BANACH–SAKS PROPERTIES OF $C^*$-ALGEBRAS AND HILBERT $C^*$-MODULES

MICHAEL FRANK$^1$ AND ALEXANDER A. PAVLOV$^2$

Communicated by M. S. Moslehian

Abstract. The investigation of $C^*$-algebras and Hilbert $C^*$-modules with respect to the classical, the weak and the uniform weak Banach–Saks properties is completed giving a full picture, in particular in the non-unital cases. This way some open questions by M. Kusuda and C.-H. Chu are answered. Criteria and structural characterizations are given. In particular, the weak and the uniform weak Banach–Saks property turn out to be invariant under strong Morita equivalence for non-unital $C^*$-algebras.

1. Introduction

The study of Banach–Saks type properties of Banach spaces was initiated by S. Banach and S. Saks in [6] in 1930. They focused on Banach spaces of type $L^p([0, 1])$ with $1 < p < \infty$. In the sequel a number of case studies appeared. Concrete results were discovered e.g. for Banach spaces of type $C(X)$ where $X$ has been assumed to be a compact metric space (N. R. Farnum, [13]), for symmetric sequence spaces and for certain compact operator algebras over them (J. Arazy, [1]), for commutative and non-commutative $C^*$-algebras deriving criterions (C.-H. Chu, [10], M. Kusuda, [19, 21, 22]), for Banach spaces of vector-valued functions (C. Nuñez, [24]), for Hilbert $C^*$-modules (M. Kusuda, [19, 22]), and for other types of Banach spaces. Concise expositions of known results for Banach

* Corresponding author.

2000 Mathematics Subject Classification. Primary 46B07; Secondary 46L08, 46L05.

Key words and phrases. Banach–Saks properties, $C^*$-algebras, Hilbert $C^*$-modules, Morita equivalence.

Partially supported by the RFBR (grant 07-01-91555) and by the DFG project “K-Theory, $C^*$-Algebras, and Index Theory”.

Date: Received: 22 May 2009; Revised: 30 August 2009; Accepted: 8 September 2009.
spaces have been published e.g. by J. Diestel ([11, 12]). Related investigations can be found in [18, 25, 27].

In the present paper we are interested in Hilbert $C^*$-modules over (non-unital, in general) $C^*$-algebras and in $C^*$-algebras as particular classes of examples for the study of various Banach–Saks type properties of Banach spaces. We are going to demonstrate the overall invariance of three types of Banach–Saks properties in the context of strong Morita equivalence in this context, in particular in the still incompletely understood case of non-unital $C^*$-algebras of coefficients of full Hilbert $C^*$-modules. For the transfer of properties from the non-unital to the unital case, and vice versa, we describe a general new method for Hilbert $C^*$-modules. We get the full picture in the cases of the classical Banach–Saks property, the weak and the uniformly weak Banach–Saks property. Also, we are able to describe the inner structure of $C^*$-algebras and of Hilbert $C^*$-modules with these properties following ideas by C.-H. Chu ([10]).

In the seek for good classes of coefficients of Hilbert $C^*$-modules the class of $C^*$-algebras of compact operators (i.e. of dual $C^*$-algebras) has been emphasized several times, [15, 16, 17]. In the present paper it is emphasized again as one of the invariant with respect to strong Morita equivalence classes with weak and uniformly weak Banach–Saks property. However, there are other invariant with respect to strong Morita equivalence classes of $C^*$-algebras built from this class by exact sequences. So it would be interesting to reveal more about the properties of Hilbert $C^*$-modules over these other classes of $C^*$-algebras in the future.

Another motivation comes from general properties of unbounded regular operators on certain classes of Hilbert $C^*$-modules, because they seem to be related to special kinds of convergence properties of nets and sequences in classes of Hilbert $C^*$-modules over $C^*$-algebras of compact operators (i.e. over dual $C^*$-algebras), cf. [16, 17]. The convergence conditions which serve as definitions of weak and uniformly weak Banach–Saks properties might give hints for a geometrical understanding of the background of the theory of unbounded modular operators.

2. Preliminaries

$C^*$-algebras can be faithfully $*$-represented as norm-closed $*$-subalgebras of sets of all bounded linear operators on fixed Hilbert spaces by the Gel’fand-Naimark-Segal theorem. We use italics for their denotation like $A$. The symbol $A^{**}$ is reserved for the bidual Banach space of $A$, a W*-algebra. The multiplier algebra $M(A)$ of a $C^*$-algebra $A$ can be defined by $M(A) = \{ a \in A^{**} : aA \subseteq A, Aa \subseteq A \}$, [26].

