Abstract. We give a topological classification of the orbit space of cohomogeneity two isometric actions on flat Riemannian manifolds.


Key words: Orbit space; isometry; Riemannian manifold.

1 Introduction

Let $G \times M \rightarrow M$ be a differentiable action of a Lie group $G$ on a differentiable manifold $M$ and consider the orbit space $M \over G$, with the quotient topology. The dimension of $M \over G$, which we will denote by $\text{Coh}(M,G)$, is called the cohomogeneity of the action of $G$ on $M$. The study of orbit spaces has many important applications in invariant function theory and $G$-invariant variational problems associated to $M$. Many $G$-invariant objects associated to $M$ can be related to similar objects associated to the orbit space.

Therefore, we can effectively reduce many problems about $G$-invariant objects of $M$ to generally easier problems on $M \over G$. Because of this motivation, many mathematicians studied topological properties of the orbit spaces of Lie group actions on manifolds. A pioneer theorem in this area is the following theorem proved by P. Mostert in 1957 ([11]): If $M$ is a differentiable manifold and $G$ is a compact Lie group acting on $M$ such that $\text{Coh}(M,G) = 1$, then the orbit space $M \over G$ is homeomorphic to one of the spaces $[0, 1], (0, 1), S^1$ or $\mathbb{R}$.

This theorem has been generalized to noncompact Lie groups with proper actions on manifolds. Moreover, if $M$ is endowed with a Riemannian metric, and $G$ is a connected and closed subgroup of the isometries of $M$, which acts by cohomogeneity one on $M$, there are more interesting results about the orbit space and orbits (see [10], [11], [13]). It is proved in [13] that if $M$ is a Riemannian manifold of negative curvature and $G$ is a connected and closed subgroup of isometries of $M$, acting on $M$ with $\text{Coh}(M,G) = 1$, then the orbit space is not homeomorphic to $[0, 1]$, so by (generalized) Mostert’s theorem, it would be homeomorphic to $(0, 1)$ or $S^1$ or $\mathbb{R}$, and if in addition $M$ is simply connected then the orbit space is homeomorphic to $(0, 1)$. 
or $\mathbb{R}$. This result, generalized to flat Riemannian manifolds in [10], and recently it is proved for Riemannian manifolds of non-positive curvature. To extend Mostert’s theorem, it is natural to ask, what may be the orbit space $\mathbb{M}$, when $\text{Coh}(\mathbb{M}, G) = 2$. There is no classification for orbit spaces of cohomogeneity two $G$-manifolds in general. Cohomogeneity two actions of compact Lie groups on $\mathbb{R}^n$, $n > 1$, are polar (in the sense of Dadok) and all such actions and their orbits are classified (see [12]). It is clear in this case that the orbit space is homeomorphic to plane or half-plane. Also, it is proved in [8] that if $G$ is a connected (compact or non-compact) group of the isometries of $\mathbb{R}^n$ such that $\text{Coh}(\mathbb{R}^n, G) = 2$, then the orbit space $\mathbb{R}^n / G$ is homeomorphic to plane or half-plane. Classification of orbit spaces of cohomogeneity two actions on the standard sphere $S^n$ has been described in [1].

This article follows a series of papers [6]-[9], where we are trying to study orbits and orbit spaces of cohomogeneity two Riemannian manifolds under conditions on curvature. In [7] the following theorem is proved which gives a topological description of cohomogeneity two flat Riemannian manifolds and their orbits.

**Theorem A.** Let $M^n$, $n \geq 3$, be a complete connected nonsimply connected and flat Riemannian manifold, which is of cohomogeneity two under the action of a closed and connected Lie group $G$ of isometries. Then, one of the following is true:

(a) $\pi_1(M) = \mathbb{Z}$ and each principal orbit is isometric to $S^{n-2}(c)$, for some $c > 0$ ($c$ depends on orbits).

(b) There is a positive integer $l$, such that $\pi_1(M) = \mathbb{Z}^l$ and one of the following is true:

(b1) There is a positive integer $m$, $2 < m < n$, such that each principal orbit is covered by $N^{m-2}(c) \times \mathbb{R}^{n-m}$, where $N^{m-2}(c)$ is a homogeneous hypersurface of $S^{m-1}(c)$ ($c > 0$ depends on orbits). There is a unique orbit diffeomorphic to $T^l \times \mathbb{R}^{n-m-l}$.

