The new Minkowski norm and integral formulas for a manifold endowed with a set of one-forms

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Abstract. Integral formulas are the power tool for obtaining global results in Analysis and Geometry. We explore the problem: Find integral formulas for a closed manifold endowed with a set of linearly independent 1-forms (or vector fields). In our recent works in common with P. Walczak, the problem was examined for a manifold endowed with a codimension-one foliation and a 1-form \( \beta \), using approach of Randers norm. Continuing this study, we introduce new Minkowski norm, determined by Euclidean norm \( \alpha \), linearly independent 1-forms \( \beta_i \), \((1 \leq i \leq p)\) and a function \( \phi \) of \( p \) variables; this produces a new class of “computable” Finsler metrics generalizing Matsumoto’s \((\alpha,\beta)\)-metric. The geometrical meaning of our Minkowski norm is that its indicatrix is a rotation hypersurface with the axis \( \bigcap_{i=1}^{p} \ker \beta_i \) passing through the origin. We explore a Riemannian structure, naturally arising from this norm and a codimension-one distribution \( \ker \omega \) of 1-form \( \omega \neq 0 \), and find the second fundamental form of \( \ker \omega \) through invariants of \( \alpha, \omega, \beta_i \) and \( \phi \). Then we apply the above to prove new integral formulas for a closed Riemannian manifold endowed with a codimension-one distribution and linearly independent 1-forms \( \beta_i \), \((1 \leq i \leq p)\), which generalize the Reeb’s integral formula and its counterpart for the second mean curvature of the distribution.


Key words: Riemannian metric; Minkowski norm, 1-form; shape operator; mean curvature; Ricci curvature; integral formula.

Integral formulas are the power tool for obtaining global results in Analysis and Geometry (e.g. generalized Gauss-Bonnet theorem and Minkowski-type formulas for submanifolds). Such formulas are usually proved applying the Divergence theorem to appropriate vector field. The first known integral formula by G. Reeb [10], for a closed Riemannian manifold \((M,\alpha)\) endowed with a 1-form \( \omega \neq 0 \) tells us that the total mean curvature \( H \) of the distribution \( \ker \omega \) vanishes:

\[
\int_M H \, d\text{vol}_\alpha = 0;
\]

thus, either $H \equiv 0$ or $H(x)H(x') < 0$ for some points $x \neq x'$. Its counterpart (6.1) for the second mean curvature of a codimension one foliation (see [9]) has been used to estimate the energy of a vector field [3] and to prove that codimension-one foliations with negative Ricci curvature are far from being totally umbilical [6]. Recently, these were extended into infinite series of integral formulas including the higher order mean curvatures of the leaves and curvature tensor, see [1, 7, 11]. The integral formulas for foliations can be used for prescribing the mean curvatures of the leaves, e.g. characterizing totally geodesic, totally umbilical and Riemannian foliations.

We explore the problem: Find integral formulas for a closed Riemannian manifold endowed with a set of linearly independent 1-forms (or vector fields). The “maximal number of pointwise linearly independent vector fields on a closed manifold” is an important topological invariant; such vector fields on a sphere $S^l$ are built using orthogonal multiplications on $\mathbb{R}^{l+1}$.

In [12, 13], the problem was examined for $(M, a)$ endowed with 1-forms $\omega \neq 0$ and $\beta$, using approach of Randers norm, that is a Euclidean norm $\alpha$ shifted by a small vector. In the paper we extend this approach for $(M, a)$ with the codimension-one distribution $\text{ker} \omega$ and $p$ linearly independent 1-forms $\beta_1, \ldots, \beta_p$, by introducing new Minkowski norm, generalizing $(\alpha, \beta)$-norm of M. Matsumoto, see [8]. Remark that navigation $(\alpha, \beta)$-norms appear when $p = 2$. The $(\alpha, \beta)$-metrics form a rich class of computable Finsler metrics and play an important role in geometry, see [2, 8, 14, 17], thus we expect that our so called $(\alpha, \vec{\beta})$-metrics will also find many applications.

The paper contains an introduction and six sections. In Section 1 we introduce and explore the $(\alpha, \vec{\beta})$-norm, determined by Euclidean norm $\alpha$, linearly independent 1-forms $\beta_1, \ldots, \beta_p$ and a function $\phi$ of $p$ variables; the indicatrix is a rotational hypersurface with $p$-dimensional rotation axis. The norm produces a class of “computable” Finsler metrics generalizing Matsumoto’s $(\alpha, \beta)$-metric. In Sections 2–4 we study a new Riemannian structure, naturally arising on $M$ endowed with $(\alpha, \vec{\beta})$-metric with $\vec{\beta} = (\beta_1, \ldots, \beta_p)$ and 1-form $\omega \neq 0$, and calculate the second fundamental form of the distribution $\text{ker} \omega$ through invariants of $\alpha, \omega, \beta_i$ and $\phi$. Sections 5–6 contain applications to proving new integral formulas for a closed $M$ endowed with a codimension-one distribution $\text{ker} \omega$ and a set of linearly independent 1-forms, which generalize the Reeb’s formula (0.1) and its counterpart for the second mean curvature of the distribution. Using our norm and assuming for simplicity $p = 1$, we get new estimates of the “non-umbilicity” of a codimension-one distribution and the energy of a vector field.

1 The $(\alpha, \vec{\beta})$-norm

In this section, we define a new Minkowski norm, generalizing the $(\alpha, \beta)$-norm of M. Matsumoto.

A Minkowski norm on a vector space $V^{m+1}$ ($m \geq 1$) is a function $F : V \to [0, \infty)$ with the properties of regularity, positive 1-homogeneity and strong convexity [14]:

- $M_1 : F \in C^\infty(V \setminus \{0\})$.
- $M_2 : F(\lambda y) = \lambda F(y)$ for $\lambda > 0$ and $y \in V$.
- $M_3 : \text{For any } y \in V \setminus \{0\}$, the following symmetric bilinear form is positive definite:

$$
(1.1) \quad g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]_{s=t=0}.
$$
By \( M_2 - M_3 \), \( g_{xy} = g_y (\lambda > 0) \) and \( g_y(y, y) = F^2(y) \). As a result of \( M_3 \), the indicatrix \( S := \{ y \in V : F(y) = 1 \} \) is a closed, convex smooth hypersurface that surrounds the origin.

The following symmetric trilinear form is called the Cartan torsion for \( F \):

\[
C_y(u, v, w) = \frac{1}{4} \frac{\partial^3}{\partial r \partial s \partial t} \left[ F^2(y + ru + sv + tw) \right]_{r=s=t=0},
\]

where \( y, u, v, w \in V \) and \( y \neq 0 \). Note that \( C_y(u, v, y) = 0 \) and \( C_{\lambda y} = \lambda^{-1} C_y \) for \( \lambda > 0 \). Vanishing of a 1-form \( I_y(u) = \text{Tr}_{g_y} C_y(u, \cdot, \cdot) \), called the mean Cartan torsion, characterizes Euclidean norms among all Minkowski norms, see e.g. [14].

**Definition 1.1.** Given \( p \in \mathbb{N} \) and \( \delta_i > 0 \) (\( i \leq p \)), let \( \phi : \Pi \to (0, \infty) \) be a smooth function on \( \Pi = \prod_{i=1}^{p} [-\delta_i, \delta_i] \), and \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \) a scalar product with the Euclidean norm \( \alpha(y) = \langle y, y \rangle^{1/2} \) on a \((m + 1)\)-dimensional vector space \( V \). Given linearly independent 1-forms \( \beta_i \) (\( 1 \leq i \leq p \)) on \( V \) of the norm \( \alpha(\beta_i) < \delta_i \), the \((\alpha, \beta)\)-norm (see below Lemma 1.3 on regularity) with \( \tilde{\beta} = (\beta_1, \ldots, \beta_p) \) is defined on \( V \setminus \{0\} \) by

\[
F(y) = \alpha(y) \phi(s), \quad s = (s_1, \ldots, s_p), \quad s_i = \beta_i(y)/\alpha(y).
\]

Usually, we assume \( \phi(0, \ldots, 0) = 1 \). We call \( \alpha \) the associated norm (or metric).

The geometrical meaning of (1.3) is that the indicatrix of \( F \) is a rotation hypersurface in \( V \) with the axis \( \bigcap_{i=1}^{p} \ker \beta_i \) passing through the origin, see below Proposition 1.1. For \( p = 1 \), (1.3) defines the \((\alpha, \beta)\)-norm. By shifting the indicatrix of an \((\alpha, \beta)\)-norm, we obtain new Minkowski norms, called navigation \((\alpha, \beta)\)-norms, [17].

The indicatrix of this norm is still a rotation hypersurface, but the rotation axis does not pass the origin in general. Meanwhile, this is a case of \((\alpha, \tilde{\beta})\)-norm with \( p = 2 \), whose indicatrix has a two-dimensional rotation axis passing through the origin.

The “musical isomorphisms” \( \sharp \) and \( \flat \) will be used for rank one and symmetric rank 2 tensors. For example, \( \langle \beta_i^g, u \rangle = \beta_i(u) = u^i(\beta^g_i) \). We will use Einstein summation convention. Set

\[
b_{ij} = \langle \beta_i, \beta_j \rangle = \langle \beta_i^g, \beta_j^g \rangle.
\]

A Minkowski norm on \( V^{m+1} \) is Euclidean if and only if it is preserved under the action of \( O(m+1) \). Next, we will clarify the geometric property about the indicatrices of \((\alpha, \tilde{\beta})\)-metrics.

**Definition 1.2** (The symmetry of a Minkowski norm, see [17]). Let \( F \) be a Minkowski norm on \( V^{m+1} \) and \( G \) a subgroup of \( GL(m + 1, \mathbb{R}) \). Then \( F \) is called \( G \)-invariant if the following holds for some affine coordinates \((y^1, \ldots, y^{m+1})\) of \( V \):

\[
F(y^1, \ldots, y^{m+1}) = F((y^1, \ldots, y^{m+1}) f), \quad y \in V, \ f \in G.
\]

The next proposition for \( p = 1 \) belongs to [17].

