in K_m} x_{n_j}^2 + |K_m|^{-1} \sum_{r,s \in K_m, r \neq s} 2x_{n_r}x_{n_s}.
\]

By our assumption on the family \((\mathcal{G}_n)\), every element in \(L_p^*(\Omega, \mathcal{G}_n, P)\) can be approximated in norm as close as required by \(\mathcal{G}_n\)-measurable simple functions for \(n\) sufficiently large. Thus for \(m\) sufficiently large for all \(r, s \in K_m, r \neq s\), \(x_{n_r}x_{n_s}\) is orthogonal to \(L_p^*(\Omega, \mathcal{G}_n, P)\) and hence \( |K_m|^{-1} \sum_{r, s \in K_m, r \neq s} 2x_{n_r}x_{n_s} \) tends to 0 weakly.

Our next step is to show that we may assume

\[(*) \ \nu \text{ is measurable with respect to some finite (cardinality) } \sigma\text{-algebra } \mathcal{G}_0 \text{ and that } (y_n) \text{ is a } p\text{-uniformly integrable martingale difference sequence such that } \mathbb{E}(y_n|\mathcal{G}_0) = 0 \text{ for all } n.\]
By another application of Lemma 1.2 the set $S$ of all linear combinations of $(y_n)$ with coefficients in the sphere of $\ell_2$ is $p$-uniformly integrable. Let $\epsilon > 0$ and choose $\delta > 0$ such that for all $z \in S$, if $B$ is measurable with $P(B) < \delta$ then $E(|z|^{p}1_B) < \epsilon$. Choose $M_0 > 0$ such that $P(\{v \leq M_0\}) < \delta/2$ and $M_1 > 0$ such that $P(\{v > M_1\}) < \delta/2$. Let $A = \{v \leq M_0\} \cup \{v > M_1\}$. Then if $z \in S$, $E(|z|^{p}1_A) < \epsilon$. Consequently we may (and do) assume by a norm perturbation that $y_n1_A = 0$ and that $v$ and $1/v$ are bounded on the support of $v$.

Let $\rho > 0$. Let $R_0$ and $R_1$ be integers such that $(1 + \rho)^{R_0} \leq M_0$ and $(1 + \rho)^{R_1} > M_1$. For each $r$, $R_0 \leq r < R_1$, let $A_r = \{(1 + \rho)^{r} \leq v < (1 + \rho)^{r+1}\}$. Let

$$w = 1_{\Omega_{\text{supp}}} + \sum_{r=R_0}^{R_1-1} \frac{(1 + \rho)^r}{v} 1_{A_r}. $$

Notice that for all $r$, $1 \leq r < \infty$,

$$E(|f|^r w^{r/2}) \leq E(|f|^r) \leq (1 + \rho)^{r/2} E(|f|^r w^{r/2})$$

for all $f \in L_r(\Omega, P)$. Thus multiplication by $w^{1/2}$ is a $(1 + \rho)^{1/2}$-isomorphism from $L_r(\Omega, P)$ onto $L_r(\Omega, P)$. Moreover $(y_nw^{1/2})$ converges weakly to 0 and $(y_n^2w)$ converges weakly to $v_0 = \sum_{r=R_0}^{R_1-1}(1 + \rho)^r 1_{A_r}$. The first assertion is immediate and the second follows by approximating $w$ by simple functions. Another perturbation gives our required reduction (*). Indeed let $F_0$ be the finite $\sigma$-algebra generated by $v_0$ and for each $n \in \mathbb{N}$ let $F_n$ be the $\sigma$-algebra generated by $F_0$ and $G_n$.

Because $(y_nw^{1/2})$ converges weakly to 0, there is a subsequence $(y_{n_j}w^{1/2})$ and a $L_p$-norm perturbation $(z_j)$, i.e., $\|y_{n_j}w^{1/2} - z_j\|_p \to 0$, such that $(z_j)$ is a martingale difference relative to $(F_{m_j})$. Moreover

$$\|y_{n_j}^2w - z_j^2\|_1 \leq \|y_{n_j}w^{1/2} - z_j\|_2 \|y_{n_j}w^{1/2} + z_j\|_2 \to 0.$$ 

Hence the weak limit of $(z_j^2)$ is $v_0$.

To summarize, we can now assume that $(y_n)$ is a martingale difference sequence where $y_n$ is $G_n$ measurable for all $n$, $G_0 \subset \ldots G_n \subset G_{n+1} \subset \ldots$, $|G_n| < \infty$ for all $n$, $(y_n)$ converges weakly to $v$, $v$ is measurable with respect to the $\sigma$-algebra $G_0$ and the set of all linear combinations of $(y_n)$ with coefficients in the unit sphere of $\ell_2$ is $p$-uniformly integrable. Because $(y_n^2)$ converges weakly to $v$ and the $\sigma$-algebras $(G_n)$ are finite, by passing to a subsequence we may assume that $\|E(y_n^2|G_{n-1}) - v\|_p < 2^{-n}$ for all $n$.

Next we will apply the Central Limit Theorem to the restriction of $(y_n)$ to each of the level sets of $v$. Let $v = \sum_{r=1}^{R} 2^1 A_r$, with $a_r > 0$ for all $r$, $a_r \neq a_s$ and $A_r \cap A_s = \emptyset$ if $s \neq r$. Fix $r$ and consider the probability space $(A_r, P_r)$ where $P_r(S) = P(S)/P(A_r)$ and the corresponding expectation is $E_r$ and let $(z_n)$ be the restriction of $(y_n/a_r)$ to $A_r$. Clearly $(z_n^2)$ converges weakly to $1_{A_r}$ and $(z_n)$ is $p$-uniformly integrable martingale difference.

