Some optimal inequalities on Bochner-Kähler manifolds with Casorati curvatures

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Abstract. The main purpose of this article is to construct optimal inequalities on some submanifolds in a Bochner-Kähler manifold involving Casorati curvatures.

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1 Introduction

The Bochner tensor was introduced by S. Bochner in Kähler manifolds analogue of the Weyl conformal curvature tensor [1]. The Bochner tensor is equal to the fourth order Chern-Moser curvature tensor of CR-manifolds by Webster [19]. Webster showed that a Bochner-Kähler surface is nothing but a self-dual Kähler surface in Penrose’s theory. A Kähler manifold is said to be Bochner-Kähler if its Bochner curvature tensor vanishes. Bochner-Kähler manifolds with constant scalar curvature are classified in [15]. Moreover, Chen and Dillen investigated geometric characterizations of Bochner-Kähler and Einstein-Kähler spaces of complex space forms by using the $\delta$-invariants $\delta(n_1, n_2, \cdots, n_k)$ and $\hat{\delta}(n_1, n_2, \cdots, n_k)$ in [4]. On the other hand, it is well known that the Casorati curvature of a submanifold in a Riemannian manifold is an extrinsic invariant defined as the normalized square of the length of the second fundamental form, introduced by F. Casorati in [2, 9]. Moreover, there are very interesting optimal inequalities involving Casorati curvatures in [5, 6, 7, 8, 10, 11, 12, 13, 14, 17, 20, 21] for several submanifolds in some space forms with various connections. In our paper, we establish optimal inequalities involving the generalized normalized $\delta$-Casorati curvatures for some submanifolds in a Bochner-Kähler manifold and also characterize those submanifolds for which the equalities hold.

2 Preliminaries

This section gives several basic definitions and notations for our framework based mainly.
Let $M^n$ be an $n$-dimensional Riemannian submanifold of a Riemannian manifold $(\bar{M}, \bar{g})$ with the Riemannian metric $\bar{g}$. Let $K(\pi)$ be the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM$, $p \in M$. Assume that $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space $T_pM$ and $\{e_{n+1}, ..., e_m\}$ is an orthonormal basis of the normal space $T_p^\perp M$. Then the scalar curvature $\tau$ at $p$ is given by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$$

and the normalized scalar curvature $\rho$ of $M$ is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$  

We denote by $H$ the mean curvature vector, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i)$$

and we also set

$$h_{ij}^\alpha = g(h(e_i, e_j), e_\alpha), \quad i, j \in \{1, ..., n\}, \quad \alpha \in \{n+1, ..., m\}.$$  

Then it is well-known that the squared mean curvature of the submanifold $M$ in $\bar{M}$ is defined by

$$||H||^2 = \frac{1}{n^2} \sum_{\alpha = n+1}^m \left( \sum_{i=1}^n h_{ii}^\alpha \right)^2$$

and the squared norm of $h$ over dimension $n$ is denoted by $C$, called the Casorati curvature of the submanifold $M$. Therefore we have

$$C = \frac{1}{n} \sum_{\alpha = n+1}^m \sum_{i,j=1}^n (h_{ij}^\alpha)^2.$$  

The submanifold $M$ is called invariantly quasi-umbilical if there exists $m - n$ mutually orthogonal unit normal vectors $\xi_{n+1}, ..., \xi_m$ such that the shape operators with respect to all directions $\xi_\alpha$ have an eigenvalue of multiplicity $n - 1$ and that for each $\xi_\alpha$ the distinguished eigendirection is the same.

Suppose now that $L$ is an $s$-dimensional subspace of $T_pM$, $s \geq 2$ and let $\{e_1, ..., e_s\}$ be an orthonormal basis of $L$. Then the scalar curvature $\tau(L)$ of the $s$-plane section $L$ is given by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq s} K(e_\alpha \wedge e_\beta).$$

and the Casorati curvature $C(L)$ of the subspace $L$ is defined as

$$C(L) = \frac{1}{s} \sum_{\alpha = n+1}^m \sum_{i,j=1}^s (h_{ij}^\alpha)^2.$$
The generalized normalized $\delta$-Casorati curvatures $\delta_C(t; n-1)$ and $\hat{\delta}_C(t; n-1)$ of the submanifold $M^n$ are defined for any positive real number $r \neq n(n-1)$ as

$$[\delta_C(t; n-1)]_p = tC_p + \frac{(n-1)(n+t)(n^2 - n - t)}{nt} \inf \{ C(L) | L \text{ a hyperplane of } T_p M \},$$

if $0 < t < n^2 - n$, and

$$[\hat{\delta}_C(t; n-1)]_p = tC_p - \frac{(n-1)(n+t)(t - n^2 + n)}{nt} \sup \{ C(L) | L \text{ a hyperplane of } T_p M \},$$

if $t > n^2 - n$.