A (left) pre-Hilbert $C^*$-module over a (not necessarily unital) $C^*$-algebra $A$ is a (left) $A$-module $E$ equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$, which is $A$-linear in the first variable and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geq 0 \quad \text{with equality if and only if } x = 0.$$

We always suppose that the linear structures of $A$ and $E$ are compatible. A pre-Hilbert $A$-module $E$ is called a Hilbert $A$-module if $E$ is a Banach space with respect to the norm $\| x \| = \| \langle x, x \rangle \|^{1/2}$. A Hilbert $A$-module is full if the range
of the $A$-valued inner product on $E$ is dense in $A$. If $E$, $F$ are two Hilbert $A$-modules then the set of all ordered pairs of elements $E \oplus F$ from $E$ and $F$ is a Hilbert $A$-module with respect to the $A$-valued inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_E + \langle y_1, y_2 \rangle_F$. It is called the direct orthogonal sum of $E$ and $F$.

Beside the Hilbert $A$-modules we shall consider $A$-linear bounded operators $T$ on them. The set $\text{End}_A(E)$ of all bounded module operators on $E$ forms a Banach algebra, whereas the set $\text{End}_A^r(E)$ of all bounded module operators which possess an adjoint operator inside $\text{End}_A(E)$ has the structure of a unital $C^*$-algebra. Note that these two sets do not coincide in general. An important subset of $\text{End}_A^r(E)$ is the set $K_A(E)$ of “compact” operators, which is defined as the norm-closure of the set $K^*_A(E)$ of all finite linear combinations of the specific operators

$$\{\theta_{x,y} \in \text{End}_A(E) : x, y \in E, \theta_{x,y}(z) = \langle z, x \rangle y \text{ for every } z \in E\}.$$

It is a $C^*$-subalgebra and a two-sided ideal of $\text{End}_A^r(E)$, and the $C^*$-algebra $\text{End}_A^r(E)$ can be isometrically identified with the multiplier algebra of $K_A(E)$ ([28]). Note, that $E$ is a right (full!) Hilbert $K_A(E)$-module at the same time, with $K_A(E)$-valued inner product $\langle x, y \rangle_{op} = \theta_{x,y}$ for $x, y \in E$. If $E$ is a full Hilbert $A$-module then the picture is symmetric - the $C^*$-algebras of coefficients $A$ and $K_A(E)$ have equal rights with respect to $E$.

Two $C^*$-algebras $A$ and $B$ are strongly Morita equivalent if there exists a left full Hilbert $A$-module $E$ which is a right full Hilbert $B$-module at the same time, with the properties (i) $\langle x, y \rangle_{A}z = \langle y, z \rangle_{B}$ for any $x, y, z \in E$, (ii) $A$ and $B$ act as the module operator $C^*$-algebras of “compact” $B$-linear and $A$-linear operators on $E$ respectively. In this situation $E$ is called an $A$-$B$ imprimitivity bimodule. For the $A$-$B$ imprimitivity bimodule $E$ there exists a related $C^*$-algebra $L = K_A(A \oplus E)$ over the orthogonal sum $A \oplus E$ of the Hilbert $A$-modules $A$ and $E$. The $C^*$-algebra $L$ is called the linking algebra, cf. [7, 9]. The multiplier algebra $M(L)$ contains two orthogonal projections $p, q$ such that $1_{M(L)} = p + q$, $pLp = K_A(A) = A$, $qLq = K_A(E) = B$ and $qLp = K_A(A, E) = E$ and $pLq = K_A(E, A) = \tilde{E}$ - the dual to $E$ $B$-$A$ imprimitivity bimodule, cf. [9, §2]. So one may write

$$L = \begin{pmatrix} A & E \\ \tilde{E} & B \end{pmatrix}.$$

3. A general construction

In the present section we are going to construct canonical extensions for any full Hilbert $C^*$-module $E$ over a non-unital $C^*$-algebra $A$ that are full Hilbert $B$-modules $E_B$ for a given $C^*$-algebra $A \subset B \subseteq M(A)$ with a canonical $A$-linear isometric embedding $\Gamma : E \to E_B$, where $AE_B \equiv \Gamma(E)$, and with $*$-isomorphic $C^*$-algebras of adjointable bounded module operators $\text{End}_A^r(E) \equiv \text{End}_B^r(E_B) \equiv \text{End}_M^*(E_M(A))$ which are identified by the respective operator extensions of bounded module operators via $\Gamma$, cf. [5, Thm. 2.3]. In particular, we associate to every Hilbert $A$-module $E$ over a non-unital $C^*$-algebra $A$ a canonical Hilbert $A_1$-module $E_{A_1} := E_c$ over its unitization $A_1$. It will be the right object to solve
the open problems with the investigation of various Banach–Saks properties of Hilbert $C^*$-modules and $C^*$-algebras in the non-unital case.