(b2) Each principal orbit is covered by $S^r \times \mathbb{R}^{n-r-2}$, for some positive integer $r$.

(b3) Each principal orbit is covered by $H \times \mathbb{R}^{n-m}$, such that $H$ is a helix in $\mathbb{R}^m$. There is an orbit diffeomorphic to $T^t \times \mathbb{R}^t$, for some non-negative integer $t$.

(c) Each orbit is diffeomorphic to $\mathbb{R}^t \times T^t$, for some non-negative integer $t$.

To complete the study of flat cohomogeneity two Riemannian manifolds, it remains to characterize the orbit space, which is the aim of the present paper. For any flat surface $S$ there exists a cohomogeneity two flat Riemannian $G$-manifold $M$ such that all orbits are flat and $\frac{M}{G}$ is homeomorphic to $S$ (put $M = S \times \mathbb{R}^n$, $G = \{I\} \times H$ such that $I$ is the identity map on $S$ and $H$ is a closed and connected subgroup of $\text{Iso}(\mathbb{R}^n)$ which acts transitively on $\mathbb{R}^n$).

Thus, study of the orbit space of cohomogeneity two flat Riemannian manifolds is interesting when there are some non-flat orbits. We will prove the following theorem.

**Theorem B.** Let $M$ be a flat Riemannian manifold and $G$ be a closed and connected subgroup of the isometries of $M$ such that $\text{Coh}(M, G) = 2$. If there are some
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non-flat orbits then $\frac{M}{G}$ is homeomorphic to one of the following spaces:

$$[0, +\infty) \times \mathbb{R}, S^1 \times \mathbb{R}, S^1 \times [0, \infty), \mathbb{R}^2$$

2 Preliminaries

In the following, $M^n$ is a Riemannian manifold of dimension $n$, $G$ is a closed and connected subgroup of $\text{Iso}(M)$, and $\pi : M \rightarrow \frac{M}{G}$ denotes the projection on to the orbit space. We know that the fixed point set of the action of $G$ on $M$, given by

$$M^G = \{ x \in M : G(x) = x \}$$

is a totally geodesic submanifold of $M$.

We will write $A = B$ if $A$ and $B$ are homeomorphic topological spaces, isomorphic groups or diffeomorphic manifolds.

**Fact 2.1.** If $\text{Coh}(G, M) = m \geq 1$ then there are two types of points in $M$ called principal and singular points (for definition and details about singular and principal points, we refer to [1] and [13]. If $x$ is a principal(singular) point then $\pi(x)$ is an interior(boundary) point of $\frac{M}{G}$. Also, if $x$ is a principal point, the orbit $G(x)$ is called a principal (singular) orbit and we have $\dim G(x) = n - m$ (dim $G(x) \leq n - m$). The union of all principal orbits is an open and dense subset of $M$.

**Remark 2.2.** If $\text{Coh}(G, \mathbb{R}^n) = 1$ then one of the following is true:

1. All orbits are isometric to $\mathbb{R}^{n-1}$. So, by suitable choice of coordinates, each orbit will be equal to $\{b\} \times \mathbb{R}^{n-1}$ for some $b \in \mathbb{R}$ related to the orbit, and $\frac{\mathbb{R}^n}{G} = \mathbb{R}$.
2. Each principal orbit is diffeomorphic to $S^{n-m-1} \times \mathbb{R}^m$ for some $m \geq 0$, there is a unique singular orbit isometric to $\mathbb{R}^m$ and $\frac{\mathbb{R}^n}{G} = [0, +\infty)$.
3. If $G$ is compact then each principal orbit is diffeomorphic to $S^{n-1}$, the unique singular orbit is a one point set, and $\frac{\mathbb{R}^n}{G} = [0, \infty)$.

**Proof.** See [10], proof of the theorems 3.1 and 3.5. □

**Definition 2.3.** If $\text{Coh}(G, \mathbb{R}^n) = 1$ then one of the following is true:

1. All orbits are isometric to $\mathbb{R}^{n-1}$. So, by suitable choice of coordinates, each orbit will be equal to $\{b\} \times \mathbb{R}^{n-1}$ for some $b \in \mathbb{R}$ related to the orbit, and $\frac{\mathbb{R}^n}{G} = \mathbb{R}$.
2. Each principal orbit is diffeomorphic to $S^{n-m-1} \times \mathbb{R}^m$ for some $m \geq 0$, there is a unique singular orbit isometric to $\mathbb{R}^m$ and $\frac{\mathbb{R}^n}{G} = [0, +\infty)$.
3. If $G$ is compact then each principal orbit is diffeomorphic to $S^{n-1}$, the unique singular orbit is a one point set, and $\frac{\mathbb{R}^n}{G} = [0, \infty)$.