**Proposition 1.1.** Let \( F \) be a Minkowski norm and \( \beta_i \) (\( 1 \leq i \leq p \)) linearly independent 1-forms on a vector space \( V^{m+1} \). Then \( F \) is an \((\alpha, \tilde{\beta})\)-norm with \( \tilde{\beta} = (\beta_1, \ldots, \beta_p) \) if and only if \( F \) is \( G \)-invariant, where \( G = \{ x \in GL(m + 1, \mathbb{R}) : x = \begin{pmatrix} C & 0 \\ 0 & I_p \end{pmatrix}, \ C \in GL(m - p + 1, \mathbb{R}) \} \).
Proof. Let \( F = \alpha(\beta_1, \ldots, \beta_p) \) be the \((\alpha, \beta)\)-norm. Let \( \{e_1, \ldots, e_{m+1}\} \) be an \( \langle \cdot, \cdot \rangle \)-orthonormal basis such that \( \bigcap_{i=1}^p \ker \beta_i = \text{span}\{e_1, \ldots, e_{m-p+1}\} \). Then \( \beta_i(y) = \sum_{j=m-p+2}^{m+1} \beta_i(e_j)y^j \) where

\[
F(y) = \sqrt{(y^1)^2 + \ldots + (y^{m+1})^2} \left( \frac{\sum_{j=m-p+2}^{m+1} \beta_i(e_j)y^j}{\sqrt{(y^1)^2 + \ldots + (y^{m+1})^2}} \right) \]

and \( y = y^i e_i \). Hence, \( F \) is \( G \)-invariant.

Conversely, let \( F \) obey (1.4) for \( G \) and affine coordinates \( y = (y^1, \ldots, y^{m+1}) \). If \( p = m+1 \) then for \( G = \{id_{m+1}\} \) one may take \( \beta_i = e_i^1 \) and use axiom \( M_2 \). Let \( p \leq m \). By restricting \( F \) on the \((m-p+1)\)-dimensional linear subspace \( U \) given by \( p \) equations \( y^{m-p+2} = \ldots = y^{m+1} = 0 \), one obtains an \( O(m-p+1) \)-invariant Minkowski norm, which must be Euclidean. Thus, there exists \( B > 0 \), such that the norm \( \alpha(y) = B\sqrt{(y^1)^2 + \ldots + (y^{m+1})^2} \) on \( V \) obeys \( \alpha|_U = F|_U \). Set

\[
\tilde{\phi}(y) = F(y)/\alpha(y) \quad (y \neq 0).
\]

Then \( \tilde{\phi} \) is \( G \)-invariant, hence \( \tilde{\phi} \) depends on \( p \) variables \( y^{m-p+2}, \ldots, y^{m+1} \) only. Since \( \tilde{\phi} \) is \( 0 \)-homogeneous, we have \( \phi(y) = \tilde{\phi}(By^{m-p+2}/\alpha(y), \ldots, By^{m+1}/\alpha(y)) \), that is \( \beta_i = B e_i^{m-p+1} \).

Define real functions \( \rho, \rho_0^{ij}, \rho_i^1 \) \((1 \leq i, j \leq p)\) of variables \( s = (s_1, \ldots, s_p) \), see also (1.3):

\[
\rho = \phi \left( \frac{\partial \phi}{\partial s_i} \right), \quad \rho_0^{ij} = \phi \tilde{\phi}_{ij} + \phi_i \tilde{\phi}_j, \quad \rho_i^1 = \phi \tilde{\phi}_i - \sum_j s_j \left( \phi \tilde{\phi}_{ij} + \phi_i \tilde{\phi}_j \right),
\]

where \( \phi_i = \frac{\partial \phi}{\partial s_i}, \phi_{ij} = \frac{\partial^2 \phi}{\partial s_i \partial s_j}, \) etc. Assume in the paper that \( \rho > 0 \), thus

\[
\phi - \sum_i s_i \phi_i > 0.
\]

The following relations hold:

\[
\dot{\rho}_i = \rho_1^i, \quad \ddot{\rho}_{ij} = (\rho_1^i)_j = -s_k (\rho_0^{ik})_j.
\]

**Proposition 1.2.** For \((\alpha, \beta)\)-norm, the bilinear form \( g_y(y \neq 0) \) in (1.1) is given by

\[
\begin{align*}
g_y(u, v) &= \rho(u, v) + \rho_0^{ij} \beta_i(u) \beta_j(v) \\
&\quad + \rho_1^i (\beta_i(u) y, v) + \beta_i(v)(y, u))/\alpha(y) - \beta_i(y) \rho_1^i (y, u) \alpha(y)/\alpha^3(y).
\end{align*}
\]

The Cartan tensor of \((\alpha, \beta)\)-norm is expressed by

\[
\begin{align*}
2 C_y(u, v, w) &= \alpha^{-1}(y) \sum_i \rho_1^i (K_y(u, v)p_{yi}(w) + K_y(v, w)p_{yi}(u) + K_y(w, u)p_{yi}(v)) \\
&\quad + \alpha^{-1}(y) \sum_{i,j,k} \left( \phi_{ij} \phi_{jk} + \phi_{ij} \phi_{ik} + \phi_{ij} \phi_{ik} + \phi_{ij} \phi_{ik} \right) p_{yi}(u)p_{yj}(v)p_{yk}(w),
\end{align*}
\]

where \( p_{yi} = \beta_i - s_i \rho/y/\alpha(y) \) \((1 \leq i \leq p)\) are 1-forms and \( K_y(u, v) = \langle u, v \rangle - \langle y, u \rangle \langle y, v \rangle/\alpha^2(y) \) is the angular metric of the associated metric \( a = \langle \cdot, \cdot \rangle \).
The new Minkowski norm and integral formulas

Proof. From (1.1) and (1.3) we find
\[ g_y(u, v) = |F|^2/2 \alpha K_y(u, v)/\alpha(y) + |F|^2/2 \alpha \langle y, u \rangle \langle y, v \rangle/\alpha^2(y) + \sum_i ([|F|^2/2]_{\alpha \beta_i}/\alpha(y)) \langle \langle y, u \rangle \beta_i(v) + \langle y, v \rangle \beta_i(u) \rangle + \sum_{i,j} [F^2/2]_{\beta_i \beta_j} \beta_i(u) \beta_j(v). \]

(1.7)

Calculating derivatives of \( \frac{1}{2} F^2 = \frac{1}{2} \alpha^2 \phi^2(\beta_1/\alpha, \ldots, \beta_p/\alpha), \)
\[ |F|^2/2 = \alpha \rho, \quad |F|^2/2 \beta_i = \alpha \phi \dot{\phi}_i, \quad |F|^2/2 \alpha \beta_i = \rho^1_i, \quad |F|^2/2 \beta_i \beta_j = \rho^2_{ij}, \]
(1.8)

and comparing (1.5) and (1.7), completes the proof of (1.5).

Then using equalities (1.8) and
\[ 2 C_y(u, v, w) = \alpha^{-1}(y) \sum_i [F^2/2]_{\alpha \beta_i}(K_y(u, v)p_{\beta_i}(w) + K_y(v, u)p_{\beta_i}(w) + K_y(w, u)p_{\beta_i}(w)) + \sum_{i,j,k} [F^2/2]_{\beta_i \beta_j \beta_k} p_{\beta_i}(u)p_{\beta_j}(v)p_{\beta_k}(w). \]
(1.9)

Then using equalities (1.8) and
\[ |F|^2/2 |\beta_i \beta_j \beta_k| = \alpha^{-1}(y) (\dot{\phi}_i \dot{\phi}_j \dot{\phi}_k + \dot{\phi}_i \dot{\phi}_j \dot{\phi}_k + \dot{\phi}_i \dot{\phi}_k \dot{\phi}_j + \dot{\phi}_i \dot{\phi}_j \dot{\phi}_k), \]
and comparing (1.9) and (1.6) completes the proof of (1.6).

Note that if \( s_i = 0 \) (1 \( \leq \) i \( \leq \) p) then \( \rho = 1. \) By Proposition 1.2, \( g_y \) (for small \( s_i \) and \( \rho > 0 \)) of \((\alpha, \vec{\beta})\)-norm can be viewed as a perturbed scalar product \( \langle \cdot, \cdot \rangle \).

Define nonnegative quantities: \( R_1 = \max_{s \in \Pi} \| \rho_1(s) \| - \) the maximal norm of the vector \( \rho_1 = (\rho_1^i), \) \( R_0 = \max_{s \in \Pi} \| \rho_0(s) \| - \) the maximal norm of the symmetric matrix \( \rho_0 = (\rho_0^{ij}), \) and \( R = \min_{s \in \Pi} \rho_0(s), \) where \( \Pi = \prod_{i=1}^p [-\delta_i, \delta_i] \) and \( \delta_i > 0. \)

Lemma 1.3 (Regularity). Let \( \delta_0 := (\delta_1^2 + \ldots + \delta_p^2)^{1/2} \) obeys the following inequality:
\[ (1.10) \quad \delta_0 < \frac{2R}{3R_1 + \sqrt{9R_1^2 + 4RR_0}}. \]

Then \( F \) in (1.3) is a Minkowski norm on \( V. \)

Proof. Since \( \alpha(\beta_i) \leq \delta_i \) (1 \( \leq \) i \( \leq \) p), the terms in (1.5) obey the inequalities when \( y \neq 0: \)
\[ |\rho_0^{ij} \beta_i \otimes \beta_j| \leq |\rho_0^{ij} \delta_i \delta_j| \leq R_0 \delta_0^2, \]
\[ \alpha^{-1}(y)|\rho_0^i(\beta_i \otimes y^\beta \otimes \beta_i)| \leq 2|\rho_0^i \delta_i| \leq 2R_1 \delta_0, \]
\[ \alpha^{-3}(y)|(\beta_i(y)\rho_0^i) y^\beta \otimes y^\gamma| \leq |\rho_0^i \delta_i| \leq R_1 \delta_0. \]

Thus, \( g_y \geq R - 3R_1 \delta_0 - R_0 \delta_0^2. \) The RHS of the last inequality (quadratic polynomial in \( \delta_0 \geq 0 \)) is positive if and only if \( \delta_0 < \sqrt[3]{9R_1^2 + 4RR_0 - 3R_1}, \) that is (1.10) holds.