If $\nabla$ is the Levi-Civita connection on $\overline{M}$ and $\nabla$ is the covariant differentiation induced on $M$, then the Gauss and Weingarten formulas are given by:

$$\nabla_X Y = \nabla_X Y + h(X,Y), \forall X,Y \in \Gamma(TM)$$

and

$$\nabla_X N = -A_N X + \nabla^\perp_X N, \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^\perp)$$

where $h$ is the second fundamental form of $M$, $\nabla^\perp$ is the connection on the normal bundle and $A_N$ is the shape operator of $M$ with respect to $N$. If we denote by $\overline{R}$ and $R$ the curvature tensor fields of $\nabla$ and $\nabla$, then we have the Gauss equation:

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + \bar{g}(h(X,W),h(Y,Z)) - \bar{g}(h(X,Z),h(Y,W)),$$

(2.1)

for all $X,Y,Z,W \in \Gamma(TM)$.

Assume now that $(\overline{M}^m, \bar{g}, J)$ is an almost Hermitian with an almost complex structure $J$ and a Riemannian metric $\bar{g}$ satisfying for

$$\bar{g}(J \cdot, J \cdot) = \bar{g}(\cdot, \cdot) \quad \text{and} \quad J^2 = -\text{Id},$$

where $\text{Id}$ denotes the identity tensor field of type $(1, 1)$ on $\overline{M}$. Moreover, if the almost complex structure $J$ is parallel with respect to the Levi-Civita connection $\nabla$ of $\bar{g}$, then $(\overline{M}, \bar{g}, J)$ is said to be a Kähler manifold.

The Bochner curvature tensor on a Kähler manifold is defined by [18]

$$B(X,Y,Z,W) = \overline{R}(X,Y,Z,W) - L(Y,Z)\bar{g}(X,W) - L(X,Z)\bar{g}(Y,W) + L(X,W)\bar{g}(Y,Z) + L(Y,W)\bar{g}(X,Z) + L(JY,W)\bar{g}(JX,Y) + L(JX,W)\bar{g}(JY,Z) + 2L(JY,W)\bar{g}(JX,Y),$$

(2.2)

where

$$L(X,Y) = \frac{1}{2n+4}Ric(X,Y) - \frac{\tau}{8(n+1)(n+2)} \bar{g}(X,Y)$$

and

$$L(X,Y) = L(Y,X), \quad L(JX,Y) = -L(X,JY),$$

(2.3)
for all \( X, Y, Z, W \in \Gamma(TM) \).

Let \((\bar{M}, \bar{g}, J)\) be a Kähler manifold. If the Bochner tensor \( B \) on \( \bar{M} \) vanishes identically, \((\bar{M}, \bar{g}, J)\) is called a Bochner-Kähler manifold. From (2.2), the curvature tensor \( \bar{R} \) of a Bochner-Kähler manifold is given by

\[
\bar{R}(X, Y, Z, W) = L(Y, Z)\bar{g}(X, W) - L(X, Z)\bar{g}(Y, W) \\
+ L(X, W)\bar{g}(Y, Z) - L(Y, W)\bar{g}(X, Z) \\
+ L(JY, Z)\bar{g}(JX, W) - L(JX, Z)\bar{g}(JY, W) \\
+ L(JX, W)\bar{g}(JY, Z) - L(JY, W)\bar{g}(JX, Z) \\
- 2L(JX, Y)\bar{g}(JZ, W) - 2L(JZ, W)\bar{g}(JX, Y).
\]

(2.4)

As a generalization of CR-submanifolds, B.-Y. Chen introduced the notion of slant submanifolds. We introduce the definition of slant submanifolds of Bochner-Kähler manifolds as follows:

**Definition 2.1.** A submanifold \( M \) of a Bochner-Kähler manifold \((\bar{M}, \bar{g}, J)\) is said to be slant if for any \( p \in M \), the angle \( \theta \) between \( JX \) and \( T_pM \) is constant. In other words, the angle does not depend on the choice of \( p \in M \) and \( X \in T_pM \). The angle \( \theta \in [0, \pi/2] \) is called the slant angle of \( M \) in \( \bar{M} \).

If \( \theta = 0 \) (\( \theta = \pi/2 \)), \( M \) is called an invariant (anti-invariant) submanifold of \( \bar{M} \), respectively. If \( 0 < \theta < \pi/2 \), \( M \) is called a proper slant submanifold of \( \bar{M} \).

