CHARACTERIZATIONS OF INNER PRODUCT SPACES BY STRONGLY CONVEX FUNCTIONS

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Abstract. New characterizations of inner product spaces among normed spaces involving the notion of strong convexity are given. In particular, it is shown that the following conditions are equivalent: (1) \((X, \| \cdot \|)\) is an inner product space; (2) \(f : X \to \mathbb{R}\) is strongly convex with modulus \(c > 0\) if and only if \(f - c\| \cdot \|^2\) is convex; (3) \(\| \cdot \|^2\) is strongly convex with modulus 1.

1. Introduction

It is well known that in a normed space \((X, \| \cdot \|)\) the following Jordan–von Neumann parallelogram law

\[\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X,\]

holds if and only if the norm \(\| \cdot \|\) is derivable from an inner product (cf.\([8], [5]\)). In the literature one can find many other conditions characterizing inner product spaces among normed spaces. A rich collection of such characterizations is contained in the celebrated book of Amir [5] (cf. also [1, Chpt. 11], [2], [3], [4], [6], [11]). The aim of this note is to present some new results of this type involving strongly convex and strongly midconvex functions.

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In what follows \((X, \| \cdot \|)\) is a real normed space, \(D\) stands for a convex subset of \(X\) and \(c\) is a positive constant. A function \(f : D \to \mathbb{R}\) is called strongly convex with modulus \(c\) if
\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - ct(1 - t)\|x - y\|^2,
\]
for all \(x, y \in D\) and \(t \in (0, 1)\). We say that \(f\) is strongly midconvex with modulus \(c\) if (1.1) is assumed only for \(t = 1/2\), that is
\[
f \left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2,
\]
\(x, y \in D\). (1.2)

Recall also that \(f\) is convex (midconvex) if it satisfies (1.1) (1.2), respectively) with \(c = 0\). Strongly convex functions have been introduced by Polyak [10] and they play an important role in optimization theory. Many properties of them can be found, among other, in [7], [9], [12], [13]. The following result gives relationships between strongly convex (strongly midconvex) and convex (midconvex) functions. In the case where \(X = \mathbb{R}^n\) the first part of this result can be found in [7, Prop. 1.1.2].

2. Main result

We start this section with a useful lemma.

**Lemma 2.1.** Let \((X, \| \cdot \|)\) be a real inner product space, \(D\) be a convex subset of \(X\) and \(c\) be a positive constant.

1. A function \(f : D \to \mathbb{R}\) is strongly convex with modulus \(c\) if and only if the function \(g = f - c\| \cdot \|^2\) is convex.
2. A function \(f : D \to \mathbb{R}\) is strongly midconvex with modulus \(c\) if and only if the function \(g = f - c\| \cdot \|^2\) is midconvex.

**Proof.** 1. Assume that \(f\) is strongly convex with modulus \(c\). Using elementary properties of the inner product and the fact that \(\|x\|^2 = \langle x | x \rangle\), we get
\[
g(tx + (1 - t)y) = f(tx + (1 - t)y) - c\|tx + (1 - t)y\|^2
\leq tf(x) + (1 - t)f(y) - ct(1 - t)\|x - y\|^2 - c\|tx + (1 - t)y\|^2
\leq tf(x) + (1 - t)f(y) - c\left(t(1 - t)\langle x | y \rangle + \|y\|^2 - 2\|x\|^2 - 2\|x\|\|y\|\right)
+ t^2\|x\|^2 + 2t(1 - t)\langle x | y \rangle + (1 - t)^2\|y\|^2
= tf(x) + (1 - t)f(y) - ct\|x\|^2 - c(1 - t)\|y\|^2
= tg(x) + (1 - t)g(y),
\]
which proves that \( g \) is convex. 
Conversely, if \( g \) is convex, then 
\[
\begin{align*}
f(tx + (1 - t)y) &= g(tx + (1 - t)y) + c\|tx + (1 - t)y\|^2 \\
&\leq tg(x) + (1 - t)g(y) + c(t^2\|x\|^2 + 2t(1 - t)\langle x | y \rangle + (1 - t)^2\|y\|^2) \\
&= t(g(x) + c\|x\|^2) + (1 - t)(g(y) + c\|y\|^2) \\
&\quad - ct(1 - t)(\|x\|^2 - 2\langle x | y \rangle + \|y\|^2) \\
&= f(x) + (1 - t)f(y) - ct(1 - t)\|x - y\|^2,
\end{align*}
\]
which shows that \( f \) is strongly convex with modulus \( c \). 