We restrict ourselves to regular \((\alpha, \vec{\beta})\)-norms alone, that is \( \det g_y \neq 0 \) (\( y \neq 0 \)).
Let \( \{e_1, \ldots, e_{m+1}\} \) be a basis of \( V \). A scalar product (metric) \( a \) on \( V \) and similarly, the metric \( g_y \) for any \( y \neq 0 \), define volume forms by

\[
d \text{vol}_a(e_1, \ldots, e_{m+1}) = \sqrt{\det b_{ij}}, \quad d \text{vol}_{g_y}(e_1, \ldots, e_{m+1}) = \sqrt{\det g_y(e_i, e_j)}.
\]

Then

\[
d \text{vol}_{g_y} = \mu_{g_y}(y) d \text{vol}_a
\]

for some function \( \mu_{g_y}(y) > 0 \). Let \( q_k = (q_k^1, \ldots, q_k^p) \in \mathbb{R}^p \) be unit eigenvectors with eigenvalues \( \lambda_k \) of the matrix \( \{\rho_i^j + \varepsilon^{-1}\rho_i^j\} \). Define vectors \( \tilde{\beta}_k = q_k^i \beta_i \) (\( 1 \leq k \leq p \)). Then (1.5) takes the form

\[
(1.11) \quad g_y(u, v) = \rho(u, v) + \sum_i \lambda^i \tilde{\beta}_i(u) \tilde{\beta}_i(v) - \varepsilon \bar{Y}(u)\bar{Y}(v),
\]

which can be used to find \( \mu_{g_y}(y) \).

Let \( M^{m+1} \) (\( m \geq 2 \)) be a connected smooth manifold with Riemannian metric \( a = \langle \cdot, \cdot \rangle \) and the Levi-Civita connection \( \nabla \). We will generalize definition in [17] for \( p = 1 \).

**Definition 1.3.** A general \((\alpha, \tilde{\beta})\)-metric \( F \) on \( M \) is a family of \((\alpha, \tilde{\beta})\)-norms \( F_x \) in tangent spaces \( T_x M \) depending smoothly on a point \( x \in M \).

The study of a sphere \( S^{m+1} \) endowed with a general \((\alpha, \tilde{\beta})\)-metric (e.g., the bounds of curvature, and totally geodesic submanifolds) seem to be interesting and is delegated to further work.

## 2 The \((\alpha, \tilde{\beta})\)-modification of a scalar product

Let \( \omega \neq 0 \) be a 1-form and \( \beta_1, \ldots, \beta_p \) linear independent 1-forms on a vector space \( V^{m+1} \) endowed with Euclidean scalar product \( \langle \cdot, \cdot \rangle \). Let \( N \) be a unit normal to a hyperplane \( W = \ker \omega \) in \( V \),

\[
\langle N, v \rangle = 0 \quad (v \in W), \quad \langle N, N \rangle = 1.
\]

If \( W \neq \ker \beta_i \) (\( 1 \leq i \leq p \)) then \( \beta_i^{\top} \neq 0 \) (the projection of \( \beta_i^2 \) onto \( W \)) and \( |\beta_i(N)| < b_i \).

For any Minkowski norm on \( V \), there are two normal directions to \( W \), opposite when this norm is reversible, see [15]. Hence, there is a unique \( \alpha \)-unit vector \( n \in V \), which is \( g_n \)-orthogonal to \( W \) and lies in the same half-space as \( N \):

\[
g_n(n, n) = 0 \quad (v \in W), \quad \alpha(n) = 1, \quad \langle n, N \rangle > 0.
\]

Remark that \( \nu = F(n)^{-1} n \) is a \( g_n \)-unit normal to \( W \), where \( F(n) = \alpha \phi(s) \), and we get \( g_n(n, n) = \phi^2(s) \), where \( s = (s_1, \ldots, s_p) \) and

\[
s_i = \beta_i(n), \quad 1 \leq i \leq p.
\]

In what follows, in all expressions with \( s_i, \phi \) and \( \rho^i \)'s we assume (2.1). Put \( g := g_n \), thus

\[
(2.1) \quad g(u, v) = \rho(u, v) + \rho_i^j \beta_i(u) \beta_j(v) + \rho_i^j (\beta_i(u) \langle n, v \rangle + \beta_i(v) \langle n, u \rangle) - (\rho_i^j s_i)(n, u) \langle n, v \rangle,
\]

\[
(2.2) \quad g(u, v) = \rho(u, v) + \rho_i^j \beta_i(u) \beta_j(v) + \rho_i^j (\beta_i(u) \langle n, v \rangle + \beta_i(v) \langle n, u \rangle) - (\rho_i^j s_i)(n, u) \langle n, v \rangle,
\]
see (1.5) with \( y = n \). Define the quantities (needed for two lemmas in what follows),

\[
\begin{align*}
\gamma_1^i &= (\rho_i^i + \rho_0^i s_i) / \rho = \ddot{\phi}_i / (\phi - \sum_j \dot{\phi}_j s_j) \quad (1 \leq i \leq p), \\
\gamma_2^{ij} &= \rho_0^{ij} - \gamma_1^i \rho_0^i - \gamma_1^j \rho_0^j - \gamma_1^i \gamma_1^j \rho_0^k s_k \quad (1 \leq i, j \leq p), \\
c_1 &= \gamma_1^i \beta_i(N) + (1 - \gamma_1^i \gamma_1^j b_{ij}^T)^{1/2},
\end{align*}
\]

where \( b_{ij}^T := b_{ij} - \beta_i(N) \beta_j(N) \). Assume that

\[
(2.4) \\
\begin{align*}
b_{ij}^T \gamma_1^i \gamma_1^j &\leq 1.
\end{align*}
\]

By (2.4), discriminant in the formula (2.3) for \( c_1 \) is nonnegative, hence \( c_1 \) is real. In the following lemma we express \( g \)-normal \( n \) to \( W \) through the \( a \)-normal \( N \) and the auxiliary functions (2.3).

**Lemma 2.1.** Let (2.4) holds, then the value of \( c_1 \) is real and

\[
(2.5) \\
n = c_1 N - \gamma_1^i \beta_i^2,
\]

\[
(2.6) \\
g(u, v) = \rho \langle u, v \rangle + \gamma_2^{ij} \beta_i(u) \beta_j(v) \quad (u, v \in W).
\]

Moreover, the values \( s_i = \beta_i(n) \) can be found from the system

\[
(2.7) \\
s_i = c_1 \beta_i(N) - \gamma_1^i b_{ij} \quad (1 \leq j \leq p).
\]

**Proof.** From (2.2) with \( u = n \) and \( v \in W \) and \( g(n, v) = 0 \) we find

\[
(2.8) \\
\langle \rho n + \gamma_1^i \beta_i^2, v \rangle = 0 \quad (v \in W).
\]

From (2.8) and \( \rho > 0 \) we conclude that \( \rho n + \gamma_1^i \beta_i^2 \rangle = c_1 N \) for some real \( c_1 \). Using

\[
1 = \langle n, n \rangle = c_1^2 - 2 c_1 \gamma_1^i \beta_i^2 + \gamma_1^i \gamma_1^j \langle \beta_i, \beta_j \rangle
\]

and \( \langle \beta_i, \beta_j \rangle = b_{ij} - \beta_i(N) \beta_j(N) \), we get two real solutions

\[
(c_1)_{1,2} = \gamma_1^i \beta_i(N) \pm (1 - \gamma_1^i \gamma_1^j b_{ij}^T)^{1/2}.
\]

The greater value (with +) provides inequality \( \langle n, N \rangle > 0 \), that proves (2.5). Thus, we get (2.7):

\[
s_i = \beta_i(n) = \beta_i(c_1 N - \gamma_1^i \beta_i^2) = c_1 \beta_i(N) - \gamma_1^i b_{ij} \quad (1 \leq i \leq p).
\]

Finally, (2.6) follows from (2.2), (2.5) and \( \langle n, u \rangle = -\gamma_1^i \beta_i(u) \) \( (u \in W) \).

**Remark 2.1** (Case \( \beta_i^2 \in W \)). An interesting particular case appears when all vectors \( \beta_i^2 \) belong to \( W \), that is \( \beta_i(N) = 0 \). Then, rather complicated system (2.7) reads

\[
(2.9) \\
\sum_i \ddot{\phi}_i / \phi (b_{ij} - s_i s_j) = -s_j \quad (1 \leq j \leq p),
\]

from which all \( \ddot{\phi}_i \) at \( s_i = \beta_i(n) \) can be expressed through \( \phi \) and \{\( s_i \}\).
Define a matrix \( P \) with elements
\[
P^j_k = \gamma_{ij}b^k_i.
\]

\( Q = \rho \text{id} + P \) is non-singular, if \( \gamma_{ij} \) are “small” relative to \( \rho > 0 \), i.e.,
\[
(2.10) \quad \det[\rho \delta^j_k + \gamma_{ij}b^k_i] \neq 0.
\]

Using the inverse matrix \( Q^{-1} \), define the quantities (needed for the following lemma),
\[
\gamma_{ij}^{\prime} = -\gamma_{kj}^{\prime}(Q^{-1})^k_i \quad (1 \leq i, j \leq p).
\]

In the following lemma, we find relation between \( u \in W \) and \( U \in W \) such that
\[
(2.11) \quad g(u, v) = \langle U, v \rangle, \quad \forall v \in W.
\]

**Lemma 2.2.** Let \((2.4)\) and \((2.10)\) hold. If the vectors \( u, U \) belong to \( W \) and obey \((2.11)\) then
\[
(2.12) \quad \rho u = U + \gamma_{ij}^{\prime} \beta_j(u) \beta^T_i.
\]

**Proof.** By \((2.6)\), \( g(u, v) = \langle \rho u + \gamma_{ij} \beta_i(u) \beta^T_j, v \rangle \) for \( u, v \in W \). By conditions, and since \( U, \beta_j^T \in W \), we find \( \rho u + \gamma_{ij} \beta_i(u) \beta^T_j = U \). Applying \( \beta_k \) and using \( \beta_k(\beta_j^T) = b_{jk}^\dagger \) yields
\[
(\rho \delta^j_k + P^j_k) \beta_j(u) = \beta_k(U) \quad (1 \leq k \leq p),
\]
and then \((2.12)\).

\[\square\]

## 3 Examples

The following lemma is used to compute the volume forms of \((\alpha, \beta)\)-norm for \( p = 1, 2 \). This extends the Silvester’s determinant identity, see \([14]\),
\[
\det(\text{id}_m + C_1 P_i^l) = 1 + C_1^l P_i
\]
where \( C_1, P_1 \) are \( m \)-vectors (columns), and \( \text{id}_m \) is the identity \( m \)-matrix.