Let $\epsilon > 0, \delta > 0$, and $M \subset \mathbb{N}, M \neq \emptyset$, and choose $K$ such that $E_r(|z_n|^p 1_{\{|z_n| \geq K\}}) < \delta^{p/2}$. Then if $c = |M|^{-1/2}$ and $|M| > K^2 \epsilon^{-2}$, by Hölder’s and
Chebychev’s inequalities

$$\begin{align*}
E_r(\|cz_n\|^2 1_{\{|cz_n| \geq \epsilon\}}) & \leq c^2 (E_r(\|z_n\|^2 1_{\{|z_n| \geq \epsilon\}}))^{2/p} P_r(\{|z_n| \geq \epsilon\})^{(p-2)/p} \\
& \leq (E_r(\|z_n\|^2 1_{\{|z_n| \geq \epsilon\}}))^{2/p} \epsilon^{-(p-2)} E_r(\|z_n\|^p)^{(p-2)/p}
\end{align*}$$

and thus

$$\sum_{n \in M} E_r(\|cz_n\|^2 1_{\{|cz_n| \geq \epsilon\}}) < \delta |M| |M|^{-p/2} \epsilon^{-(p-2)} \max_{n \in M} E_r(\|z_n\|^p)^{(p-2)/p}.$$ 

It follows that if \((M_j)\) is a sequence of non empty subsets of \(\mathbb{N}\) with \(\max M_j < \min M_{j+1}\) and \(\lim_{j \to \infty} |M_j| = \infty\) then for every \(\epsilon > 0\),

$$\sum_{n \in M_j} E_r(\|M_j\|^{-1/2} |z_n|^2 1_{\{|M_j|^{-1/2} |z_n| \geq \epsilon\}}) \text{ converges in probability to 0. Thus the Lindeberg condition is satisfied.}$$

Our assumption that \(\|E_r(y_n^2|G_{n-1}) - v\|_p < 2^{-n}\) for all \(n\) implies that \(\sum_{n \in M_j} E_r(y_n^2|G_{n-1})\) converges in probability, \(P_r\), to 1. By the Central Limit Theorem \(|M_j|^{-1/2} \sum_{n \in M_j} z_n\) converges in distribution to \(\mathcal{N}(0,1)\).

Let \(w_j = |M_j|^{-1/2} \sum_{n \in M_j} z_n\) for all \(j\). We claim that for any \(\epsilon_1 > 0\) there is a \(J\) such that if \(w\) is a linear combination of \((w_j)_{j \geq J}\) with coefficients in the sphere of \(\ell_2\), then \((1 + \epsilon_1)^{-1} E_r(|g_r|^p) \leq E_r(|w|^p) \leq (1 + \epsilon_1) E_r(|g_r|^p)\) Here \(g_r\) is normally distributed with mean 0 and variance 1 on \((\Omega, P_r)\).

Suppose that for some \(\epsilon_0 > 0\) there is no such \(J\). Then we can find a sequence \((x_k)\) of linear combinations of \((w_j)_{j \geq J}\) with coefficients in the sphere of \(\ell_2\), max \(J_k < \min J_{k+1}\) and \(\epsilon_0\) such that

\[(1 + \epsilon_0)^{-1} E_r(|g_r|^p) > E_r(|x_k|^p),\]

or

\[E_r(|x_k|^p) > (1 + \epsilon_0) E_r(|g_r|^p),\]

for all \(k\). Because \((z_n)\) is \(p\)-uniformly integrable, so is \((x_k)\). The argument above shows that \((x_k)\) converges in distribution to \(\mathcal{N}(0,1)\). Thus \(\lim_{k \to \infty} E_r(|x_k|^p) = E_r(|g_r|^p)\), a contradiction.

There are only finitely many sets \(A_r\) in the representation of \(v\) as a simple function, so we can choose sets \(M_j\) and \(J\) as above so that \((1 + \epsilon_1)^{-1} E_r(|g_r|^p) \leq E_r(|w|^p) \leq (1 + \epsilon_1) E_r(|g_r|^p)\) for all \(r\) and all linear combinations of \((w_j)_{j \geq J}\) with coefficients in the sphere of \(\ell_2\). Thus if \(u_j = |M_j|^{-1/2} \sum_{n \in M_j} y_n\) for \(j \geq J\) and \(u\) is a linear combination of \((u_j)_{j \geq J}\) with coefficients in the sphere of \(\ell_2\), then

\[\sum_{r=1}^R (1 + \epsilon_1)^{-1} a_r^p P(A_r) E_r(|g_r|^p) \leq \sum_{r=1}^R E_r(|u|^p 1_{A_r}) \leq \sum_{r=1}^R (1 + \epsilon_1) a_r^p P(A_r) E_r(|g_r|^p)\]

The orthogonal projection from \(L_p\) onto the closed span of \((u_j)_{j \geq J}\) is bounded by \(\sup\{\|u\|_p/\|u\|_2 : u \in [u_j : j \geq J]\} \leq (1 + \epsilon_1)^{1/p} \|v\|_p^{1/p}\). With a suitable choice of \(\epsilon_1\) the sequence \((u_j/\|u_j\|_p)_{j \geq J}\) satisfies the conclusion of Theorem 1.3.
Remark 2.1. In the last estimate in the proof the comparison is to $\|v\|_p^p \gamma_p^p$ but in fact is really an approximation in distribution. Thus the argument above gives a fairly explicit limiting distribution for the elements of the subspace and the approximating basic sequence is obtained by at most two $\ell_2$-averages of the original basic sequence. If $1 \leq p < 2$ and $(x_n) \subset L_p$ is equivalent to unit vector basis of $\ell_r$, $p < r < 2$, is there a sequence of linear combinations of $(x_n)$ which are close in distribution to a sum of multiples of disjointly supported $r$-stable random variables?

REFERENCES


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