REVERSE OF THE GRAND FURUTA INEQUALITY
AND ITS APPLICATIONS

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\textit{This paper is dedicated to Professor J.E. Pečarić}

Submitted by A. R. Villena

Abstract. We shall give a norm inequality equivalent to the grand Furuta inequality, and moreover show its reverse as follows: Let $A$ and $B$ be positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then

$$
\| A^{\frac{1}{2}} \{ A^{-\frac{1}{2}} (A^{-\frac{1}{2}} B^{\frac{1}{2}} (A^{-\frac{1}{2}})^{1-s} A^{\frac{1}{2}})^{p} A^{\frac{1}{2}} \} \| \\
\leq K(h^{r-t}, \frac{(p-t)s + r}{1-t+r})^{\frac{1}{ps}} \| A^{\frac{1-t+r}{2}} B^{r-t} A^{\frac{1-r+s}{2}} \| \frac{(p-t)s + r}{1-t+r}
$$

for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$ and $r \geq t \geq 0$, where $K(h, p)$ is the generalized Kantorovich constant. As applications, we consider reverses related to the Ando-Hiai inequality.

1. Introduction

The origin of reverse inequalities is the Kantorovich inequality. It says that if a positive operator $A$ on a Hilbert space $H$ satisfies $0 \leq m \leq A \leq M$, then

$$
\langle A^{-1}x, x \rangle \leq \frac{(M + m)^2}{4Mm} \langle Ax, x \rangle^{-1} \quad \text{for all unit vectors } x \in H.
$$

(K)
The point in $\mathbb{K}$ is the convexity of the function $t \mapsto t^{-1}$. Mond and Pečarić turned their attention to the convexity of functions, and established the so-called Mond-Pečarić method in the theory of reverse inequalities, see [13] in detail. The subject of this note is just on the line of Mond-Pečarić’s idea, and our target is the grand Furuta inequality.

Let $A$ and $B$ be positive (bounded linear) operators acting on a Hilbert space. The grand Furuta inequality [10] says that

$$A \geq B \geq 0 \Rightarrow A^{1-t+r} \geq \{A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-r)(1+r)}}$$

for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$ and $r \geq t$.

The inequality (GFI) is considered as a parametric formula interpolating the Furuta inequality (FI) and Ando-Hiai one (1.1), respectively [9] and [1]:

$$A \geq B \geq 0 \Rightarrow A^{1+r} \geq (A^r B^p A^r)^{\frac{1+r}{p+r}} \quad (r \geq 0, p \geq 1) \quad (FI)$$

and

$$A \geq B \geq 0 \Rightarrow A^r \geq \{A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\}^\frac{1}{s} \quad (p, r \geq 1). \quad (1.1)$$

Now the Furuta inequality appeared as a useful extension of the so-called Löwner-Heinz inequality (cf. [14]):

$$A \geq B \geq 0 \Rightarrow A^\alpha \geq B^\alpha \quad (0 \leq \alpha \leq 1). \quad (1.2)$$

This Löwner-Heinz inequality (1.2) is equivalent to the Araki-Cordes inequality ([2], [4]):

$$\| A^{\frac{p}{2}} B^p A^{\frac{p}{2}} \| \leq \| A^{\frac{1}{2}} B A^{\frac{1}{2}} \|^p \quad (0 \leq p \leq 1). \quad (1.3)$$

M.Fujii and Y.Seo [8] gave a reverse inequality of the Araki-Cordes inequality: If $A$ and $B$ are positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} (> 1)$, then

$$K(h, p) \| A^{\frac{p}{2}} B A^{\frac{p}{2}} \|^p \leq \| A^{\frac{1}{2}} B^p A^{\frac{1}{2}} \| \quad (0 \leq p \leq 1) \quad (1.4)$$

where a generalized Kantorovich constant $K(h, p)$ is defined as follows:

$$K(h, p) := \frac{1}{h-1} \left( p \left( \frac{h^p - h}{h^p - h} \right)^{p-1} \left( \frac{p-1}{h^p - h} \right)^p \right)$$

for all $h(\neq 1), p \in \mathbb{R}$ and $K(h, 0) = K(h, 1) = 1$, see [11] and [13].

In this note, we first give a norm inequality equivalent to the grand Furuta inequality (GFI). Based on this, we show a reverse inequality of (GFI), in which the generalized Kantorovich constant (1.5) is used. As an application, we obtain reverses of a generalization of Ando-Hiai inequality (1.1).