**Lemma 3.1.** Let \( C_i, P_i \) \((1 \leq i \leq j \leq m)\) be \( m \)-vectors. Then \( \text{Tr}(C_i P_j^l) = C_i^l P_j = P_j^l C_i \) and
\[
(3.1) \quad \det(\text{id}_m + C_1 P_1^l + C_2 P_2^l) = 1 + C_1^l P_1 + C_1^l P_1 P_2 + C_1^l P_1 \cdot C_1^l P_2 - C_1^l P_2 \cdot C_1^l P_1,
\]
\[
\det(\text{id}_m + C_1 P_1^l + C_2 P_2^l + C_3 P_3^l) = 1 + C_1^l P_1 + C_2^l P_2 + C_3^l P_3 + C_1^l P_1 \cdot C_1^l P_2 + C_2^l P_2 \cdot C_1^l P_1 + C_3^l P_3 \cdot C_2^l P_2 + C_1^l P_1 \cdot C_3^l P_3 + C_2^l P_2 \cdot C_3^l P_3 - C_1^l P_2 \cdot C_3^l P_3 - C_3^l P_3 \cdot C_1^l P_2 - C_2^l P_2 \cdot C_3^l P_3 - C_3^l P_3 \cdot C_2^l P_2 + C_3^l P_3 \cdot C_1^l P_2 + C_2^l P_2 \cdot C_1^l P_3 + C_3^l P_3 \cdot C_2^l P_3 + C_1^l P_1 \cdot C_2^l P_3 + C_2^l P_2 \cdot C_3^l P_1 + C_1^l P_1 \cdot C_3^l P_2 + C_2^l P_2 \cdot C_3^l P_1 + C_3^l P_3 \cdot C_2^l P_1 + C_3^l P_3 \cdot C_2^l P_1 + C_3^l P_2 \cdot C_3^l P_1 + C_3^l P_2 \cdot C_3^l P_1,
\]
(3.2) \[-C_1^l P_1 \cdot C_3^l P_3 \cdot C_3^l P_2 - C_1^l P_1 \cdot C_1^l P_2 \cdot C_3^l P_3 - C_1^l P_1 \cdot C_2^l P_2 \cdot C_3^l P_3 - C_1^l P_1 \cdot C_3^l P_2 \cdot C_3^l P_1 \cdot C_2^l P_3 \cdot C_3^l P_2 \cdot C_3^l P_1 \cdot C_3^l P_2, \text{ and so on.}
\]
For $p = 1$, (1.3) defines $(\alpha, \beta)$-norm $F = \alpha \phi(s)$ for $s = \beta/\alpha$. This function $F$ is a Minkowski norm on $V$ for any $\alpha$ and $\beta$ with $\alpha(\beta) < \delta_0$ if and only if $\phi(s)$ satisfies

$$
\phi - s \dot{\phi} + (b^2 - s^2) \ddot{\phi} > 0,
$$

where real $s, b$ obey $|s| < b$, see [14]. Taking $s \to b$ in (3.3), we get $\phi - s \dot{\phi} > 0$. By (1.5),

$$
g_y(u, v) = \rho\langle u, v \rangle + \rho_0 \beta(u) \beta(v) + \rho_1 \beta(u) \beta(v) \langle y, u \rangle / \alpha(y)
$$

(3.4)

$$
- \rho_1 \beta(y) \langle y, u \rangle / \alpha^3(y).
$$

Here $\rho > 0$ and $\rho_0, \rho_1$ are the following functions of $s$:

$$
\rho = \phi - s \dot{\phi}, \quad \rho_0 = \phi \ddot{\phi} + \dot{\phi}^2, \quad \rho_1 = \phi \dot{\phi} - s(\phi \ddot{\phi} + \dot{\phi}^2).
$$

The following relations hold: $\dot{\rho} = \rho_1$, $\ddot{\rho} = s \rho_0$. Set $\tilde{Y} = s^{-1} \beta - y^2 / \alpha(y)$ and $\varepsilon = s \rho_1$. Then (3.4) takes the form

$$
g_y(u, v) = \rho\langle u, v \rangle + (\rho_0 + \rho_1^2 \varepsilon) \beta(v) - \varepsilon \tilde{Y}(u) \tilde{Y}(v),
$$

(3.5)

From (3.5) and (3.1) with $C_1 = (\rho_0 + \rho_1^2 / \varepsilon) \rho^{-1} \beta^2$, $P_1 = \beta^2$, $C_2 = -\varepsilon \rho^{-1} \tilde{Y}^2$, $P_2 = \tilde{Y}^2$, for the volume form $d\text{vol}_y = \mu_{y_2}(y) d\text{vol}_y$ we obtain, see also [14],

$$
\mu_{y_2}(y) = \rho^{m+1}(\rho + \rho_0 \rho_1 s + \rho_1^2 s^2 + (\rho - \rho_0 b^2) + \rho_0 b^2 + \rho_1 b^2)
$$

(3.6)

$$
= \rho^{m+2}((\phi - s \dot{\phi} + (b^2 - s^2) \ddot{\phi})^{-1} [\phi - s \dot{\phi} + (b^2 - s^2) \ddot{\phi}]).
$$

Set $p_y = \beta^2 - sy/\alpha(y)$. The Cartan tensor of $(\alpha, \beta)$-norm has an interesting special form [8]:

$$
2 C_y(u, v, w) = \rho_1 \alpha^{-1}(y) (K_y(u, v) \langle p_y, w \rangle + K_y(v, w) \langle p_y, u \rangle + K_y(w, u) \langle p_y, v \rangle)
$$

$$
+ (3 \phi \ddot{\phi} + \phi \dot{\phi} \rho) \alpha^{-1}(y) \langle p_y, u \rangle \langle p_y, v \rangle \langle p_y, w \rangle,
$$

see (1.6) for $p = 1$. For a hyperplane $W \subset V$ we have $s = \beta(n)$ and

$$
c_1 = \gamma_1 \beta(N) + (1 - \gamma_1^2 (b^2 - \beta(N)^2))^{1/2},
$$

$$
\gamma_1 = (\rho_1 + \rho_0 \beta(n)) / \rho = \dot{\phi} / (\phi - s \dot{\phi}),
$$

$$
\gamma_2 = \rho_0 - \gamma_1 \rho_1 \beta(n) \gamma_1 + 2 = \phi (\phi^2 \ddot{\phi} - \phi \dot{\phi}^2 + s \dot{\phi}^2) \gamma_2 / (\phi - s \dot{\phi})^2,
$$

$$
\gamma_3 = - \gamma_2 / \rho + (b^2 - \beta(N)^2) \gamma_2.
$$

Then (2.7) reads

$$
\frac{\dot{\phi}}{\phi} = - \frac{s \sqrt{b^2 - s^2} + \beta(N) \sqrt{b^2 - \beta(N)^2}}{(b^2 - s^2 - \beta(N)^2) \sqrt{b^2 - s^2}},
$$

which for $\beta^2 \in W$ reads $\frac{\dot{\phi}}{\phi} = - \frac{s}{b^2 - s^2}$, see also (2.9) for $p = 1$.

**Example 3.1** ($p = 1$). Some progress was achieved for particular cases of $(\alpha, \beta)$-norms. Below we consult some of $(\alpha, \beta)$-norms to illustrate the above metric $g$ on $V$. 


(i) For φ(s) = 1 + s, |s| < b < δ0 = 1, we have the norm F = α + β, introduced by a physicist G. Randers to consider the unified field theory. We have ρ = 1 + s, ρ0 = 1 and ρ1 = 1. For a hyperplane W ⊂ V and g = g_n, we get n = c_1 N = β, s = β(n) = c c_1 − 1, φ(s) = c c_1, where c_1 = c + β(N) and c = √1 − b^2 + β(N)^2 ∈ (0, 1], see also [13]. Then

γ_1 = 1, \quad γ_2 = −c c_1, \quad γ_3 = c^2.

Conditions (2.4) and (2.10) become trivial: c > 0. Next, μ_g(n) = (c c_1)^{m+2} and

\[ g(u, v) = (1 + s) \langle u, v \rangle − s(n, u)(n, v) + β(u)(n, v) + β(v)(n, u) + β(u)β(v). \]

(ii) The (α, β)-norms F = α^{l+1}/β^l (l > 0), i.e., φ(s) = 1/s^l (0 < s < b), are called generalized Kropina metrics, see [8], and have applications in general dynamical systems. The Kropina metric, i.e., l = 1, first introduced by L. Berwald in connection with a Finsler plane with rectilinear extremals, and investigated by V.K. Kropina in 1961. We have ρ = 2/s^2, ρ_0 = 3/s^4 and ρ_1 = −4/s^3. For a hyperplane W ≠ ker β in V and g = g_n we get

\[ c_1 = (b − 2β(N))/\sqrt{2b(b − β(N))}, \quad β(n) = s = \sqrt{b(b − β(N))/2}; \]

γ_1 = −1/(2s) = −1/2b(b − β(N)), \quad γ_2 = γ_3 = 0,

and μ_g(n) = b^{m+1}/(b−β(N))^{m+1}. Note that conditions (2.4) and (2.10) become trivial.

(iii) The (α, β)-norm F = s^2/α−β, i.e., φ(s) = 1/(1−s^2) with |s| < b < δ0 = 1/2, (called slope-metric) was introduced by M. Matsumoto to study the time it takes to negotiate any given path on a hillside. We have ρ = (1−2s)/(1−s^2), ρ_0 = 3/(1−s^2) and ρ_1 = (1−4s)/(1−s^2). For a hyperplane W ≠ ker β and g = g_n, from (2.7) we find that s = β(n) obeys 4th-order equation

\[ 4s^4 − 4s^3 + (1 − 4b^2)s^2 + 2(b^2 + β(N)^2)s + b^4 − (b^2 + 1)β(N)^2 = 0, \]

and s = 1/(4 + 8b^2) if β isn’t W, see (2.9). We find μ_g(n) = (1−2s)^m/(1−s^2)^{m+1}(2b^2 − 3s + 1) and

\[ c_1 = \frac{β(N) + \sqrt{1 − 2s^2 − b^2 + β(N)^2}}{1 − 2s}; \]

γ_1 = 1/(1−2s), \quad γ_2 = 1/(1−2s)(1−s)^3, \quad γ_3 = 1/(1−2s)^3 + b^2 − β(N)^2.