**Proof.** See [10], proof of the theorems 3.1 and 3.5. □

**Definition 2.4.**
(a) Let $G$ be a connected subgroup of $\text{Iso}(\mathbb{R}^n)$ and $d,e$ be positive integers such that $d + e = n$. If $G$ is not compact and it is a subgroup of $SO(d) \times \mathbb{R}^e$ (direct product), then we say that $G$ is $d$-helicoidal on $\mathbb{R}^n$.

(b) Following (a), let

\[ K = \{ A \in SO(d) : (A, b) \in G, \text{ for some } b \in \mathbb{R}^e \} \]
\[ T = \{ b \in \mathbb{R}^e : (A, b) \in G, \text{ for some } A \in SO(d) \} \]

If $x = (x_1, x_2) \in (\mathbb{R}^d - \{0\}) \times \mathbb{R}^e$, $T(x_2) = \mathbb{R}^e$ and $K(x_1) = S^{d-1}(\|x_1\|)$, then $G(x)$ is called a $d$-helix around $S^{d-1}(\|x_1\|) \times \mathbb{R}^e$.

**Definition 2.5.** Let $G$ be a closed and connected subgroup of $\text{Iso}(\mathbb{R}^n)$, $n \geq 3$. We say that $G$ has compact (or helicoidal) factor, if there is an integer $0 < m < n$ and there are Lie groups $G_1 \subset \text{Iso}(\mathbb{R}^{n-m})$, $G_2 \subset \text{Iso}(\mathbb{R}^m)$, such that

1. $G_2$ is compact (or helicoidal on $\mathbb{R}^n$).
2. $G \cong G_2 \times G_1$.
3. For some (so each) $x \in \mathbb{R}^{n-m}$, $G_1(x) = \mathbb{R}^{n-m}$.

**Corollary 2.6 ([7]).** If $G$ is a connected and closed subgroup of $\text{Iso}(\mathbb{R}^n), n \geq 3$, and $\text{Coh}(G, \mathbb{R}^n) = 2$. Then one of the following is true:

1. $G$ is compact. (II) $G$ has compact factor on $\mathbb{R}^n$. (III) $G$ is helicoidal on $\mathbb{R}^n$.
2. $G$ has helicoidal factor on $\mathbb{R}^n$. (V) All $G$-orbits are Euclidean.

3 **Orbit spaces**

By Lemma 3.6 in [7] and its proof, we get the following fact.

**Fact 3.1.** If the action of $G$ on $\mathbb{R}^n$ is helicoidal then one of the following assertions is true:

1. $G$ action on $\mathbb{R}^n$ is orbit equivalent to the action of a product $H \times T \subset SO(d) \times \mathbb{R}^e$ on $\mathbb{R}^d \times \mathbb{R}^e$, $d + e = n$, such that each principal $H$-orbit in $\mathbb{R}^d$ is isometric to $S^{d-1}(r), r > 0$, and $T$ acts by cohomogeneity one on $\mathbb{R}^e$ such that all $T$-orbits on $\mathbb{R}^e$ are isometric to $\mathbb{R}^{e-1}$.
2. Each principal $G$-orbit is isometric to a $d$-helix around $S^{d-1}(r) \times \mathbb{R}^e$, $e > 1$, $r > 0$, and $G$ acts transitively on $\{0\} \times \mathbb{R}^e = \mathbb{R}^e$.