2. Assume now that \( f \) is strongly midconvex with modulus \( c \). Using the parallelogram law we get 
\[
g\left(\frac{x + y}{2}\right) = f\left(\frac{x + y}{2}\right) - c\left\|\frac{x + y}{2}\right\|^2
\]
\[
\leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2 - \frac{c}{4}\|x + y\|^2
\]
\[
= \frac{f(x) + f(y)}{2} - \frac{c}{4}(2\|x\|^2 + 2\|y\|^2) = \frac{g(x) + g(y)}{2}.
\]

Similarly, if \( g \) is midconvex, then 
\[
f\left(\frac{x + y}{2}\right) = g\left(\frac{x + y}{2}\right) + c\left\|\frac{x + y}{2}\right\|^2
\]
\[
\leq \frac{g(x) + g(y)}{2} + \frac{c}{4}\|x + y\|^2
\]
\[
= \frac{g(x)}{2} + \frac{\|x\|^2}{2} + \frac{g(y)}{2} + \frac{\|y\|^2}{2} + \frac{c}{4}(\|x - y\|^2 - 2\|x\|^2 - 2\|y\|^2)
\]
\[
= \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2;
\]
\[
\square
\]

The following example shows that the assumption that \( X \) is an inner product space is essential in the above lemma. 

**Example 2.2.** Let \( X = \mathbb{R}^2 \) and \( \| x \| = |x_1| + |x_2| \), for \( x = (x_1, x_2) \). Take \( f = \| \cdot \|^2 \). Then \( g = f - \| \cdot \|^2 \) is convex being the zero function. However, \( f \) is neither strongly convex nor strongly midconvex with modulus 1. Indeed, for \( x = (1, 0) \) and \( y = (0, 1) \) we have 
\[
f\left(\frac{x + y}{2}\right) = 1 > 0 = \frac{f(x) + f(y)}{2} - \frac{1}{4}\|x - y\|^2,
\]
which contradicts (1.2). 

It appears that something stronger can be proved: the assumption that \( X \) is an inner product space is necessary in Lemma 2.1. Namely, the following characterizations of inner product spaces hold. 

**Theorem 2.3.** Let \((X, \| \cdot \|)\) be a real normed space. The following conditions are equivalent to each other: 

1. For all \( c > 0 \) and for all functions \( f : D \to \mathbb{R} \), \( f \) is strongly convex with modulus \( c \) if and only if \( g = f - c\| \cdot \|^2 \) is convex.
2. For all $c > 0$ and for all functions $f : D \to \mathbb{R}$, $f$ is strongly midconvex with modulus $c$ if and only if $g = f - c \| \cdot \|^2$ is midconvex;
3. There exists $c > 0$ such that, for all functions $f : D \to \mathbb{R}$, $g$ is convex if and only if $f = g + c \| \cdot \|^2$ is strongly convex with modulus $c$;
4. There exists $c > 0$ such that, for all functions $f : D \to \mathbb{R}$, $g$ is midconvex if and only if $f = g + c \| \cdot \|^2$ is strongly midconvex with modulus $c$;
5. $\| \cdot \|^2 : X \to \mathbb{R}$ is strongly convex with modulus $1$;
6. $\| \cdot \|^2 : X \to \mathbb{R}$ is strongly midconvex with modulus $1$;
7. $(X, \| \cdot \|)$ is an inner product space.

Proof. We will show the following chains of implications: $1 \Rightarrow 3 \Rightarrow 5 \Rightarrow 7 \Rightarrow 1$ and $2 \Rightarrow 4 \Rightarrow 6 \Rightarrow 7 \Rightarrow 2$.

Implications $1 \Rightarrow 3$ and $2 \Rightarrow 4$ are obvious. To show $3 \Rightarrow 5$ and $4 \Rightarrow 6$ take $g = 0$. Then $f = c \| \cdot \|^2$ is strongly convex (resp. strongly midconvex) with modulus $c$. Consequently, $\frac{1}{c} f = \| \cdot \|^2$ is strongly convex (resp. strongly midconvex) with modulus 1.

To see that $5 \Rightarrow 7$ and $6 \Rightarrow 7$ also hold, observe that, by the strong convexity or strong midconvexity with modulus 1 of $\| \cdot \|^2$ we have

$$\left\| \frac{x + y}{2} \right\|^2 \leq \frac{\|x\|^2 + \|y\|^2}{2} - \frac{1}{4} \|x - y\|^2,$$

and hence

$$\|x + y\|^2 + \|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$. Now, putting $u = x + y$ and $v = x - y$ in (2.1), we get

$$2\|u\|^2 + 2\|v\|^2 \leq \|u + v\|^2 + \|u - v\|^2,$$

$u, v \in X$. (2.2)

Conditions (2.1) and (2.2) mean that the norm $\| \cdot \|$ satisfies the parallelogram law, which implies that $(X, \| \cdot \|)$ is an inner product space.

Implications $7 \Rightarrow 1$ and $7 \Rightarrow 2$ follow by Lemma 2.1.

REFERENCES


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