Thus, (2.10) becomes trivial and (2.4) reads as (1−2s)^2 ≥ b^2 − β(N)^2.

(iv) A Finsler metric is a polynomial (α, β)-norm if φ(s) = \sum_{k=0} C_k s^k, C_0 = 1, C_k ≠ 0. The quadratic metric F = (α + β)^2/α, i.e., φ(s) = (1 + s)^2 with |s| < b < δ0 = 1, appears in many geometrical problems, [14]. We have ρ = (1−s)(1+s)^3, ρ_0 = 6(1+s)^2 and ρ_1 = 2(1−2s)(1+s)^2. For a hyperplane W ≠ ker β in V and g = g_n, from (2.7) we find that s obeys 4th-order equation

\[ s^4 − 2s^3 + (1 − 4b^2 + 3β(N)^2)s^2 + 2(2b^2 − β(N)^2)s + 4b^4 − (4b^2 + 1)β(N)^2 = 0, \]
and \( s = (1 - \sqrt{1 + 8b^2})/2 \) if \( \beta \in W \), see (2.9). Then we obtain

\[
\begin{align*}
c_1 &= \frac{2\beta(N) + \sqrt{(1-s)^2 - 4(b^2 - \beta(N)^2)}}{1-s}, \\
\gamma_1 &= \frac{2}{1-s}, \quad \gamma_2 = \frac{2(3s-1)(1+s)^3}{(1-s)^2}, \quad \gamma_3 = \frac{2(3s-1)}{(1-s)^3 - 2(1-3s)(b^2 - \beta(N)^2)}
\end{align*}
\]

and \( \mu_g(n) = (1+s)^{3m+3}(1-s)^{m-1}(2b^2 - 3s^2 + 1) \). Conditions (2.4) and (2.10) read

\[
(1-s)^2 \geq 4(b^2 - \beta(N)^2), \quad (1-s)^3 \neq 2(1-3s)(b^2 - \beta(N)^2).
\]

(v) Define by \( \phi(s) = e^{s/k} \), \( |s| < b < \delta_0 := |k| \), the exponential metric \( F = e^{\phi/k} \). Condition (3.3) reads as a quadratic inequality \( s^2 + ks - (b^2 + k^2) < 0 \). Taking \( s = b \) in (3.3) yields \( k(s-k) < 0 \) when \( |s| < |k| \). Thus, (3.3) is satisfied for arbitrary numbers \( s \) and \( b \) with \( |s| \leq b < |k| \). We have \( \rho = e^{2s/k}(k-s)/k > 0 \), \( \rho_0 = 2e^{2s/k}/k^2 \) and \( \rho_1 = e^{2s/k}(k-2s)/k^2 \). For a hyperplane \( W \neq \ker \beta \) in \( V \) and \( g = g_n \), by (2.7), \( s = \beta(n) \) obeys 4th-order equation

\[
s^4 - 2ks^3 + (k^2 - 2b^2 + \beta(N)^2)s^2 + 2b^2ks + b^4 - (b^2 + k^2)\beta(N)^2 = 0,
\]

and \( s = (k - \sqrt{k^2 + 4b^2})/2 \) if \( \beta \) is tangent to the foliation, see (2.9). Then we get

\[
\begin{align*}
c_1 &= \frac{\beta(N) + ((k-s)^2 - b^2 + \beta(N)^2)^{1/2}}{k-s}, \\
\gamma_1 &= \frac{1}{k-s}, \quad \gamma_2 = \frac{s e^{2s/k}}{k(k-s)^2}, \quad \gamma_3 = \frac{s}{(k-s)^3 + s(b^2 - \beta(N)^2)}
\end{align*}
\]

and \( \mu_g(n) = \frac{(k-s)^{m-1}}{k^{m-1}} (b^2 + k^2 - ks - s^2) e^{(2m+2)s/k} \). Conditions (2.4) and (2.10) read, respectively,

\[
(1-s)^2 \geq b^2 - \beta(N)^2, \quad (k-s)^3 \neq -s(b^2 - \beta(N)^2).
\]

Fig. 3.1 shows the dependence of \( s \) on \( \beta(N) \in [-b, b] \), see (2.7), for four of above metrics. For \( \beta(N) = 0 \) we obtain the values of \( s \): a) 0.64, b) -0.13, c) -0.26, d) -0.53.

For \( p = 2 \), we can use (1.11) to find \( \mu_g(y) \). By (1.5) we get

\[
\begin{align*}
g_g(u, v) &= \rho(u, v) + (\rho_0^y + \varepsilon^{-1}\rho_1^y)\beta_i(u)\beta_j(v) - \varepsilon Y(u)\tilde{Y}(v), \\
\tilde{Y} &= \varepsilon^{-1}\rho_1^y\beta_i - y^j/\alpha(y), \quad \varepsilon = s_i\rho_1^y.
\end{align*}
\]
From (3.2) with

\[ C_1 = \lambda^1 \rho^{-1} \tilde{\beta}_1^1, \quad P_1 = \tilde{\beta}_1^1, \quad C_2 = \lambda^2 \rho^{-1} \tilde{\beta}_2^1, \quad P_2 = \tilde{\beta}_2^1, \quad C_3 = -\varepsilon \rho^{-1} \tilde{Y}^4, \quad P_3 = \tilde{Y}^4, \]

using \( \tilde{Y} \) from (3.7), \( \tilde{b}_{ij} = \langle \tilde{\beta}_i, \tilde{\beta}_j \rangle, \tilde{\beta}_i = q_i^1 \beta_1 + q_i^2 \beta_2 \) and \( \varepsilon = \rho_1^1 s_1 + \rho_1^2 s_2 \), we obtain

\[
\mu_{\theta_2}(y) = \rho^{-1} (p^2 + \rho(\lambda_1 \tilde{b}_{11} + \lambda_2 \tilde{b}_{22}) - \rho \varepsilon (\tilde{Y}, \tilde{Y}) + \lambda_1 \lambda_2 (\tilde{b}_{11} \tilde{b}_{22} - \tilde{b}_{12}^2) \\
+ \varepsilon (\tilde{Y}, \tilde{Y}) \lambda_1 \tilde{b}_{11} + \lambda_2 \tilde{b}_{22} + \lambda_1 \varepsilon (\tilde{b}_1, \tilde{Y}) + \lambda_2 \varepsilon (\tilde{b}_2, \tilde{Y}) + \lambda_1 \lambda_2 \varepsilon / \rho (\tilde{b}_{11} (\tilde{b}_2, \tilde{Y})^2 \\
+ \tilde{b}_{22} (\tilde{b}_1, \tilde{Y})^2 + \tilde{b}_{12} (\tilde{Y}, \tilde{Y})^2 - \tilde{b}_{11} \tilde{b}_{22} (\tilde{Y}, \tilde{Y}) - 2 \tilde{b}_{12} (\tilde{b}_1, \tilde{Y}) \langle \tilde{b}_2, \tilde{Y} \rangle)].
\]

**Example 3.2** \( (p = 2) \). A navigation \((\alpha, \beta)\)-norm is the \((\alpha, \tilde{\beta})\)-norm with \( p = 2 \).

(a) For shifted Kropina norm \( \phi = 1 + \frac{1}{s_1} + s_2 \) for \( s_1 > 0 \), hence \( F = \alpha (1 + \frac{\beta_1}{\alpha} + \frac{\beta_2}{\alpha}) \), we have

\[
\rho = (2 + s_1)(1 + s_1 + s_1 s_2)/s_1^4, \quad \rho_1^1 = -(4 + 3 s_1 + 2 s_1 s_2)/s_1^3, \quad \rho_1^2 = (2 + s_1)/s_1, \quad \rho_0^1 = (3 + 2 s_1 + 2 s_1 s_2)/s_1^4, \quad \rho_0^2 = -1/s_1^2, \quad \rho_0^3 = 1.
\]

For a hyperplane \( W \neq \ker \beta_i \) \((i = 1, 2)\) in \( V \) and the metric \( g = g_n \) we get

\[
\begin{align*}
\gamma_1^1 &= \frac{s_1 t_2(N - \beta_1(N))}{s_1 (2 + s_1)} + \left( 1 - \frac{b_{11} - \beta_1(N)}{s_1 (2 + s_1)} \right)^2 + 2 b_{12} - \beta_2(N) - 2 b_{22} (\beta_1(N)) - \frac{\beta_1(N)}{s_1 (2 + s_1)}, \\
\gamma_1^2 &= \frac{s_1}{2 s_1}, \quad \gamma_1^{11} = \frac{2 s_1 - s_2}{s_1 (2 + s_1)}, \quad \gamma_1^{22} = \frac{2 s_1 - s_2}{s_1 (2 + s_1)}.
\end{align*}
\]

If \( \beta_i^1 \in W \) then \( s_1, s_2 \) obey the system

\[
(1 + 2 s_2) s_1^3 - b_{12} s_1^2 + b_{11} = 0, \quad (1 + 2 s_1) s_1 s_2^2 - b_{22} s_1^2 + b_{12} = 0.
\]

Thus \( s_2 = \frac{1}{2} [(b_{11} - s_1^2 b_{12})/s_1^3 - 1] \), where \( s_1 \) is a positive root of the 6th-order polynomial:

\[
2 b_{22} s_1^6 + b_{12} s_1^5 - (b_{12}^2 + 2 b_{12}) s_1^4 - b_{11} s_1^3 + 2 b_{11} b_{12} s_1^2 - b_{11}^2 = 0;
\]

for example, if \( b_{12} = 0 \) then \( s_1 = (\frac{b_{11}}{4 b_{22}} (1 + \sqrt{1 + 8 b_{22}}))^{1/3} \) and \( s_2 = \frac{1}{2} (b_{11}/s_1^3 - 1) \).