**Fact 3.2.** Let $M$ be a Riemannian manifold and $\tilde{M}$ be the Riemannian universal covering of $M$, by the covering map $k : \tilde{M} \to M$, and let $G$ be a closed and connected subgroup of $\text{Iso}(M)$. Then there is a connected covering $\tilde{G}$ for $G$ such that $\tilde{G}$ acts isometrically on $\tilde{M}$ and the following assertions are true:

1. $\text{Coh}(G, M) = \text{Coh}(\tilde{G}, \tilde{M})$.
2. If $D = \tilde{G}(x)$ is a $\tilde{G}$-orbit in $\tilde{M}$ then $k(D)$ is a $G$-orbit in $M$, and each $G$-orbit in $M$ is equal to $k(D)$ for some $G$-orbit $D$ in $\tilde{M}$.
3. If $\Delta$ is the deck transformation group of the covering $k : \tilde{M} \to M$ then for each $\delta \in \Delta$ and each $g \in \tilde{G}$, $\delta \circ g = g \circ \delta$. Thus $\delta$ maps $\tilde{G}$-orbits in $\tilde{M}$ on to $G$-orbits.
4. $\tilde{M}^{\tilde{G}} = \kappa^{-1}(M^G)$. 
Proof. See [1], pages 63-64.

**Fact 3.3.** Let \( \Delta \) be a discrete subgroup of the isometries of \( \mathbb{R}^m \), \( m > 1 \), and suppose that for each \( a \in \mathbb{R} \), there is \( a_1 \in \mathbb{R} \) such that \( \Delta(\{a\} \times \mathbb{R}^{m-1}) = \{a_1\} \times \mathbb{R}^{m-1} \). Put

\[
\Gamma = \{ \delta \in \Delta : \delta(\{a\} \times \mathbb{R}^{m-1}) = \{a\} \times \mathbb{R}^{m-1} \text{ for all } a \in \mathbb{R} \}.
\]

Then, \( \Gamma \) is a normal subgroup of \( \Delta \) and we have \( \frac{\Delta}{\Gamma} = \mathbb{Z} \).

**Proof.** It is clear from the definition of \( \Gamma \) that \( \Gamma \) is normal in \( \Delta \). Consider the function \( p : \mathbb{R}^m (= \mathbb{R} \times \mathbb{R}^{m-1}) \to \mathbb{R} \) defined by \( p(a, x) = a \), and put

\[
\theta : \Delta \times \mathbb{R} \to \mathbb{R}, \quad \theta(\delta, a) = p\delta(a, o), \quad o = (0, ..., 0) \in \mathbb{R}^{m-1}.
\]

Since for all \( a \in \mathbb{R} \), \( \Delta(\{a\} \times \mathbb{R}^{m-1}) = \{a_1\} \times \mathbb{R}^{m-1} \) for some \( a_1 \) related to \( a \), then for each \( x = (a, b) \in \mathbb{R} \times \mathbb{R}^{m-1} \) and \( \delta \in \Delta \), we have \( p\delta(a, b) = p\delta(a, o) \), so

\[
p\delta(x) = p\delta(px, o) \quad (*)
\]

Therefore, if \( \delta_1, \delta_2 \in \Delta \) then

\[
\theta(\delta_1, \theta(\delta_2, a)) = \theta(\delta_1, p\delta_2(a, o)) = p\delta_1(p\delta_2(a, o), o).
\]

We get from (*) that

\[
p\delta_1(p\delta_2(a, o), o) = p\delta_1\delta_2(a, o).
\]

Thus, \( \theta(\delta_1, \theta(\delta_2, a)) = \theta(\delta_1\delta_2, a) \). This means that \( \theta \) is an action of \( \Delta \) on \( \mathbb{R} \). The action of \( \Delta \) induces an effective action of \( \frac{\Delta}{\Gamma} \) on \( \mathbb{R} \), which is clearly an isometric action and no element of \( \frac{\Delta}{\Gamma} \) has a fixed point in \( \mathbb{R} \). So, \( \frac{\Delta}{\Gamma} \) can be considered as a discrete subgroup of \( (\mathbb{R}, +) \) and must be isomorphic to \( (\mathbb{Z}, +) \). \( \square \)

**Lemma 3.4 ([9]).** If \( M \) is a connected and complete cohomogeneity \( k \) Riemannian \( G \)-manifold then \( k > \dim M^G \).

**Theorem 3.5 ([8]).** If \( G \) is a closed and connected subgroup of \( \text{Iso}\mathbb{R}^n \), \( n \geq 2 \), and \( \text{Coh}(G, \mathbb{R}^n) = 2 \), then \( \mathbb{R}^n_G = [0, \infty) \times \mathbb{R} \) or \( \mathbb{R}^2 \).