The following lemma plays a key role in the proof of our theorems.

**Lemma 2.1.** [16] Let 

\[ \Gamma = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = k\} \]

be a hyperplane of \( \mathbb{R}^n \), and \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) a quadratic form given by

\[ f(x_1, x_2, \cdots, x_n) = a \sum_{i=1}^{n-1} (x_i)^2 + b(x_n)^2 - 2 \sum_{1 \leq i < j \leq n} x_ix_j, \quad a > 0, \ b > 0. \]

Then, by the constrained extremum problem, \( f \) has the global extreme as follows:

\[ x_1 = x_2 = \cdots = x_{n-1} = \frac{k}{a+1}, \quad x_n = \frac{k}{b+1} = \frac{k(n-1)}{(a+1)b} = \frac{(a-n+2)k}{a+1}, \]

provided that

\[ b = \frac{n-1}{a-n+2}. \]

### 3 Inequalities involving Casorati curvatures

Let \( M \) be a submanifold of a Bochner-Kähler manifold \((\bar{M}, \bar{g}, J)\). Let \( p \in M \) and the set \{\( e_1, \ldots, e_n \)\} and \{\( e_{n+1}, \ldots, e_m \)\} be orthonormal bases of \( T_pM \) and \( T_p^\perp M \), respectively. From (2.4), we have

\[
\bar{R}(e_i, e_j, e_j, e_i) = L(e_j, e_j)\bar{g}(e_i, e_i) + L(e_i, e_i)\bar{g}(e_j, e_j) \\
+ 6L(e_i, Je_j)\bar{g}(e_i, Je_j) - 2L(e_i, Je_j)\bar{g}(e_i, e_i).
\]

(3.1)
From (3.1), we have
\[\sum_{i,j=1}^{n} R(e_i, e_j, e_j, e_i) = (2n - 2) \sum_{i=1}^{n} L(e_i, e_i) + 6 \sum_{i,j=1}^{n} L(e_i, J e_j) \bar{g}(e_i, J e_j)\]

Combining (2.1) and (3.2), we obtain
\[2\tau = n^2 ||H||^2 - ||h||^2 + (2n - 2) \sum_{i=1}^{n} L(e_i, e_i) + 6 \sum_{i,j=1}^{n} L(e_i, J e_j) \bar{g}(e_i, J e_j)\]
\[= n^2 ||H||^2 - nC + (2n - 2) \sum_{i=1}^{n} L(e_i, e_i) + 6 \sum_{i,j=1}^{n} L(e_i, J e_j) \bar{g}(e_i, J e_j)\]

We now consider the following quadratic polynomial in the components of the second fundamental form:
\[P = tC + \frac{(n-1)(n+t)(n^2-n-t)}{nt} C(L) - 2\tau + (2n - 2) \sum_{i=1}^{n} L(e_i, e_i) + 6 \sum_{i,j=1}^{n} L(e_i, J e_j) \bar{g}(e_i, J e_j),\]

where \(L\) is a hyperplane of \(T_p M\). Without loss of generality we can assume that \(L\) is spanned by \(e_1, \ldots, e_{n-1}\). Then we derive
\[P = \sum_{\alpha = n+1}^{m} \sum_{i=1}^{n-1} \left[ \frac{n^2 + n(t-1) - 2t}{t} (h_{ii}^\alpha)^2 + \frac{2(n+t)}{n} (h_{nn}^\alpha)^2 \right] + \frac{2(n+t)(n-1)}{t} \sum_{i=1}^{n-1} (h_{ij}^\alpha)^2 - 2 \sum_{i<j}^{n} h_{ii}^\alpha h_{jj}^\alpha + \frac{t}{n} (h_{nn}^\alpha)^2 \]
\[\geq \sum_{\alpha = n+1}^{m} \sum_{i=1}^{n-1} \left[ \frac{n^2 + n(t-1) - 2t}{t} (h_{ii}^\alpha)^2 - 2 \sum_{i<j}^{n} h_{ii}^\alpha h_{jj}^\alpha + \frac{t}{n} (h_{nn}^\alpha)^2 \right] .\]

For \(\alpha = n+1, \ldots, m\), let us consider the quadratic form \(f_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}\) defined by
\[f_\alpha(h_{11}^\alpha, \ldots, h_{nn}^\alpha) = \frac{n^2 + n(t-1) - 2t}{t} \sum_{i=1}^{n-1} (h_{ii}^\alpha)^2 - 2 \sum_{i<j}^{n} h_{ii}^\alpha h_{jj}^\alpha + \frac{t}{n} (h_{nn}^\alpha)^2 ,\]
and the constrained extremum problem
\[\min f_\alpha\]
subject to \(F^\alpha : h_{11}^\alpha + \cdots + h_{nn}^\alpha = c^\alpha\),
where \(c^\alpha\) is a real constant. Comparing (3.5) with the quadratic function in Lemma 2.1, we see that
\[a = \frac{n^2 + n(t-1) - 2t}{t}, \quad b = \frac{t}{n} .\]