(b) For shifted Matsumoto norm \( \phi = \frac{1}{1 - s_1} + s_2 \) with \( \delta_i < 1 \), hence \( F = \alpha (1 + \frac{\beta_1}{\alpha} + \frac{\beta_2}{\alpha}) \), we have

\[
\begin{align*}
\rho &= \frac{(1 - 2 s_1)(1 + s_2 - s_1 s_2)}{(1 - s_1)^3}, \quad \rho_1^1 = \frac{1 + 2 s_1 (s_1 s_2 - s_2 - 2)}{(1 - s_1)^3}, \quad \rho_1^2 = \frac{1 - 2 s_1}{(1 - s_1)^2}, \\
\rho_0^1 &= (3 - 2 s_1 s_2 + 2 s_2)/(1 - s_1)^4, \quad \rho_0^2 = \frac{2 s_1}{(1 - s_1)^2}, \quad \rho_0^3 = 1/(1 - s_1)^2, \quad \rho_0^{22} = 1.
\end{align*}
\]
The new Minkowski norm and integral formulas

For a hyperplane \( W \neq \ker \beta_i \) \((1 \leq i \leq p)\) in \( V \) and the metric \( g = g_n \) we get

\[
c_1 = \frac{(1-s_1)^2}{1-2s_1} \frac{\beta_2(N)+\beta_1(N)}{(1-2s_1)^2} + \left(1 - \frac{(1-s_1)^2}{1-2s_1} \frac{(b_{12} - \beta_1(N)\beta_2(N))}{(1-2s_1)^2} \right)^{1/2} \frac{b_{11} - \beta_1(N)^2}{(1-2s_1)^2},
\]

\[
\gamma_1^1 = \frac{1}{1-2s_1}, \quad \gamma_2^1 = \frac{(1-s_1)^2}{1-2s_1}, \quad \gamma_2^1 = \frac{1 + 2s_1 + 8s_1^2 + s_2(1 + 5s_1 - 6s_1^2)}{(1-s_1)^4(1-2s_1)},
\]

\[
\gamma_2^2 = -\frac{3s_1 + 2s_1^2 - 4s_1 + s_2(1 - 4s_1 + 3s_1^2)}{(1-s_1)^4},
\]

\[
\gamma_2^{12} = -\frac{1 - 5s_1 + 3s_1^2 + 4s_1^2 + s_2(1 - 8s_1 + 17s_1^2 - 12s_1^3 + 2s_1^4)}{(1-2s_1)(1-s_1)^4}.
\]

If \( \beta_i^1 \in W \) then \( s_1 \) and \( s_2 \) obey the system

\[
b_{11} + (1-s_1)^2(b_{12} - 2s_1s_2) = s_1, \quad b_{12} + (1-s_1)^2(b_{22} - 2s_1^2) = s_2.
\]

Then \( s_1 = (2b_{11}s_2^2 - 2b_{12}s_2 - b_{11}b_{22} + b_{12}^2)/(2b_{12}s_2 - b_{22}) \), where \( s_2 \) is a root of a 6th-order polynomial.

Similarly to graphs on Fig. 3.1, one may calculate and graph pairs of surfaces in \( \mathbb{R}^3 \), showing dependence of \( s_1 \) and \( s_2 \) on variables \((\beta_1(N), \beta_2(N))\) for the above navigation \((\alpha, \beta)\)-metrics. For \( \beta_i(N) = 0 \) we obtain the values: a) \( s_1 \approx -0.79 \) and \( s_2 = -1.5 \) for Kropina norm; b) \( s_1 \approx -0.42 \) and \( s_2 = s_1^2 - 2s_1^3 + s_1 \approx -0.84 \) for Matsumoto norm.

## 4 The shape operator and the curvature of normal curves

Let \((M^{m+1}, a = \langle \cdot, \cdot \rangle)\) \((m \geq 2)\) be a connected Riemannian manifold with the Levi-Civita connection \( \nabla \). Let \( N \) be a unit normal field to a codimension-one distribution \( \mathcal{D} := \ker \omega \) on \((M, a)\). Due to Section 2, there exists a \( g_n \)-normal (to \( \mathcal{D} \)) vector field \( n \) such that \( \langle n, N \rangle > 0 \) and \( \langle n, n \rangle = 1 \). Define a new Riemannian metric \( g := g_n \) on \( M \), see (2.2), with the Levi-Civita connection \( \nabla \). Let \( \ker \beta_i \neq \mathcal{D} \) everywhere for all \( i \), hence \( |\beta_i(N)| < \sqrt{|\nu|} \). By (2.7), \( s_i = \beta_i(n) \) are smooth functions on \( M \), and \( \nu = n/\phi(s) \) is a \( g \)-unit normal to the leaves.

The shape operators \( \bar{A} \) and \( A^g \) of \( \mathcal{D} \) and the curvature vectors of \( \nu \)- and \( N \)- curves for both metrics \( \langle \cdot, \cdot \rangle \) and \( g \) belong to \textit{Extrinsic Geometry} and are defined by

\[
\bar{A}(u) = -\nabla_u N, \quad A^g(u) = -\nabla_u \nu \quad (u \in \mathcal{D}),
\]

\[
Z = \nabla_u \nu, \quad \bar{Z} = \nabla_N N.
\]

Let \( \bar{T} : \mathcal{D} \to \mathcal{D} \) be a linear operator adjoint to the integrability tensor \( \bar{T} \) of \( \mathcal{D} \) with respect to \( a \),

\[
2 \bar{T}(u,v) = \langle [u,v], N \rangle \quad (u,v \in \mathcal{D}).
\]
Note that $\bar{T}^u = \frac{1}{2} (\bar{A} - \bar{A}^*)$, where $\bar{A}^*$ is a linear operator adjoint to $\bar{A}$. The deformation tensor,

$$\bar{D} \bar{u} = (\bar{\nabla} u + (\bar{\nabla} u)^t)/2,$$

measures the degree to which the flow of a vector field $u$ distorts $\langle \cdot , \cdot \rangle$. Here, $\bar{\nabla} u$ and $(\bar{\nabla} u)^t$ are

$$(\bar{\nabla} u)(v) = \bar{\nabla}_v u, \quad \langle (\bar{\nabla} u)^t(v) , w \rangle = \langle v , (\bar{\nabla} u)(w) \rangle \quad (v,w \in TM).$$

In the next proposition, we express $A^\theta$ through $\bar{A}$ and invariants of $D$ with respect to $a$.

**Proposition 4.1** (The shape operator). Let $(M^{m+1}, a)$ be a Riemannian manifold with a form $\omega \neq 0$ and linear independent 1-forms $\beta_1, \ldots, \beta_p$ obeying conditions (2.4) and (2.10). Let $g$ be a Riemannian metric (2.2) determined by a distribution $D = \ker \omega$, $\bar{\beta} = (\beta_1, \ldots, \beta_p)$ and a smooth function $\phi(x,s)$ on $M \times \mathbb{R}^p$. Then

$$\rho \phi A^\theta = -A - \gamma^{ij}_3 (\beta_i \circ A) \otimes \beta^T_j,$$

where the linear operator $A : D \to D$ is given by

$$(4.3) \quad A = -\rho c_1 \bar{A} - \rho \gamma^i_3 (\bar{D}^T_{\beta^T_i})^T + \frac{1}{2} n(\rho) \text{id}^T + \text{Sym}(U^j \otimes \beta^T_j),$$

and the vector fields $U^j$ are given by

$$(4.5) \quad U^j = \frac{1}{2} [n(\gamma^{ij}_3) \beta^T_i + \gamma^{ij}_3 (\beta^T_i \circ \bar{\nabla}^T_j) - \rho \bar{\nabla}^T_i \gamma^j_3]
+ (\rho^0_1 - \gamma^1_1 \rho^j_1) (\beta_i(N) \bar{\nabla}_v c_1 - (\gamma^i_1 \rho^j_1) \bar{\nabla}_j b_{ik} - b_{ik} \bar{\nabla}^T \beta^T_j)
+ (c_1 - \beta_k(N) \gamma^j_1) ((\rho^0_1 - \gamma^1_1 \rho^j_1) \beta_i(N) + c_1 \rho^j_1 (1 + s_k \gamma^1_1)) \bar{Z}
\quad \rho \phi A^\theta =\n\quad \
+ (c_1 \rho^j_1 (1 + s_k \gamma^1_1) \gamma^j_3 - (\rho^0_1 - \gamma^1_1 \rho^j_1) (c_1 - \beta_k(N) \gamma^1_1)) \bar{A}^* (\beta^T_j).$$

**Proof.** By known formula for the Levi-Civita connection $\nabla$ of $g$,

$$(4.6) \quad 2 g(\nabla u,v,w) = u(g(v,w)+v(g(u,w))-w(g(u,v))+g([u,v],w)-g([u,w],v)-g([v,w],u),$$

where $u,v,w \in C^\infty(TM)$, we have

$$(4.7) \quad 2 g(\nabla u,v,w) = n(g(u,v)) + g([u,n],v) + g([v,n],u) - g([u,v],n) \quad (u,v \in D).$$