2. Norm Inequality equivalent to the Grand Furuta Inequality

The grand Furuta inequality (GFI) is equivalent to the following norm inequality:
Lemma 2.1. Let $A$ and $B$ be positive operators. Then the grand Furuta inequality (GFI) is equivalent to
\[
\| A^{\frac{1-s}{2}} B^{t-s} A^{\frac{1-t}{2}} \|_{\frac{p}{p-1}+\frac{r}{r-1}} \leq \| A^\frac{1}{2} \{ A^{\frac{1}{2}} B \frac{(p-1)(r-1)}{p(r-1)} A^{\frac{1}{2}} \} \frac{1}{2} A^\frac{1}{2} \| \tag{2.1}
\]
for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$ and $r \geq t$.

Proof. Replace $A$ to $A^{-1}$ and put
\[
C = \{ A^{\frac{1}{2}} B \frac{(p-1)(r-1)}{p(r-1)} A^{\frac{1}{2}} \} \frac{1}{2}
\]
in (2.1). Since $B^{-t} = \{ A^{\frac{1}{2}} C^p A^{-\frac{1}{2}} \} \frac{1}{p} A^{\frac{1}{2}}$, we have
\[
\| A^{\frac{1-s}{2}} \{ A^{\frac{1}{2}} C^p A^{-\frac{1}{2}} \} \frac{1}{s} \leq \| A^{\frac{1}{2}} C A^{\frac{1}{2}} \|.
\]
This is equivalent to the inequality
\[
A \geq C \Rightarrow A^{1-t+r} \geq \{ A^{\frac{1}{2}} C^p A^{-\frac{1}{2}} \} \frac{1}{2},
\]
that is, (2.1) is equivalent to the grand Furuta inequality (GFI).

Corollary 2.2. Let $A$ and $B$ be positive operators. Then
\[
\| A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1}{2}} \|_{\frac{p+s}{p+s-1}} \leq \| A^\frac{1}{2} (A^{\frac{1}{2}} B^{p+s} A^{\frac{1}{2}}) \frac{1}{2} A^\frac{1}{2} \| \tag{2.2}
\]
for $p \geq 1$ and $s \geq 0$.

Moreover
\[
\| A^{\frac{1+s}{2}} B^{t} A^{\frac{1}{2}} \| \leq \| A^\frac{1}{2} (A^{\frac{1}{2}} B^{s} A^{\frac{1}{2}}) \frac{1}{2} A^\frac{1}{2} \| \tag{2.3}
\]
for $s \geq t \geq 0$.

Proof. Put $t = 0$, $s = 1$ in (2.1). Then replacing $r$ and $B$ to $s$ and $B^{\frac{1+s}{2}}$, respectively, (2.1) implies (2.2).

Moreover, let $t$ be a real number satisfying $s \geq t \geq 0$. Then (2.2) implies
\[
\| A^{\frac{1+s}{2}} B^{t+s} A^{\frac{1}{2}} \|_{\frac{p+s}{p+s-1}} \leq \| A^{\frac{1+s}{2}} B^{s} A^{\frac{1}{2}} \|_{\frac{p+s}{p+s-1}} \leq \| A^\frac{1}{2} (A^{\frac{1}{2}} B^{p+s} A^{\frac{1}{2}}) \frac{1}{2} A^\frac{1}{2} \|
\]
by $\frac{1+s}{1+s} \in [0,1]$ and the Araki-Cordes inequality (1.3). Furthermore, replacing $B$ to $B^{\frac{1+s}{t+s}}$ and putting $p = \frac{2}{t}$, we have (2.3).

Remark 2.3. The inequality (2.3) is originated by Bebiano-Lemos-Providência in [3]. In our previous note [7], we call it the BLP inequality and we showed (2.2) as a generalization of the BLP inequality (2.3). Incidentally it is equivalent to (FI). For convenience, we give a proof of (2.2) $(\Rightarrow$ (FI)). The inequality (2.2) is rephrased by replacing $A$ to $A^{-1}$ as follows:
\[
\| A^{-\frac{1-s}{2}} B^{t} A^{-\frac{1-t}{2}} \|_{\frac{p}{p-1}+\frac{r}{r-1}} \leq \| A^{-\frac{1}{2}} (A^{-\frac{1}{2}} B^{\frac{(p+s)}{p+s-1}} A^{-\frac{1}{2}}) \frac{1}{2} A^{-\frac{1}{2}} \|.
\]
Moreover, putting
\[
C = (A^{-\frac{1}{2}} B^{\frac{(p+s)}{p+s-1}} A^{-\frac{1}{2}}) \frac{1}{2}, \text{ or } B^{t} = (A^{\frac{1}{2}} C^{p} A^{\frac{1}{2}}) \frac{1+s}{t+s},
\]
it is also rephrased as
\[ \| A^{-\frac{1}{2}} (A^\frac{1}{2} C^p A^\frac{1}{2})^{\frac{1}{p^2}} A^{-\frac{1}{2}} \| \leq \| A^{-\frac{1}{2}} C A^{-\frac{1}{2}} \| \]
which obviously implies the Furuta inequality (FI) by taking \( s = t = r \).