Therefore, we have the critical point \((h_{11}^\alpha, \ldots, h_{nn}^\alpha)\), given by
\[h_{11}^\alpha = h_{22}^\alpha = \cdots = h_{n-1}^\alpha = \frac{tc^\alpha}{(n+t)(n-1)}, \quad h_{nn}^\alpha = \frac{nc^\alpha}{n+t} .\]
is a global minimum point by Lemma 2. Moreover, \( f_\alpha (h_{a_1}^{\alpha}, \cdots, h_{a_n}^{\alpha}) = 0 \). Therefore, we have

\[
\text{(3.6)} \quad \mathcal{P} \geq 0,
\]

which implies

\[
2 \tau(p) \leq t \mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt} \mathcal{C}(L) + (2n-2) \sum_{i=1}^{n} L(e_i, e_i) + 6 \sum_{i,j=1}^{n} L(e_i, Je_j) \tilde{g}(e_i, Je_j)\]

\[
= t \mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt} \mathcal{C}(L)
\]

\[
+ \frac{(2n-2)(3n+4)-6||P||^2}{2(2n+2)(2n+4)} \tau - \frac{6}{2n+4} \sum_{i,j=1}^{n} \text{Ric}(e_i, Je_j) \tilde{g}(e_i, Je_j),
\]

where \( ||P||^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j) \) for \( JX = PX + QX, X \in \Gamma(TM) \) whose \( PX \) and \( QX \) are the tangential and normal components of \( JX \), respectively.

From (2.3), we derive

\[
\frac{5n^2+23n+20+3||P||^2}{4(n+1)(n+2)} \tau \leq t \mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt} \mathcal{C}(L)
\]

\[- \frac{3}{n+2} \sum_{i,j=1}^{n} \text{Ric}(e_i, Je_j) \tilde{g}(e_i, Je_j).
\]

Therefore, we derive

\[
\rho \leq \frac{8(n+1)(n+2)}{n(n-1)(5n^2+23n+20+3||P||^2)} \left( t \mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt} \mathcal{C}(L) \right)
\]

\[- \frac{6(n+1)}{n(n-1)(5n^2+23n+20+3||P||^2)} \sum_{i,j=1}^{n} \text{Ric}(e_i, Je_j) \tilde{g}(e_i, Je_j).
\]

Therefore, we have the following theorem:

**Theorem 3.1.** Let \( M^n \) be an \( n \)-dimensional Riemannian submanifold of a Bochner-Kähler manifold \((\bar{M}, \bar{\mathcal{g}}, J)\). When \( 0 < t < n^2 - n \), the generalized normalized \( \delta \)-Casorati curvature \( \delta_{\mathcal{C}}(t, n-1) \) on \( M^n \) satisfies

\[
\rho \leq \frac{8(n+1)(n+2)}{n(n-1)(5n^2+23n+20+3||P||^2)} \delta_{\mathcal{C}}(t, n-1)
\]

\[- \frac{6(n+1)}{n(n-1)(5n^2+23n+20+3||P||^2)} \sum_{i,j=1}^{n} \text{Ric}(e_i, Je_j) \tilde{g}(e_i, Je_j).
\]

Moreover, the equality case holds if and only if \( M^n \) is an invariantly quasi-umbilical submanifold with trivial normal connection in a Bochner-Kähler manifold \((\bar{M}, \bar{\mathcal{g}}, J)\).
such that with respect to suitable orthonormal tangent frame \(\{\xi_1, \cdots, \xi_n\}\) and normal orthonormal frame \(\{\xi_{n+1}, \cdots, \xi_m\}\), the shape operators \(A_r \equiv A_{\xi_r}, r \in \{n+1, \cdots, m\}\), take the following forms:

\[
A_{n+1} = \begin{pmatrix}
  a & 0 & 0 & \cdots & 0 & 0 \\
  0 & a & 0 & \cdots & 0 & 0 \\
  0 & 0 & a & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a & 0 \\
  0 & 0 & 0 & \cdots & 0 & n(n-1) \alpha
\end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0.
\]