Assume $\bar{\nabla}^T_X u = \bar{\nabla}^T_X v = 0$ for $X \in T_x M$ at a given point $x \in M$. Using (2.2) and (2.6), we get

$$n(g(u,v)) = n(\rho(u,v)) + n(\gamma^{ij}_3 \beta(u) \beta_j(v))
= n(\rho) \langle u,v \rangle + [n(\gamma^{ij}_3) \beta_i(u) \beta_j(v) + \gamma^{ij}_3 (\beta_i(u)(\bar{\nabla}_v (\beta^T_j))(v) + \beta_i(v)(\bar{\nabla} n(\beta^T_j))(u))]g([u,v],n) = 2 \rho c_1 \bar{T}(u,v),
\quad
\quad g([u,v],n) = \rho(\bar{\nabla} \omega u,v) + \rho^0_1 \beta_i([u,n]) \beta_j(v) + \rho^1_1 (\beta_i([u,n]) \nabla n(\beta^T_j))(v) + \beta_i(v)(\nabla n(\beta^T_j))(u)
\quad \rho(\bar{\nabla} \omega u,v) + \rho^0_1 \beta_i([u,n]) \beta_j(v) + \rho^1_1 (\beta_i([u,n]) \nabla n(\beta^T_j))(v) + \beta_i(v)(\nabla n(\beta^T_j))(u)
- \rho^1_1 s_i ([u,n][u,n]) \langle n,v \rangle,$$
where \( u, v \in \mathcal{D} \). Using equalities

\[
\langle \nabla_u n, v \rangle = -\langle c_1 \bar{A}(u), v \rangle - \gamma_1^i(\nabla_u \beta_i^k, v) - \beta_i(v)(\nabla \gamma_1^i, u) = \langle U_3, v \rangle,
\]
\[
\beta_i([u, u]) = -\gamma_1^i(\nabla_u \beta_i^k, \beta_i^k) + \beta_i(N)\nabla \gamma_1^i - \beta_i(N) [1 - \gamma_i^k \beta_i(N)] \tilde{Z} + \gamma_1^k A^*(\beta_i^k) \rangle, v \rangle = \langle U_{2i}, u \rangle - \gamma_1^i(\nabla_u \beta_i^k, \beta_i^k),
\]
\[
(n, [u, u]) = ((1 - \gamma_1^i \beta_i(N)) \nabla \gamma_1^i - c_1 \gamma_i^k \beta_i(N) \tilde{Z}, u) = \langle U_1, u \rangle,
\]
we then obtain

\[
g([u, n], v) = -\rho c_1(\bar{A}(u), v) - \rho(\gamma_1^i(\nabla_u \beta_i^k, v) + \beta_i(v)(\nabla \gamma_1^i, u))
\]
\[
+ \rho_0^i(\beta_i)(\langle \beta(N) \nabla \gamma_1^i - b_{ki} \nabla \gamma_1^k + \beta_i(N) [1 - \gamma_i^k \beta_i(N)] \tilde{Z} + \gamma_i^k A^*(\beta_i^k) \rangle, u) \rangle
\]
\[
- \gamma_1^k(\nabla_u \beta_i^k, \beta_i(N)) - \gamma_1^i \beta_i(v) \rho_1^i(\langle \beta_i(N) \nabla \gamma_1^i - b_{ki} \nabla \gamma_1^k + \beta_i(N) [1 - \gamma_i^k \beta_i(N)] \tilde{Z} + \gamma_i^k A^*(\beta_i^k) \rangle, u) \rangle
\]
\[
+ \rho_1^i(\beta_i)(\langle c_1 - \gamma_i^k \beta_i(N) \rangle \nabla \gamma_1^i - c_1 \gamma_i^k \nabla (\beta_i(N)) - c_1 \gamma_i^k \beta_i(N) \tilde{Z}, u) \rangle
\]
\[
= -\rho c_1(\bar{A}(u), v) - \rho(\gamma_1^i(\nabla_u \beta_i^k, v) + \beta_i(v)(\nabla \gamma_1^i, u))
\]
\[
+ (\rho_0^i - \rho_1^i \gamma_1^i)(\langle \beta_i(N) \nabla \gamma_1^i - \frac{1}{2} \gamma_i^k \nabla b_{ki} - b_{ki} \nabla \gamma_1^k \rangle \triangledown \nabla \gamma_1^i, u) \rangle
\]
\[
+ (c_1 - \beta_i(N) \gamma_i^k)(\beta_i(N) \tilde{Z} - A^*(\beta_i^k)), u) \rangle \beta_i(v) \rangle
\]
\[
+ c_1 \rho_1^i(1 + s_k \gamma_i^k)(\langle c_1 - \beta_i(N) \gamma_i^k \rangle \tilde{Z} + \gamma_i^k A^*(\beta_i^k), u) \rangle \beta_j(v),
\]
where \( u, v \in \mathcal{D} \). Formula for \( g([v, n], u) \) is obtained from \( g([u, n], v) \) after change \( u \leftrightarrow v \). Substituting the above into (4.7), we find \( g(\nabla_u n, v) = \langle \mathcal{A}(u), v \rangle \), where \( \mathcal{A} \) is given in (4.4)–(4.5). In particular,

\[
2 \langle \mathcal{A}(u), \beta_i^k \rangle = -2 \rho c_1(\bar{A}^*(\beta_i^k), u) - 2 \rho \gamma_1^i(\text{Det}_{\beta_1}\rangle \beta_i^k), u \rangle
\]
\[
+ n(\rho) \beta_i(u) + \beta_j(u) \beta_i(U^j) + U^j(u) b_{ij},
\]
By Lemma 2.2 and \( g(\nabla_u n, v) = -\phi g(A^p(u), v) \), see (4.1), we get (4.3).

The elementary symmetric functions \( \sigma_k(A) \) of a \( m \times m \)-matrix \( A \) (or a linear transformation) are defined by equality \( \text{det}(id + tA) = \sum_{t \leq m} \sigma_k(A) t^k \) and are called mean curvatures in the case of shape operator. Thus, \( \sigma_0(A) = 1, \sigma_1(A) = \text{Tr} A, \ldots, \sigma_m(A) = \text{det} A \).

**Corollary 4.2** (The mean curvature of \( \mathcal{D} \). Let conditions of Proposition 4.1 are satisfied. Then

\[
\rho \phi \sigma_1(A^p) = \rho c_1 \sigma_1(A) - \frac{m}{2} n(\rho) + \rho \gamma_1^i(\text{Div} \beta_i^k + \beta_i(\tilde{Z}) + N(\beta_i(N)))
\]
\[
- \beta_j(U^j) - \gamma_0^i(\langle A(\beta_i^k), \beta_j^k \rangle,\beta_j^k),
\]
where \( U^j \) are given in (4.5) and
\[
\langle A(\beta^T_j), \beta^T_j \rangle = \rho c_1 \langle \bar{\alpha}(\beta^T_j), \beta^T_j \rangle + \rho \gamma_1 \frac{1}{2} (b_{ij} \beta_k (b_{kj})/2 - \beta_k (N) \langle \bar{\alpha}(\beta^T_j), \beta^T_j \rangle)
\]
(4.9)
\[
- b_{ij} (\frac{1}{2} n(\rho) + \beta_k (U^k)).
\]

**Proof.** Let \( \{e_i\} \) be a local \( g \)-orthonormal frame of \( D \). We calculate
\[
\langle \text{Def} \beta^T_i (\beta^T_j), \beta^T_j \rangle = \frac{1}{2} \langle \nabla b^T_{ij}, \beta^T_j \rangle - \beta_k (N) \langle \bar{\alpha}(\beta^T_j), \beta^T_j \rangle,
\]
see (4.4)–(4.5). Tracing of (4.3), we obtain
\[
\rho \phi \sigma_1 (A^g) = -\sigma_1 (A) - \gamma_1 \langle A(\beta^T_j), \beta^T_j \rangle.
\]
Then, using
\[
\text{Tr} (\text{Def} \beta^T_i )_{T_{\mathcal{F}}} = \text{div} \beta^T_i - \beta_i (\bar{Z}) + N(\beta_i (N)),
\]
(4.9) and Lemma 2.2, we get (4.8)–(4.9).

**Example 4.1.** (i) One may ask the question: “When \( D \) is totally geodesic with respect to \( g \), i.e., \( A^g = 0 \)” ? In this case, when \( \nabla \beta_i = 0 \) and \( \beta_i (N) = 0 \), by Proposition 4.1, \( \bar{A} \) has a special form
\[
\bar{A} = W^i \otimes \beta_i + \omega^i \otimes \beta^T_i,
\]
for some vector fields \( W^i \) and 1-forms \( \omega^i \). If \( p = 1 \) then, necessarily, \( \text{rank} \bar{A} \leq 2 \).

In next corollary and proposition, for simplicity, we assume that \( D \) is integrable and \( p = 1 \).

**Corollary 4.3** (The second mean curvature). If \( p = 1 \) and \( \nabla \beta^T = 0 \) then
\[
(\rho \phi)^2 \sigma_2 (A^g) = (\rho c_1)^2 \sigma_2 (\bar{A}) + \frac{1}{8} m (m - 1) n(\rho)^2 - \frac{1}{2} (m - 1) c_1 \rho n(\rho) \sigma_1 (\bar{A})
\]
\[
+ \frac{1}{4} \beta (U) (2 \gamma_3 A(\beta^T_j) + U, \beta^T_j) - \frac{1}{4} (b^2 - \beta (N)^2) (2 \gamma_3 A(\beta^T_j) + U, U)
\]
(4.10)
\[
+ \frac{(m - 1)}{2} n(\rho) - \rho c_1 \sigma_1 (\bar{A}) \rangle (\gamma_3 \rho A(\beta^T_j) + U, \beta^T_j) + \rho c_1 (\gamma_3 A(\beta^T_j) + U, \bar{A}(\beta^T_j)),
\]
where \( A = -\rho c_1 \bar{A} + \text{Sym}(U \otimes \beta^T) \) and \( U \) is given in (4.5).