We rely on the respective canonical construction of a multiplier module $E_{M(A)} := E_d$ for any full Hilbert $C^*$-module $E$ introduced by D. Bakić and B. Guljaš in [4, 5]. Let $E_d$ denote the Hilbert $C^*$-module $\text{End}_A^*(A, E)$ over the multiplier algebra $M(A)$ of $A$ consisting of all adjointable bounded $A$-linear maps from $A$ into $E$. Its $M(A)$-valued inner product is defined by the formula $\langle x, y \rangle = x^*y$. In [3, Thm. 1.2] $E_d$ is proved to be the largest essential extension of $E$ in full analogue to the respective property of multiplier algebras for $C^*$-algebras. Moreover, the map $\Gamma : E \to E_d$ defined by $a\Gamma(x) = ax$ with $a \in A$, $x \in E$ is an isometric $A$-linear embedding, and the image $\Gamma(E)$ coincides with the subset $AE_d$. The sets of adjointable bounded $C^*$-linear operators on both $E_d$ and on $\Gamma(E)$ coincide, i.e. $\text{End}_A^*(E) \equiv \text{End}_{M(A)}^*(E_d)$. The Hilbert $M(A)$-module $E_d$ is a full Hilbert $M(A)$-module in case $E$ has been a full Hilbert $A$-module. Now, define

$$E_{A_1} := E_c := \{ x \in E_d : \langle x, x \rangle \in A_1 \},$$

$$E_B := \{ x \in E_d : \langle x, x \rangle \in B \}.$$

Obviously, the sets $E_B \subseteq E_d$ are invariant under the action of $B \subseteq M(A)$. Furthermore, for any positive functional $f : M(A) \to \mathbb{C}$ vanishing on $B \subseteq M(A)$ the bilinear form $f(\langle ., . \rangle)$ is a semi-inner product on $E_d$. Therefore, the triangle inequality gives

$$f((ax + by, ax + by))^{1/2} \leq f((ax, ax))^{1/2} + f((by, by))^{1/2}$$

for any $a, b \in B$, $x, y \in E_c$. Since the set of all positive functionals on $M(A)$ that vanish on $B \subseteq M(A)$ characterizes precisely elements of the subset $B$ in $M(A)$ the set $E_B$ turns out to be a $B$-module. So $E_B$ is a Hilbert $B$-module which contains the isometric copy $\Gamma(E)$ of the Hilbert $A$-module $E$ in such a way that $AE_B = \Gamma(E)$. In terms of [3, Def. 1.1] the triple $(E_B, B, \Gamma)$ is an essential extension of the Hilbert $A$-module $E$.

Denote by $\pi : B \to B/A$ and by $q : E_B \to E_B/E$ the quotient maps. The left action of the $C^*$-algebra $B/A$ on the $B/A$-module $E_B/E$ is defined by $\pi(a)q(x) = q(ax)$ for any $a \in B$, $x \in E_B$, cf. [5, Def. 1.4]. Moreover, $E_B/E$ can be equipped with a $B/A$-valued inner product setting $\langle q(x), q(y) \rangle = \pi(\langle x, y \rangle)$ for any $x, y \in E_B$, turning $E_B/E$ into a Hilbert $B/A$-module, [5, Thm. 1.6].

Now, let us resort to the case of $B = A_1$, the unitization of $A$. Consider the exact sequence $0 \to E \to E_c \to E_c/E \to 0$ of Hilbert $C^*$-modules. In this final step we shall prove that the Banach space $E_c$ always splits into a direct sum of the Banach spaces $E$ and an isometric copy of the Hilbert space $E_c/E$. Consider the short exact sequence

$$0 \to E \to E_c \to E_c/E \to 0$$

and an orthonormal basis $\{ e_\alpha : \alpha \in I \}$ of the Hilbert space $E_c/E$. Let $\rho : A_1 \to A$ be the linear bounded map which is the projection onto the first summand in the direct sum decomposition $A_1 = A + \mathbb{C}1$. Define a candidate for a new norm on $E_c$ setting

$$\|x\|_c := \|\rho(\langle x, x \rangle_{E_c})\|_A^{1/2} + |(\text{id}_{A_1} - \rho)(\langle x, x \rangle_{E_c})|^{1/2}.$$
The only non-trivial part is to demonstrate subadditivity of \( \| \cdot \|_c \). Since the second summand is generated by a single state on \( A_1 \) it fulfills the triangle inequality. So we focus on the first summand. Let \( f \) be a state on \( A \). Then \( f \circ \rho \) is a state on \( A_1 \), and \( f(\rho(\langle \cdot , \cdot \rangle_{E_c}))^{1/2} \) is a semi-norm on \( E_c \) fulfilling the triangle inequality. For given elements \( x, y \in E_c \) fix a (existing by [8, Lemma 2.3.23]) state \( f_0 \) on \( A_1 \) such that \( f_0(\rho(\langle x + y, x + y \rangle_{E_c}))^{1/2} = \| \rho(\langle x + y, x + y \rangle_{E_c}) \| \). Then