**Corollary 3.2.** Let \(M^n\) be an \(n\)-dimensional Einstein submanifold of a Bochner-Kähler manifold \((M^m, \tilde{g}, J)\). Then, for a Ricci curvature \(\lambda\), we obtain

\[
\rho \leq \frac{8(n+1)(n+2)}{n(n-1)(5n^2 + 23n + 20 + 3||P||^2)} \delta_C(t, n-1)
- \frac{6(n+1)||P||^2}{n(n-1)(5n^2 + 23n + 20 + 3||P||^2)} \lambda.
\]

Moreover, the equality case holds if and only if with respect to a suitable frames \(\{e_1, \cdots, e_n\}\) on \(M\) and \(\{e_{n+1}, \cdots, e_m\}\) on \(T_p^1 M, p \in M\), the components of \(h\) satisfy

\[
h_{11}^\alpha = h_{22}^\alpha = \cdots = h_{n-1,n-1}^\alpha = \frac{n}{(n-1)} h_{nn}^\alpha, \quad \alpha \in \{n+1, \cdots, m\},
\]

\[
h_{ij}^\alpha = 0, \quad i, j \in \{1, 2, \cdots, n\}(i \neq j), \quad \alpha \in \{n+1, \cdots, m\}.
\]

For a slant submanifold of a Bochner-Kähler manifold, we have following corollaries.

**Corollary 3.3.** Let \(M^n\) be an \(n\)-dimensional slant submanifold of a Bochner-Kähler manifold \((M^m, \tilde{g}, J)\). When \(0 < t < n^2 - n\), we obtain

\[
\rho \leq \frac{8(n+1)(n+2)}{n(n-1)(5n^2 + 23n + 20 + 3\cos^2 \theta)} \delta_C(t, n-1)
- \frac{6(n+1)}{n(n-1)(5n^2 + 23n + 20 + 3\cos^2 \theta)} \sum_{i,j=1}^{n} \text{Ric}(e_i, Je_j) \cos^2 \theta,
\]

where \(\theta\) is a slant function. Moreover, the equality case holds if and only if with respect to a suitable frames \(\{e_1, \cdots, e_n\}\) on \(M\) and \(\{e_{n+1}, \cdots, e_m\}\) on \(T_p^1 M, p \in M\), the components of \(h\) satisfy

\[
h_{11}^\alpha = h_{22}^\alpha = \cdots = h_{n-1,n-1}^\alpha = \frac{t}{(n-1)} h_{nn}^\alpha, \quad \alpha \in \{n+1, \cdots, m\},
\]

\[
h_{ij}^\alpha = 0, \quad i, j \in \{1, 2, \cdots, n\}(i \neq j), \quad \alpha \in \{n+1, \cdots, m\}.
\]

**Corollary 3.4.** Let \(M^n\) be an \(n\)-dimensional invariant submanifold of a Bochner-Kähler manifold \((M^m, \tilde{g}, J)\). When \(0 < t < n^2 - n\), we obtain

\[
\rho \leq \frac{8(n+1)(n+2)}{n(n-1)(5n^2 + 23n + 23)} \delta_C(t, n-1)
- \frac{6(n+1)}{n(n-1)(5n^2 + 23n + 23)} \sum_{i,j=1}^{n} \text{Ric}(e_i, Je_j),
\]
Moreover, the equality case holds if and only if with respect to a suitable frames \( \{e_1, \ldots, e_n\} \) on \( M \) and \( \{e_{n+1}, \ldots, e_m\} \) on \( T^\perp_p M, p \in M \), the components of \( h \) satisfy
\[
 h_{11}^\alpha = h_{22}^\alpha = \cdots = h_{n-1,n-1}^\alpha = \frac{\ell}{n(n-1)} h_{nn}^\alpha, \quad \alpha \in \{n + 1, \cdots, m\}, \\
 h_{ij}^\alpha = 0, \quad i, j \in \{1, 2, \cdots, n\} (i \neq j), \quad \alpha \in \{n + 1, \cdots, m\}.
\]

**Corollary 3.5.** Let \( M^n \) be an \( n \)-dimensional anti-invariant submanifold of a Bochner-Kähler manifold \((\bar{M}^m, \bar{\gamma}, J)\). When \( 0 < t < n^2 - n \), we obtain
\[
 \rho \leq \frac{8(n + 1)(n + 2)}{n(n - 1)(5n^2 + 23n + 20)} \delta_C(t, n - 1),
\]

Moreover, the equality case holds if and only if \( M \) is an invariantly quasi-umbilical submanifold of Bochner-Kähler manifold.

**Remark 3.1.** In the case for \( t > n^2 - n \), the methods of finding the above inequalities is analogous. Thus, we leave the problems for readers.

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