**Remark 2.4.** In [12], Furuta gave a similar inequality to (2.1).

### 3. A reverse grand Furuta inequality and its applications

In this section, we give a reverse inequality of (2.1) by using the generalized Kantorovich constant (1.5).

**Theorem 3.1.** Let \( A \) and \( B \) be positive operators such that \( 0 < m \leq B \leq M \) for some scalars \( 0 < m < M \) and \( h := \frac{M}{m} > 1 \). Then
\[
\| A^{\frac{1}{2}} \{ A^{-\frac{1}{2}} (A^\frac{1}{2} B^{\frac{r-t}{1-t+r}} A^\frac{1}{2})^{\frac{1}{p^2}} A^{-\frac{1}{2}} \}^{\frac{1}{p^2}} A^{\frac{1}{2}} \|
\leq K \left( h^{\frac{1}{1-t+r'}} (p-t)s + r \right) \| A^{\frac{1-t+r'}{2}} B^{\frac{1-t+r'}{1-t+r}} A^{\frac{1-t+r'}{2}} \| \left( \frac{p-t+s}{p^2(1-t+r')} \right)
\]
for \( 0 \leq t \leq 1 \), \( p \geq 1 \), \( s \geq 1 \) and \( 1+r > 1+r' > t \), where \( K(h, p) \) is the generalized Kantorovich constant defined by (1.5).

**Proof.** For \( p \geq 1 \) and \( s \geq 1 \), the Araki-Cordes inequality (1.3) implies that
\[
\| A^{\frac{1}{2}} \{ A^{-\frac{1}{2}} (A^\frac{1}{2} B^{\frac{r-t}{1-t+r}} A^\frac{1}{2})^{\frac{1}{p^2}} A^{-\frac{1}{2}} \}^{\frac{1}{p^2}} A^{\frac{1}{2}} \|
\leq \| A^\frac{1}{2} \| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{\frac{r-t}{1-t+r}} A^\frac{1}{2} \right)^{\frac{1}{p^2}} A^\frac{1}{2} \|
= \| A^{\frac{1}{2}} \| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{\frac{r-t}{1-t+r}} A^\frac{1}{2} \right)^{\frac{1}{p^2}} A^\frac{1}{2} \|
\leq \| A^\frac{1}{2} \| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{\frac{r-t}{1-t+r}} A^\frac{1}{2} \right)^{\frac{1}{p^2}} A^\frac{1}{2} \|
\leq \| A^\frac{1}{2} \| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{\frac{r-t}{1-t+r}} A^\frac{1}{2} \right)^{\frac{1}{p^2}} A^\frac{1}{2} \|
= \| A^{\frac{1}{2}} \| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{\frac{r-t}{1-t+r}} A^\frac{1}{2} \right)^{\frac{1}{p^2}} A^\frac{1}{2} \|
\leq K \left( h^{\frac{1}{1-t+r'}} (p-t)s + r \right) \| A^{\frac{1-t+r'}{2}} B^{\frac{1-t+r'}{1-t+r}} A^{\frac{1-t+r'}{2}} \| \left( \frac{p-t+s}{p^2(1-t+r')} \right)
\]
Moreover, since \( (p-t)s + r \geq 1-t+r' > 0 \), it follows from the reverse Araki-Cordes inequality (1.4) that
\[
\| A^{\frac{1}{2}} \| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{\frac{r-t}{1-t+r}} A^\frac{1}{2} \right)^{\frac{1}{p^2}} A^\frac{1}{2} \|
\leq \| A^{\frac{1}{2}} \| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{\frac{r-t}{1-t+r}} A^\frac{1}{2} \right)^{\frac{1}{p^2}} A^\frac{1}{2} \|
\leq K \left( h^{\frac{1}{1-t+r'}} (p-t)s + r \right) \| A^{\frac{1-t+r'}{2}} B^{\frac{1-t+r'}{1-t+r}} A^{\frac{1-t+r'}{2}} \| \left( \frac{p-t+s}{p^2(1-t+r')} \right)
\]
Combining them, we have the desired inequality (3.1). □

From the reverse grand Furuta inequality (3.1) we have the following reverse Furuta inequality (see [7]):
Corollary 3.2. Let $A$ and $B$ be positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then

$$
\| A^\frac{1}{2} (A^\frac{1}{2} B^{\alpha + s} A^\frac{1}{2}) \| \leq K \left( h^{1+\frac{p}{2}} B^{1+t} A^\frac{1}{2} \right)^{\frac{1}{p}} \parallel A^\frac{1}{2} B \parallel^{\frac{p+s}{p}} (3.2)
$$

for all $p \geq 1$ and $s \geq t > -1$.