**Lemma 3.6.** Let \( M \) be a flat Riemannian manifold, \( \dim M > 2 \), and let \( G \) be a closed and connected subgroup of the isometries of \( M \). If \( \text{Coh}(M, G) = 2 \) and \( M^G \neq \emptyset \), then \( \frac{M}{G} \) is homeomorphic to one of the following spaces:

\[
[0, +\infty) \times \mathbb{R}, S^1 \times [0, \infty), \mathbb{R}^2
\]

**Proof.** Consider \( \widetilde{M} = \mathbb{R}^n \) the universal Riemannian covering manifold of \( M \), and use the symbols used in Fact 3.2. Since \( M^G \neq \emptyset \) then by Fact 3.2(4), \( \widetilde{M}^G \neq \emptyset \). Put \( L = \widetilde{M}^G \) and let \( m = \dim L \). By Lemma 3.4, we have \( 2 > m \), so \( m = 0 \) or \( m = 1 \).

If \( m = 0 \) then from the fact that \( \widetilde{M}^G \) is a (connected) totally geodesic submanifold
of \(\mathbb{R}^n\), we get that \(\overline{M^G}\) is a one point set and by Fact 3.2(4), \(M\) is simply connected, so \(M = \mathbb{R}^n\), \(G = \tilde{G}\). Then, by Theorem 3.5, \(\overline{M^G} = [0, \infty) \times \mathbb{R}\) or \(\mathbb{R}^2\).

If \(m = 1\) and \(M\) is not simply connected, then \(L\) is a line in \(\mathbb{R}^n\). Since the elements of \(\tilde{G}\) and \(\Delta\) are commutative, then \(\Delta(L) = L\). If \(a \in L\), denote by \(W_a\) the hyperplane of \(\mathbb{R}^n\) which is perpendicular to \(L\) at \(a\). Without lose of generality we can suppose that \(L = \{a\} \times \mathbb{R} \subset \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n\). Since \(\tilde{G}\) leaves \(L\) invariant point wisely, then \(\tilde{G}\) decomposes as \(\tilde{G} = \	ilde{G} \times \{I\}\), where \(\tilde{G} \subset SO(n-1)\) and \(I\) is the identity map on \(\mathbb{R}\). So, for all \(a \in L\) and all \(x \in W_a\), \(\tilde{G}(x) \subset W_a\). Now, it is easy to show that the following map is a homeomorphism:

\[
\left\{ \begin{array}{ll} 
\psi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}^2 \\
\psi(G(x)) = (\tilde{G}(x_1), x_2), \quad x = (x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} 
\end{array} \right.
\]

Since \(\text{Coh}(\mathbb{R}^{n-1}, \tilde{G}) = 1\) then by Remark 2.2(3), \(\mathbb{R}^{n-1}_G = [0, \infty)\), so \(\mathbb{R}^n_G = [0, \infty) \times \mathbb{R}\).

Since the members of \(\Delta\) map \(\tilde{G}\)-orbits to \(\tilde{G}\)-orbits, then by curvature reasons, for each \((x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}\), \(\Delta(\tilde{G}(x_1), x_2) = (\tilde{G}(x_1), y_2)\) for some \(y_2 \in \mathbb{R}\). So, we get from \(\Delta(L) = L\) that \(\Delta\) decomposes as \(\Delta = \{I\} \times \Gamma \subset \text{Iso}(\mathbb{R}^{n-1}) \times \text{Iso}(L)\). Thus \(\Delta\) can be considered as a discrete subgroup of the isometries of \(L = \mathbb{R}\) without fixed point, then \(\Delta = \Gamma\), and we have

\[
\frac{M}{G} = \frac{[0, \infty) \times \mathbb{R}}{\Delta} = [0, \infty) \times \frac{\mathbb{R}}{\Gamma} = [0, \infty) \times S^1.
\]

\(\Box\\

Remark 3.7.

(1) Let \(E = \mathbb{R}^2\) or \([0, \infty) \times \mathbb{R}\), and \(\Gamma\) be a nontrivial discrete subgroup of the isometries of \(E\) such that \(\Gamma(o) = o\), then \(\frac{E}{\Gamma}\) is homeomorphic to \(\mathbb{R}^2\) or \([0, \infty) \times \mathbb{R}\).