\[
\| \rho(\langle x + y, x + y \rangle_{E_c}) \|^{1/2} = \| \rho(\langle x + y, x + y \rangle_{E_c}) \|^{1/2} \\
\leq f_0(\rho(\langle x, x \rangle_{E_c}))^{1/2} + f_0(\rho(\langle y, y \rangle_{E_c}))^{1/2} \\
\leq \| \rho(\langle x, x \rangle_{E_c}) \|^{1/2} + \| \rho(\langle y, y \rangle_{E_c}) \|^{1/2} \\
= \| \rho(\langle x + x \rangle_{E_c}) \|^{1/2} + \| \rho(\langle y + y \rangle_{E_c}) \|^{1/2}
\]

So the mapping \( \| \cdot \|_c \) is really a norm the equivalence of which to the Hilbert norm on \( E_c \) follows from the way of construction of it as a special norm on \( A_1 \) equivalent to the \( C^* \)-norm on \( A_1 \), combined with the \( A_1 \)-valued inner product on \( E_c \).

For any \( \alpha \in I \) the set \( M_\alpha = \{ y \in E_c : q(y) = e_\alpha \} \) is not empty and, moreover, the difference of any two elements of \( M_\alpha \) belongs to \( E_c \setminus \Gamma(E) \subseteq E_c \), i.e. these sets are sufficiently rich. Now, for any \( \alpha \in I \) select an element \( y_\alpha \in M_\alpha \) such that \( q(y_\alpha) = e_\alpha \). By construction

\[
\langle e_\alpha, e_\beta \rangle_{E_c/E} = (\text{id}_{A_1} - \rho)(\langle y_\alpha, y_\beta \rangle_{E_c}) = \pi(\langle y_\alpha, y_\beta \rangle_{E_c}) = \langle q(y_\alpha), q(y_\beta) \rangle_{E_c/E}
\]

for any \( \alpha, \beta \in I \). Therefore, fixing an index \( \alpha \in I \), we have

\[
1 = \langle e_\alpha, e_\alpha \rangle_{E_c/E}^{1/2} \\
= \inf_{y \in M_\alpha} \| y \|_c \\
= \inf_{z \in \Gamma(E)} \| y_\alpha - z \|_c \\
= \inf_{z \in \Gamma(E)} \| \rho(\langle y_\alpha - z, y_\alpha - z \rangle_{E_c}) \|^{1/2} + \langle e_\alpha, e_\alpha \rangle_{E_c/E}^{1/2}
\]

forcing \( \inf_{z \in \Gamma(E)} \| \rho(\langle y_\alpha - z, y_\alpha - z \rangle_{E_c}) \| = 0 \). Thus, there has to exist a sequence \( \{ z_{\alpha,k} : k \in \mathbb{N} \} \subseteq \Gamma(E) \subseteq E_c \) with the property \( \lim_{k \to \infty} \| \rho(\langle y_\alpha - z_{\alpha,k}, y_\alpha - z_{\alpha,k} \rangle_{E_c}) \| = 0 \). In other words, the sequence \( \{ z_{\alpha,k} : k \in \mathbb{N} \} \subseteq \Gamma(E) \) is a Cauchy sequence, and since \( \Gamma(E) \) is complete in \( E_c \) there exists a norm-limit \( z_\alpha \) of \( \{ z_{\alpha,k} \} \) inside \( \Gamma(E) \subseteq E_c \). However, \( \rho(\langle y_\alpha - z_\alpha, y_\alpha - z_\alpha \rangle) = 0 \) in \( A \). So for every element \( e_\alpha \) of the selected orthonormal basis of \( E_c/E \) we obtain an element \( y'_\alpha \in M_\alpha \) such that \( \rho(\langle y'_\alpha, y'_\alpha \rangle_{E_c}) = 0 \) and \( (\text{id}_{A_1} - \rho)(\langle y'_\alpha, y'_\alpha \rangle_{E_c}) = \langle e_\alpha, e_\alpha \rangle_{E_c/E} \). Now, consider the linear subspace \( H_c \) of \( E_c \) which arises as the norm-closed linear hull of the selected elements \( \{ y'_\alpha : \alpha \in I \} \subseteq E_c \). By construction, \( E_c = \Gamma(E) + H_c \) and \( H_c \) is isometrically isomorphic to the Hilbert space \( E_c/E \), since \( \pi(\langle y, y \rangle_{E_c}) = \langle q(y), q(y) \rangle_{E_c/E} \) for any \( y \in H_c \) by construction.