Proof. In [3.1], if we put $t = 0$, $s = 1$, and replace $r$, $r'$, $B$ and $h$ to $t$, $B^{1+\frac{s}{t}}$ and $h^{\frac{1}{1+\frac{s}{t}}}$, respectively, then the desired inequality (3.2) holds. $\square$

On the other hand, Ando and Hiai [1] proved

$$
A_\alpha^\sharp B \leq 1 \Rightarrow A_\alpha^\sharp r B^r \leq 1 \text{ for } 0 \leq \alpha \leq 1, \, r \geq 1
$$

where $A_\alpha^\sharp B := A^\frac{1}{2} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^\frac{1}{2}$. This inequality is equivalent to

$$
\| A_\alpha^\sharp B \| \leq \| A_\alpha^\sharp B \|^r. \quad (AH)
$$

M.Fujii and E.Kamei [6] proved that (AH) is equivalent to (FI). Also they extended (AH) as follows:

$$
\| A_\alpha^\sharp \| B^s \| \left( \frac{(1+\alpha)s+\alpha r}{s-r} \right) \leq \| A_\alpha^\sharp B \| \quad (GAH)
$$

for $r, s \geq 1$ and $0 \leq \alpha \leq 1$. It is easy to see that the inequality (2.1) equivalent to the grand Furuta inequality is rewritten as follows:

$$
\| A^\frac{1}{2} (A^{-\frac{1}{2}} B^s A^{-\frac{1}{2}}) \| \left( \frac{(1+\alpha)s+\alpha r}{s-r} \right) \leq \| A^\frac{1}{2} (A^{-\frac{1}{2}} B^s A^{-\frac{1}{2}}) \| \quad (3.3)
$$

for $0 \leq t \leq 1, \, p \geq 1, \, s \geq 1$ and $r \leq r' \geq t \geq 0$. Here if we put $\alpha = \frac{1}{p}$, then we have

$$
\| A^\frac{1}{2} (A^{-\frac{1}{2}} B^s A^{-\frac{1}{2}}) ^\frac{a(t-r)+s}{s(t-r)} \| \left( \frac{(1+\alpha)s+\alpha r}{s-r} \right) \leq \| A^\frac{1}{2} (A^{-\frac{1}{2}} B^s A^{-\frac{1}{2}}) ^\frac{a(t-r)}{t-r} \| \quad (3.3)
$$

This inequality (3.3) implies (GAH) by $t = 1$.

From the viewpoint of the Ando-Hiai inequality, we consider the following inequality related to a reverse inequality of (3.3) which is equivalent to (3.1). Theorem 3.3. Let $A$ and $B$ be positive operators such that $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then

$$
K \left( \frac{h^{1+t}}{1-\alpha t} \right) \| A^\frac{1}{2} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^\frac{1}{2} \| \| B^\frac{s}{r} A^{\frac{1}{2}} \| \quad (3.4)
$$

for $0 \leq t \leq 1, \, s \geq 1, \, 1 + r \geq 1 + r' \geq t$ and $0 \leq \alpha \leq 1$ where $K(h, p)$ is the generalized Kantorovich constant defined by (1.5).
Proof. In (3.1), we replace $B^{-t}$, $h^{-t}$ and $p$ to $(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha(1-t+r)}$, $h^{\frac{\alpha(1-t+r)}{(1-\alpha)s+\alpha r}}$, $h^{\frac{\alpha(1+t+r)}{(1-\alpha)s+\alpha r}}$ and $\frac{1}{\alpha}$, respectively. Then we have

$$
\| A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \| \leq K \left( h^{\frac{\alpha(1-t+r)}{(1-\alpha)s+\alpha r}}, \frac{1-\alpha t}{\alpha(1-t+r)} \right) \frac{s}{(1-\alpha)s+\alpha r}.
$$

By the inversion formula (i.e., $K(h, \frac{1}{r}) = K(h, r)^{-\frac{1}{2}}$ for all $r \neq 0$), it implies

$$
K \left( h^{\frac{\alpha(1-t+r)}{(1-\alpha)s+\alpha r}}, \frac{1-\alpha t}{\alpha(1-t+r)} \right) = K \left( h^{r+s}, \frac{1-\alpha t}{(1-\alpha)s+\alpha r} \right) \frac{(1-\alpha)t+\alpha r}{s},
$$

and hence (3.4) holds.