(2) If \(\Gamma = \Gamma\) and \(E = [0, \infty) \times \mathbb{R}\), then \(\frac{E}{\Gamma} = [0, \infty) \times S^1\).

Proof. (1) Let \(E = \mathbb{R}^2\) and consider the circles \(S^1(r)\) of radius \(r > 0\) around the origin of \(\mathbb{R}^2\), and put \(S^1(o) = o\). Since \(\Gamma \subset O(2)\) is compact and discrete, it is finite. Consider a point \(a \in S^1(1)\) and let \(\Gamma(a) = \{a_1 = a, a_2, ..., a_n\}\) ordered in clockwise. Then, we have

\[
\Gamma(ra) = \{ra_1, ra_2, ..., ra_n\}, \quad ra \in S^1(r).
\]

If \(b\) is the length of the arc \(\overline{a_1a_2}\) (clockwise arc) on \(S^1(1)\) then the length of the arc \(\overline{ra_1ra_2}\) on \(S^1(r)\) is equal to \(rb\) and we have \(S^1(r) = S^1(rb)\). So,

\[
\frac{\mathbb{R}^2}{\Gamma} = \bigcup_{r \geq 0} S^1(r) = \bigcup_{rb \geq 0} S^1(rb) = \mathbb{R}^2.
\]

Now, let \(E = [0, \infty) \times \mathbb{R}\). We know that the isometries of plane are combinations of three kind of isometries called rotations, reflections respect to lines, gelid reflections (see[3]). Since \(\Gamma(E) = E\) and \(\Gamma(o) = o\) then \(\Gamma\) can only contain a reflection respect to the line \([0, \infty) \times \{0\}\) and the identity, then \(\frac{E}{\Gamma}\) is equal to \([0, \infty) \times [0, \infty)\), which is homeomorphic to \([0, \infty) \times \mathbb{R}\).

(2) Proof is similar to (1). \(\Box\)
4 Theorem B

Proof. Consider \( \tilde{M} = \mathbb{R}^n \) the universal covering manifold of \( M \) and use the symbols of Fact 3.2. Put
\[
\Delta' = \{ \delta \in \Delta : \delta(D) = D \text{ for all } \tilde{G} - \text{orbits } D \text{ in } \mathbb{R}^n \}.
\]

Since \( \Delta' \) is normal in \( \Delta \), we can consider the quotient group \( \tilde{\Delta} = \frac{\Delta}{\Delta'} \). It is not hard to show that \( \tilde{\Delta} \) acts effectively on the orbit space \( \tilde{\Omega} = \mathbb{R}^n / \tilde{G} \) and \( M \tilde{G} = \frac{\Omega}{\tilde{G}} = \frac{\Omega}{\Delta} \). By Corollary 2.6, one of the following cases is true:

a) \( \tilde{G} \) is compact
b) \( \tilde{G} \) is helicoidal

c) \( \tilde{G} \) has compact factor
d) \( \tilde{G} \) has helicoidal factor
e) All orbits are Euclidean.

a) Since \( \tilde{G} \) is compact then \( \tilde{M} \tilde{G} \neq \emptyset \), so \( M \tilde{G} \neq \emptyset \) and we get the result from Theorem 3.6.

b) By Fact 3.1 and by suitable choice of ordinates, two cases may occur:

(1) \( G \) action is orbit equivalent to the action of a product \( S \times T \subset \text{So}(d) \times \mathbb{R}^e \), \( d + e = n \), on \( \mathbb{R}^d \times \mathbb{R}^e \) such that each principal \( S \)-orbit in \( \mathbb{R}^d \) is isometric to \( S^{d-1}(r) \), \( r > 0 \), and \( T \) acts by cohomogeneity one on \( \mathbb{R}^e \) such that all \( T \)-orbits are isometric to \( \mathbb{R}^{e-1} \).

(2) Each principal \( \tilde{G} \)-orbit is isometric to a helicoid around \( S^{d-1}(r) \times \mathbb{R}^e \), \( e > 1 \), \( r > 0 \), and \( \tilde{G} \) acts transitively on \( \{o\} \times \mathbb{R}^e = \mathbb{R}^e \).