It is not clear to us, whether the splitting works always for general \( C^* \)-algebras \( A \subseteq B \subseteq M(A) \) and general Hilbert \( A \)-modules \( E \), or not, since the Hilbert
$B/A$-module $E_B/E$ might not contain neither orthonormal basises nor module frames, cf. [23]. Here some further investigations have to be made in the future.

We would like to shed some light on the construction of the Hilbert $A_1$-module $E_c$ from the Hilbert $A$-module $E$ in the case of non-unital $C^*$-algebras $A$ and full Hilbert $C^*$-modules. If $A = E$ it is just the construction of an unitization of the $C^*$-algebra $A$, i.e. $A_1 = E_c$ as a Hilbert $A_1$-module. However, in more general situations the construction of $E_c$ from $E$ has the character of a maximally possible extension of $E$ keeping the coefficients of the extended $C^*$-valued inner product still in $A_1$. Full Hilbert $A$-modules $E$ give rise to full Hilbert $A_1$-modules $E_c$ by this construction.

**Example 3.1.** Let $A$ be a non-unital $C^*$-algebra and $E$ be a full Hilbert $A$-module. The key idea of the previous proof was to complete $E$ to the extend to force the extension to be a full Hilbert $A_1$-module, where $A_1 = A + \mathbb{C}1$. The first idea which comes to mind is an operation like “adding an identity to a Hilbert $C^*$-module” analogous to the existing minimal $C^*$-extension of $C^*$-algebras which is unique up to isometric $*$-isomorphisms. However, the construction of $E_c$ from $E$ celebrated in the previous proof has more of a minimax principle as the following example shows. For the $C^*$-algebras $K(l_2)$ and $B(l_2)$ of all compact and bounded linear operators on the separable Hilbert space $l_2$, respectively, set

$$A = \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix},$$

$$E = \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & K(l_2) \end{pmatrix} \oplus \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & K(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix}.$$

Two corresponding minimal extensions of the Hilbert $A$-module $E$ to Hilbert $A_1$-modules are

$$E_1 = \begin{pmatrix} K(l_2) + \mathbb{C}1 & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & K(l_2) \end{pmatrix} \oplus \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & K(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix},$$

$$E_2 = \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & K(l_2) \end{pmatrix} \oplus \begin{pmatrix} K(l_2) + \mathbb{C}1 & 0 & 0 \\ 0 & K(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix}.$$

Obviously, $E_1$ and $E_2$ are minimal extensions of the sought kind, however they are non-isomorphic as Hilbert $A_1$-modules. So the construction of

$$E_c = \begin{pmatrix} K(l_2) + \mathbb{C}1 & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & K(l_2) \end{pmatrix} \oplus \begin{pmatrix} K(l_2) + \mathbb{C}1 & 0 & 0 \\ 0 & K(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix}$$

gives a more correct result, the maximum of all possible minimal essential extensions of the Hilbert $A$-module $E$ to a Hilbert $A_1$-module. The Hilbert space
$H_c \subseteq E_c$ constructed in the proof above equals to

$$H_c = \begin{pmatrix} C1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} C1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the present example.

4. The classical Banach–Saks property

The aim of the reminder of the present paper is to complete the classification of $C^*$-algebras and Hilbert $C^*$-modules which are Banach spaces with (classical, weak, uniform weak) Banach–Saks properties. Special emphasis is put on the invariance of Banach–Saks properties with respect to strong Morita equivalence. We focus on the still open non-unital case since the appropriate results for the unital case have been discovered by Cho-Ho Chu and M. Kusuda in [10, 19, 20, 21, 22].

A Banach space $E$ has the Banach–Saks property if every bounded sequence $\{x_n\}_n \subset E$ has a subsequence $\{x_{n(k)}\}_k$ such that the derived from it sequence of partial arithmetic means converges in norm, i.e.

$$\lim_{k \to \infty} \left\| \frac{x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)}}{k} - y \right\| = 0$$

with some $y \in E$. It is known that Banach spaces $E$ with the Banach–Saks property have to be reflexive as normed spaces, [11, p. 85]. Therefore, $C^*$-algebras with the Banach–Saks property have to be finite-dimensional linear spaces, i.e. a finite direct sum of unital matrix algebras ([22, Lemma 3.1]). The following proposition has been proved by M. Kusuda for the unital case, cf. [22, Thm. 3.6]:

**Proposition 4.1.** Let $A$ be a (non-unital, in general) $C^*$-algebra and $E$ be a full Hilbert $A$-module. Suppose, that $E$ has the Banach–Saks property. Then $A$ has to be finite-dimensional as a linear space, i.e. $A$ is a finite direct sum of unital matrix algebras. In particular, any full Hilbert $A$-module over a non-trivial non-unital $C^*$-algebra $A$ does not possess the Banach–Saks property, neither such $C^*$-algebras $A$ themselves.