Remark 3.4. If $r = r'$ in (3.4), then we have the following reverse inequality of (3.3):

$$
K \left( h^{r+s}, \frac{\alpha(1-t+r)}{(1-\alpha)s+\alpha r} \right) A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \| A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \| \| A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \|
$$

for $0 \leq t \leq 1$, $s \geq 1$, $1 + r = t$ and $0 \leq \alpha \leq 1$. Moreover, let $t = 1$ in Theorem 3.3. As a reverse inequality of (GAH), we have

$$
K \left( h^{r+s}, \frac{\alpha r}{(1-\alpha)s+\alpha r} \right) A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \| A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \|,
$$

that is,

$$
K \left( h^{r+s}, \frac{\alpha r}{(1-\alpha)s+\alpha r} \right) \| A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \| \| A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \|,
$$

for $s \geq 1$, $r \geq 0$ and $0 \leq \alpha \leq 1$.

Under the conditions of $0 \leq s \leq 1$ and $r' = r$, we prove the following inequality as in Theorem 3.3.

Theorem 3.5. Let $A$ and $B$ be positive operators on a Hilbert space $H$ such that $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then

$$
\| A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \| \leq K(h^{1+t}, \alpha)^{-\frac{s(1+t+r)}{(1-\alpha)s+\alpha r}} \| A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \| \| A^{\frac{1}{2}}(A^{−\frac{r}{2}}B^sA^{−\frac{r}{2}})^{\alpha} A^{\frac{1}{2}} \|
$$

for $0 \leq s, t \leq 1$, $1 + r = t$ and $0 \leq \alpha \leq 1$ with $\alpha(1-t) \leq (1-\alpha)t$ where $K(h, p)$ is the generalized Kantorovich constant defined by (1.5).
Proof. We use the Hölder-McCarthy inequality and its reverse: Let $A$ be a positive operator with $0 < m \leq A \leq M$. Then for every vector $y \in \mathcal{H}$

$$K(h, \beta)(Ay, y)^\beta \leq \| A^\beta y, y \| (1 - \beta) \leq \| A y \| \| y \| (1 - \beta)$$

for $0 \leq \beta \leq 1$.

Since $\frac{m}{M} \leq m A^{-t} \leq A^{-t} BA^{-t} \leq M A^{-t} \leq \frac{M}{m}$ and $\| A^\gamma x \| \leq \| A \| \gamma \leq M \gamma$ for all unit vectors $x \in \mathcal{H}$ and $\gamma > 0$, we have for any $0 \leq s \leq 1$

$$\langle A^{1-t} (A^{-\frac{1}{2}} B^s A^{-\frac{1}{2}})^{\alpha(1-t)+s(1-t)} A^{1-t} t, x \rangle \leq \langle A^{t} B^s A^{-\frac{1}{2}} \| A t \| (1 - \alpha(1-t)+s) \rangle$$

$$\leq \langle A^{t} B A^{\frac{1}{2}} \| A \| (1 - \alpha(1-t)+s) \rangle M^{1-t+\gamma}(s-\alpha t-\alpha+\alpha t)$$

$$\leq \langle K(h^{1-t}, \alpha)^{-1} (A_{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} x, x) \rangle (1 - \alpha(1-t)+s) \rangle M^{1-t+\gamma}(s-\alpha t-\alpha+\alpha t)$$

$$\leq \langle K(h^{1-t}, \alpha)^{-1} (A_{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} x, x) \rangle (1 - \alpha(1-t)+s) \rangle M^{1-t+\gamma}(s-\alpha t-\alpha+\alpha t)$$

$$= \langle K(h^{1-t}, \alpha)^{-1} (A_{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} x, x) \rangle (1 - \alpha(1-t)+s) \rangle M^{1-t+\gamma}(s-\alpha t-\alpha+\alpha t)$$

Hence we obtain the desired inequality (3.5). 

Putting $t = 1$ in (3.5), we have an inequality given in [15]:

$$\| A_{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} \| (1 - \alpha(1-t)+s) \rangle M^{1-t+\gamma}(s-\alpha t-\alpha+\alpha t)$$

for $0 \leq s \leq 1$, $r \geq 0$ and $0 \leq \alpha \leq 1$.

References

[9] T. Furuta, A $\geq B \geq 0$ assures $(B^p A^p B^p)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$, Proc. Amer. Math. Soc., 101 (1987), 85–88.


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