In the case (1), we have \( \tilde{\Omega} = \mathbb{R}^n / \tilde{G} = \mathbb{R}^d / S \times \mathbb{R}^e / T \). Thus, by Remark 2.2 (3,1), \( \tilde{\Omega} = [0, \infty) \times \mathbb{R} \). If \( x \in \{o\} \times \mathbb{R} \) then \( \dim \tilde{G}(x) = e - 1 \) and if \( x \notin \{o\} \times \mathbb{R} \) then \( \dim \tilde{G}(x) = d - 1 + e - 1 = d + e - 2 \). Since \( d > 1 \), by dimensional reasons and the fact that each \( \delta \in \Delta \) maps orbits to orbits, we get that \( \Delta(\mathbb{R}^e) = \mathbb{R}^e \). Since \( \tilde{G} \) acts by cohomogeneity one on \( \mathbb{R}^e \) and all orbits are Euclidean, then by Remark 2.2(1), and without lose of generality, we can suppose that each \( \tilde{G} \)-orbit in \( \mathbb{R}^e \) is equal to \( \{b\} \times \mathbb{R}^{e-1} \) for some \( b \in \mathbb{R} \) related to the orbit. Put
\[
\Gamma = \{ \delta \in \Delta : \delta(D) = D, \text{ for all orbits } D \text{ in } \mathbb{R}^e \}.
\]

By Fact 3.3, we have \( \frac{\Gamma}{\Delta} = Z \). It is not hard to show that \( \Gamma = \Delta' \), so \( \tilde{\Delta} = \frac{\Gamma}{\Delta} = \frac{\Omega}{\tilde{G}} = \frac{\Omega}{\tilde{G}} = [0, \infty) \times \mathbb{R} \).

Then \( \frac{M \tilde{G}}{\tilde{G}} = \frac{\Omega}{\tilde{G}} = \frac{\Omega}{\tilde{G}} = [0, \infty) \times S^1 \).

In the case (2), First note that by Theorem 3.5, \( \tilde{\Omega} = \mathbb{R}^n / \tilde{G} = \mathbb{R}^2 \) or \( [0, \infty) \times \mathbb{R} \). Since the elements of \( \Delta \) are isometries which map orbits to orbits, then by curvature reasons, \( \Delta(\{o\} \times \mathbb{R}^e) = \{o\} \times \mathbb{R}^e \). Without lose of generality we can suppose that the corresponding point of the orbit \( \{o\} \times \mathbb{R}^e \) on the orbit space \( \mathbb{R}^n / \tilde{G} (= \mathbb{R}^2 \text{ or } [0, \infty) \times \mathbb{R}) \) is the point \( o \) the origin of \( \mathbb{R}^2 \) or \( [0, \infty) \times \mathbb{R} \). Then \( o \) is a fixed point of the action of
Then, by Remark 3.7(1), \( \frac{\tilde{\Omega}}{\tilde{\Delta}} = [0, \infty) \times \mathbb{R} \) or \( \mathbb{R}^2 \).

c, d) If \( \tilde{G} \) has compact factor or helicoidal factor, then we have \( \tilde{G} = G_1 \times G_2 \) and \( \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) such that \( G_1 \) is compact or helicoidal on \( \mathbb{R}^{n_1} \) and \( G_2 \) acts transitively on \( \mathbb{R}^{n_2} \). So, we have

\[
\frac{\mathbb{R}^n}{G} = \frac{\mathbb{R}^{n_1}}{G_1} \times \frac{\mathbb{R}^{n_2}}{G_2} = \mathbb{R}^{n_1}.
\]

The effective action of \( \tilde{\Delta} \) on \( \frac{\mathbb{R}^n}{G} \) induces an effective action of \( \tilde{\Delta} \) on \( \frac{\mathbb{R}^{n_1}}{G_1} \) in the following way:

Each \( G \)-orbit is in the form \( D \times \mathbb{R}^{n_2} \) such that \( D \) is a \( G_1 \)-orbit in \( \mathbb{R}^{n_1} \). For each \( \tilde{\delta} \in \tilde{\Delta} \), we have \( \tilde{\delta}(D \times \mathbb{R}^{n_2}) = D' \times \mathbb{R}^{n_2} \). Put \( \tilde{\delta}(D) = D' \). Then we get the from previous arguments that theorem is true in this case.

e) In this case all \( \tilde{G} \)-orbits in \( \mathbb{R}^n \) are isometric to \( \mathbb{R}^{n-2} \) then each \( G \)-orbit is flat, which is contradiction by assumptions of the theorem.

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References


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