**Proof.** Let $A$ be a non-unital $C^*$-algebra and $E$ be a full Hilbert $A$-module. Suppose, $E$ has the Banach–Saks property. Construct the Hilbert $A_1$-module $E_c$ from $A$ and from $E$. By the results of the previous section the Hilbert $A_1$-module $E_c$ can be written as a direct sum of the Banach subspace $\Gamma(E)$ of $E_c$ and of a Banach space $H_c$ which is isometrically isomorphic to the Hilbert space $E_c/E$. Both the isometric copy $\Gamma(E) \subset E_c$ and the Hilbert space $H_c$ have the Banach–Saks property, by supposition or as a matter of fact, respectively. So the (full) Hilbert $A_1$-module $E_c$ has the Banach–Saks property, too, since it is the direct sum of two Banach spaces with that property. By [22, Thm. 3.6] the $C^*$-algebra $A_1$ has to be finite-dimensional as a Banach space, i.e. it has to be a matrix algebra with finite center. However, for non-trivial non-unital $C^*$-algebras $A$ their unitization $A_1$ is never a finite-dimensional $C^*$-algebra, a contradiction. \[\square\]
Corollary 4.2. Let $A$ be a (non-unital, in general) $C^*$-algebra. Then $A$ has the Banach–Saks property if and only if $A$ is reflexive as a Banach space.

The fact follows immediately from the general representation theory of $C^*$-algebras because a $C^*$-algebra is reflexive as a Banach space if and only if it is finite-dimensional as a linear space. It is non-trivial since for Banach spaces reflexivity does not imply the Banach–Saks property, in general.

Corollary 4.3. Let $A$ be a $C^*$-algebra with Banach–Saks property and $E$ be a Hilbert $A$-module. Then $E$ has the Banach–Saks property.

Proof. We know that $A$ has to be finite-dimensional and unital. Since any two-sided ideal of $A$ is finite-dimensional, too, and has the Banach–Saks property, we can assume $E$ to be full, without loss of generality. Moreover, the $C^*$-algebra admits a faithful trace functional $tr(\cdot)$, and the norms $\|\langle \cdot, \cdot \rangle\|^{1/2}$ and $tr(\langle \cdot, \cdot \rangle)^{1/2}$ are equivalent on $E$. So $E$ admits a $C$-valued inner product $tr(\langle \cdot, \cdot \rangle)$ and, hence, the structure of a Hilbert space. However, Hilbert spaces are known to possess the Banach–Saks property, so does $E$. \qed

5. The weak Banach–Saks property

More interesting is the weak Banach–Saks property which is defined in the following way: if for any given weakly null sequence $\{x_n\}_n$ of a Banach space $E$, one can extract a subsequence $\{x_{n(k)}\}_k$ such that the derived from it sequence of partial arithmetic means converges in norm to zero, i.e.

$$\lim_{k \to \infty} \left\| \frac{x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)}}{k} \right\| = 0,$$

then $E$ is said to admit the weak Banach–Saks property. Note, that the weak Banach–Saks property inherits to any (closed) subspace of a Banach space with weak Banach–Saks property by definition. Beside this, if $A$ is a non-unital $C^*$-algebra and $A_1 = A + C1$ is its unitization, then $A$ has the weak Banach–Saks property if and only if $A_1$ has the weak Banach–Saks property, [10]. For $C^*$-algebras and Hilbert $C^*$-modules as classes of Banach spaces we prove the following fact relying on a key result by M. Kusuda [22, Thm. 2.2] and on a new technique for a certain standard extension of full Hilbert $C^*$-modules over non-unital $C^*$-algebras:

Theorem 5.1. Let $A$ and $B$ be two strongly Morita equivalent $C^*$-algebras and $E$ be an $A$-$B$ imprimitivity bimodule. The following four conditions are equivalent:

(i) $A$ has the weak Banach–Saks property.
(ii) $B$ has the weak Banach–Saks property.
(iii) $E$ has the weak Banach–Saks property.
(iv) $L$ has the weak Banach–Saks property.

Proof. By [22, Thm. 2.2] the first three conditions are equivalent if either $A$ or $B$ are unital. If neither $A$ nor $B$ are unital then the conditions (i) and (ii) are equivalent and imply condition (iii). So we are going to show that for non-unital $A$ and $B$ condition (iii) implies both conditions (i) and (ii).
Let $A$ be a $C^*$-algebra and $E$ be a full Hilbert $A$-module with the weak Banach–Saks property. Consider the derived Hilbert $A_1$-module $E_c$ of $E$, cf. third section. As demonstrated, $E_c$ can be decomposed into the direct sum of two Banach spaces, $\Gamma(E) \subset E_c$ isometrically isomorphic to the Hilbert $A$-module $E$ and $H_c$ isometrically isomorphic to the Hilbert space $E_c/E$. Both these summands admit the weak Banach–Saks property by supposition or by the equivalence of the conditions (i) and (iii) above for the $C^*$-algebra of all complex numbers $\mathbb{C}$, so does their direct sum $E_c$. Because $E_c$ is a full Hilbert $A_1$-module, the already proven equivalence (i)$\equiv$(iii) for unital $C^*$-algebras forces $A_1$ to admit the weak Banach–Saks property, and so its Banach subspace $A$ has the weak Banach–Saks property, too, what is to demonstrate.

The equivalent conditions (i)-(iii) imply $L$ to admit the weak Banach–Saks property since $L$ has a block structure consisting of these building blocks and of their (anti-)isomorphic copies. Conversely, if $L$ has the weak Banach–Saks property then each of its linear subspaces admits the same property. □

6. The uniform weak Banach–Saks property and structure theorems

Relying on the classical results by N. R. Farnum [13] and by C.-H. Chu [10] we can derive a number of concrete results from Theorem 5.1. The key role is played by dual $C^*$-algebras, i.e. by $C^*$-algebras that admit a faithful $*$-representation in some $C^*$-algebra $K_C(H)$ of all linear compact operators on some Hilbert space $H$, cf. [2]. C.-H. Chu proved in [10] that a $C^*$-algebra $A$ has the weak Banach–Saks property if and only if $A$ admits a finite chain $\{I_i\}$ of two-sided norm-closed ideals such that $I_1 = \{0\}$, $I_n = A$ and any $I_{i+1}/I_i$, $i = 0, ..., n - 1$, is a dual $C^*$-algebra. So the class of dual $C^*$-algebras is one large class of $C^*$-algebras with weak Banach–Saks property, cf. [16, 17]. Note, that the class of dual $C^*$-algebras is invariant under strong Morita equivalence. Another class of such $C^*$-algebras can be constructed by taking finite block-diagonal direct sums of $C^*$-algebras with weak Banach–Saks property. However, there are far more $C^*$-algebras with weak Banach–Saks property, even unital ones, which can be easily constructed:

Example 6.1. Let $A$ be any non-unital dual $C^*$-algebra, for example $A = \ell_2$ or $A = K_C(l_2)$. Then the unitization $A_1$ of $A$ always serves as an example of a non-dual, unital $C^*$-algebra with weak Banach–Saks property. Indeed, $\{0\} \subset A \subset A_1 = A + \mathbb{C} \cdot 1$. The example is non-trivial since the short exact sequence $0 \to A \to A_1 \to A_1/A = \mathbb{C} \to 0$ does not split as an exact sequence of $C^*$-algebras for any non-unital dual $C^*$-algebra $A$. (It always splits as an exact sequence of Banach spaces.)

A third Banach–Saks type property of Banach spaces has been introduced by C. Nuñez in [24]. A Banach space $E$ has the uniform weak Banach–Saks property if there is a null sequence $\{\delta_n\}_n$ of positive real numbers such that, for any weakly null sequence $\{x_n\}_n$ in $E$ with uniform bound $\|x_n\| \leq 1$ and for any natural number $k$, there exist natural numbers $n(1) < n(2) < \cdots < n(k)$ such
\[
\frac{\|x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)}\|}{k} < \delta_k.
\]

C.-H. Chu has shown in [10, Thm. 2] that \(C^*\)-algebras are Banach spaces for which the uniform weak and the weak Banach–Saks properties are equivalent. In [22, Thm. 2.2, Cor. 2.3] M. Kusuda found that for full Hilbert \(C^*\)-modules over unital \(C^*\)-algebras both these properties are equivalent. Moreover, for full Hilbert \(C^*\)-modules over \(C^*\)-algebras with weak Banach–Saks property again both these properties hold at the same time, cf. [19, Thm. 2.3] and [22, Thm. 2.2]. So we can formulate an analog to Theorem 5.1:

**Theorem 6.2.** Let \(A\) and \(B\) be two strongly Morita equivalent \(C^*\)-algebras and \(E\) be an \(A\)-\(B\) imprimitivity bimodule. The following four conditions are equivalent:

\[(i)\ A\ has\ the\ uniform\ weak\ Banach–Saks\ property.
(ii)\ B\ has\ the\ uniform\ weak\ Banach–Saks\ property.
(iii)\ E\ has\ the\ uniform\ weak\ Banach–Saks\ property.
(iv)\ L\ has\ the\ uniform\ weak\ Banach–Saks\ property.
\]

In particular, conditions (i)-(iv) hold in case either \(A\) or \(B\) or \(E\) or \(L\) have the weak Banach–Saks property. Conversely, either of conditions (i)-(iv) implies \(A\), \(B\), \(E\) and \(L\) to have the weak Banach–Saks property.

Theorem 6.2 gives the opportunity to describe the inner structure of Hilbert \(C^*\)-modules with the weak or uniform weak Banach–Saks property.

**Proposition 6.3.** Let \(A\) be a \(C^*\)-algebra and \(E\) be a full Hilbert \(A\)-module with the weak or uniform weak Banach–Saks property. Then there exist a finite sequence \(\{E_i : i = 0, ..., l\}\) of norm-closed \(A\)-submodules of \(E\) and a sequence \(\{I_i : i = 0, ..., l\}\) of two-sided norm-closed ideals of \(A\) such that

\[(i)\ I_1 = A, I_{i-1} \subset I_i\ and\ I_{i-1}\ is\ a\ two-sided\ ideal\ of\ I_i\ for\ any\ i = 1, ..., l.
(ii)\ The\ \(C^*\)-algebra \(I_0\) and the factor \(C^*\)-algebras \(\{I_i/I_{i-1} : i = 1, ..., l\}\) are dual \(C^*\)-algebras.
(iii)\ \(E_i = E, E_{i-1} \subset E_i\ and\ the\ Hilbert\ \(A\)-modules\ \(E_i\)\ are\ full\ Hilbert\ \(I_i\)-modules\ for\ any\ i = 0, ..., l.\ In\ particular,\ the\ values\ \langle x, y \rangle\ belong\ to\ \(I_i\)\ for\ any\ x \in E_i\ and\ any\ y \in E_i\ with\ j \geq i, i, j = 0, ..., l.\ The\ factor\ modules\ \(E_i/E_{i-1}\)\ are\ Hilbert\ \(C^*\)-modules\ over\ the\ dual\ \(C^*\)-algebras\ \(I_i/I_{i-1}\).\]

**Proof.** Since the full Hilbert \(A\)-module \(E\) has the weak or the uniform weak Banach–Saks property, the \(C^*\)-algebra of coefficients has the same property by Theorem 6.2, as well as the linking algebra \(L\). By [10] there exists a finite sequence \(\{J_i : i = 0, ..., l\}\) of two-sided norm-closed ideals of \(L\) such that the \(C^*\)-algebra \(J_0\) and the factor \(C^*\)-algebras \(J_i/J_{i-1}, i = 1, ..., l\), are dual \(C^*\)-algebras, \(J_{i-1} \subset J_i\), and \(J_{i-1}\) is a two-sided ideal of \(J_i\) for any \(i = 1, ..., l\), and \(J_l = L\). For the pair of orthogonal projections \(p, q\) associated to the linking algebra \(L\) of \(E\) set \(E_i = pJ_iq\) and \(I_i = pJ_iq\) for \(i = 0, ..., l\). The sets \(I_i\) are \(C^*\)-algebras and the sets \(E_i\) are full Hilbert \(I_i\)-modules, \(i = 0, ..., l\). The demonstration of the particular properties of these sets listed in the proposition above is an easy exercise. \(\square\)
As a conclusion we have characterized two strongly Morita equivalent classes of $C^*$-algebras which admit the weak and uniform weak Banach–Saks property: the $C^*$-algebras of compact operators (i.e. the dual $C^*$-algebras) and the $C^*$-algebras constructed by finite chains of growing ideals the pairwise quotients of which are again $C^*$-algebras of compact operators. The latter class may be divided into a countable set of strongly Morita invariant subclasses by the length of the decomposing chains of ideals.

In a forthcoming paper we will study modular analogues of the Schur and of various types of Banach–Saks properties for Hilbert $C^*$-modules ([14]).

Acknowledgement. The authors are grateful to the referee for his/her valuable comments.

References


1 Hochschule für Technik, Wirtschaft und Kultur (HTWK) Leipzig, Fachbereich IMN, PF 301166, D-04251 Leipzig, F.R. Germany
E-mail address: mfrank@imn.htwk-leipzig.de

2 Moscow State University, 119 922 Moscow, Russia, and Università degli Studi di Trieste, I-34127 Trieste, Italy
E-mail address: axpavlov@mail.ru