**Proof.** By conditions, \( \text{Def} \beta = 0 \). Thus, by Proposition 4.1,
\[
\rho \phi A^g = \rho c_1 \bar{A} - \frac{1}{2} n(\rho) \text{id}^T - A_1 - A_2,
\]
where \( A_1 = \frac{1}{2} U \otimes \beta^T \) and \( A_2 = (\frac{1}{2} U^b + \gamma_3 (\beta \circ A)) \otimes \beta^T \) are rank 1 matrices (thus \( \sigma_2 (A_i) = 0 \)) and
\[
\bar{A} = -\rho c_1 \bar{A} + \frac{1}{2} n(\rho) \text{id}^T + \text{Sym}(U \otimes \beta^T)
\]
is symmetric. Applying the identity
\[
\sigma_2 (\sum_i P_i) = \sum_i \sigma_2 (P_i) + \sum_{i < j} (\sigma_1 (P_i) \sigma_1 (P_j) - \sigma_1 (P_i P_j)),
\]
to matrices $P_1 = \rho c_1 \bar{A}$, $P_2 = -\frac{1}{2} n(\rho) \text{id}^T$, $P_3 = -A_1$ and $P_4 = -A_2$, and using equalities $\langle (\beta \circ A)^2, u \rangle = \langle A(u^T), \beta^2 \rangle$ and $\sigma_2(\text{id}^T) = m(m-1)/2$, we get

$$
(\rho \phi)^2 \sigma_2(A^p) = (\rho c_1)^2 \sigma_2(\bar{A}) + m(m-1) n(\rho)^2 / 8
- \frac{1}{2} (m-1) c_1 \rho n(\rho) \sigma_1(\bar{A}) + \sigma_1(A_1) \sigma_1(A_2) - \sigma_1(A_1 A_2)
+ \left( (m-1) n(\rho) / 2 - \rho c_1 \sigma_1(\bar{A}) \right) \sigma_1(A_1 + A_2) + \rho c_1 \sigma_1(\bar{A}(A_1 + A_2)),
$$

where

$$
\sigma_1(A_1) = \beta(U)/2, \quad \sigma_1(A_2) = (2 \gamma_3 A(\beta^{2T}) + U, \beta^2)/2,
\sigma_1(A_1 A_2) = (b^2 - \beta(N)^3)(2 \gamma_3 A(\beta^2) + U, U)/4,
\sigma_1(\bar{A}(A_1 + A_2)) = \langle \gamma_3 A(\beta^{2T}) + U, \bar{A}(\beta^{2T}) \rangle.
$$

From the above (4.10) follows. \hfill \square

In the next proposition, we express $Z$ through $\bar{Z}$, see (4.2), and invariants of $D$ with respect to $a$.

**Proposition 4.4.** Let $g$ be a new Riemannian metric determined by an integrable distribution $D$, a 1-form $\beta$ and a function $\phi(s)$ on $(M, a)$ with conditions (2.4), (2.10). Then

$$
\rho Z = Z + \gamma_3 \beta(\bar{Z}) \beta^{2T},
$$

where the vector field $Z$ is given by

$$
Z = \left[p_1 \nabla^T(\gamma_1 / \phi) + p_2 \nabla^T(c_1 / \phi(s))\right] \phi(s)^{-1} + [p_3 \bar{Z} + p_4 \bar{A}(\beta^{2T}) + p_5 \nabla^T(\beta(N))] \phi^{-2},
$$

$$
p_1 = c_1 \left( (4 \rho_1 \gamma_1 - \rho_0 + 3 \rho_1 s \gamma_1^3)^2 - \rho + c_1^2 \rho_1 s \right) \beta(N) - \rho_1 (2 \gamma_1 \rho_1 + 1)c_1^2 \beta(N)^2
- \gamma_1 (2 \gamma_1 \rho_1 + 1)^2 \beta(N) - c_1^2 \rho_1 s + (\rho - \gamma_1 \rho_1 (s \gamma_1 + 1)^2) c_1,

p_2 = (\rho_0 - 2 \gamma_1 \rho_1 + 1)^2 \beta(N) - c_1^2 \rho_1 s + (\rho - \gamma_1 \rho_1 (s \gamma_1 + 1)^2) c_1,

p_4 = (\rho_0 - 2 \gamma_1 \rho_1 + 1)^2 \beta(N) + (\rho_0 - 2 \gamma_1 \rho_1 + 1)c_1^2 \beta(N)^2
+ \gamma_1 (2 \gamma_1 \rho_1 + 1)^2 \beta(N) - c_1^2 \rho_1 s + (\rho - \gamma_1 \rho_1 (s \gamma_1 - 1) \beta(N) - c_1^2 \rho_1 s + (\rho - \gamma_1 \rho_1 (s \gamma_1 - 1)^2) c_1,

p_5 = \gamma_1 [c_1^2 \rho_1 s - (\rho_0 + 3 \rho_1 s \gamma_1^3) c_1^2 + (\rho_0 - 2 \gamma_1 \rho_1 - \gamma_1^2 \rho_1 s - \rho_0) \beta(N) - c_1^2 \rho_1 s + (\rho - \gamma_1 \rho_1 (s \gamma_1 - 1)^2) c_1],

\rho_0 = \gamma_1 (\rho_0 - 2 \gamma_1 \rho_1 + 1)^2 \beta(N) + (\rho_0 - 2 \gamma_1 \rho_1 + 1)c_1^2 \beta(N)^2
+ \gamma_1 (2 \gamma_1 \rho_1 + 1)^2 \beta(N) - c_1^2 \rho_1 s + (\rho - \gamma_1 \rho_1 (s \gamma_1 - 1) \beta(N) - c_1^2 \rho_1 s + (\rho - \gamma_1 \rho_1 (s \gamma_1 - 1)^2) c_1,

\rho_0 = \gamma_1 (\rho_0 - 2 \gamma_1 \rho_1 + 1)^2 \beta(N) + (\rho_0 - 2 \gamma_1 \rho_1 + 1)c_1^2 \beta(N)^2
+ \gamma_1 (2 \gamma_1 \rho_1 + 1)^2 \beta(N) - c_1^2 \rho_1 s + (\rho - \gamma_1 \rho_1 (s \gamma_1 - 1) \beta(N) - c_1^2 \rho_1 s + (\rho - \gamma_1 \rho_1 (s \gamma_1 - 1)^2) c_1],

Moreover, if $\beta^s$ is tangent to $D$ and $b = \text{const}$ then

$$
\beta = \phi^{-2} \left[ c_1^2 |\rho - c_1^2 \rho_1 s - \gamma_1 \rho_1 (s \gamma_1 + 1)^2| \beta^2 Z
+ c_1^2 |\rho_0 - 2 \gamma_1 \rho_1 - \gamma_1^2 \rho_1 s| \beta^2 (\beta^{2T}) \right].
$$

**Proof.** Extend $X \in T_x F$ onto a neighborhood of a point $x \in M$ with the property $(\nabla_Y X)^T = 0$ for any $Y \in T_x M$. By formula (4.6), we obtain at $x$:

$$
(4.11) \quad g(Z, X) = g([X, \nu], \nu).
$$
Using equalities $\nu = \phi^{-1}(c_1N - \gamma_1\beta^2)$ and $[X, fY] = X(f)Y + f[X, Y]$ we get

$$
g([X, \nu], \nu) = (c_1/\phi)X(c_1/\phi)\phi(N) - X(c_1\gamma_1/\phi^2)\phi(N) + (c_1\gamma_1/\phi^2)\phi(N, \beta^2) + (c_1/\phi)^2\phi([X, N], N)\]

(4.12)

$$
g([X, \nu], \nu) = (c_1/\phi)X(c_1/\phi)\phi(N) - X(c_1\gamma_1/\phi^2)\phi(N, \beta^2) + (c_1/\phi)^2\phi([X, N], N)\]

To compute first three terms in (4.12), by (2.2) for $p = 1$,

$$
g(u, v) = \rho(u, v) + \rho_0(\beta(u), \beta(v)) + \rho_1(\beta(u)(n, v) + \beta(v)(n, u) - \beta(n)(n, u)(n, v)),
$$

and Lemma 2.1, we find

$$
g(\beta^2, \beta^2) = \rho b^2 + \rho_0 b^2 + 2\rho_1 b^2 s - \rho_1 s^3,
$$

$$
g(N, \beta^2) = (\rho + \rho_0 b^2 + \rho_1 s)\beta(N) + \rho_1(b^2 - s^2)(n, N),
$$

$$
g(N, N) = \rho + \rho_0\beta(N)^2 + 2\rho_1\beta(N)(n, N) - \rho_1 s(n, N)^2.
$$

To compute last four terms in (4.12), we will use

$$
[X, \beta^2] = [X, [\beta^2]] + X(\beta(N))N + (\beta(N)(X, N) - \bar{A}(X)),
$$

$$
[X, N] = -\nabla_X N - \nabla_N X = -\bar{A}(X) - \langle \nabla_N X, N \rangle = \langle Z, X \rangle N - \bar{A}(X),
$$

and by (4.13) and Lemma 2.1, obtain the equalities

$$
g([X, N], \beta^2) = (\rho + \rho_0 b^2 + \rho_1 s)\phi([X, N], \beta^2) + \rho_1(b^2 - s^2)(n, N),
$$

$$
g([X, \beta^2], \beta^2) = (\rho + \rho_0 b^2 + \rho_1 s)([X, \beta^2], \beta^2) + \rho_1(b^2 - s^2)([X, \beta^2], n),
$$

$$
g([X, N], N) = \rho([X, N], N) + (\rho_0\beta(N) + \rho_1(n, N))\phi([X, N], \beta^2) + \rho_1(\beta(N) - s(n, N))\phi([X, N], n),
$$

$$
g([X, \beta^2], N) = \rho([X, \beta^2], N) + (\rho_0\beta(N) + \rho_1(n, N))\phi([X, \beta^2], \beta^2) + \rho_1(\beta(N) - s(n, N))\phi([X, \beta^2], n).
$$

Thus,

$$
g([X, \nu], \nu) = (c_1/\phi)X(c_1/\phi)\phi(N) - X(c_1\gamma_1/\phi^2)\phi(N) + (c_1\gamma_1/\phi^2)\phi(N, \beta^2) + (c_1/\phi)^2\phi([X, N], N)\]

(4.13)

$$
g([X, \nu], \nu) = (c_1/\phi)X(c_1/\phi)\phi(N) - X(c_1\gamma_1/\phi^2)\phi(N) + (c_1\gamma_1/\phi^2)\phi(N, \beta^2) + (c_1/\phi)^2\phi([X, N], N)\]

(4.13)
Note that $\langle n, N \rangle = c_1 - \gamma_1 \beta(N)$ and $\beta(n) = c_1 \beta(N) - \gamma_1 b^2$, see (2.5), and
\[
\begin{align*}
\langle [X, N], N \rangle &= \langle \bar{Z}, X \rangle, \\
\langle [X, N], \beta \rangle &= \langle \beta(N)\bar{Z} - \bar{A}(\beta^T), X \rangle, \\
\langle [X, N], n \rangle &= c_1 \langle [X, N], N \rangle - \gamma_1 \langle [X, N], \beta \rangle \\
&= \langle (c_1 - \gamma_1 \beta(N))\bar{Z} + \gamma_1 \bar{A}(\beta^T), X \rangle, \\
\langle [X, \beta], N \rangle &= \langle \bar{\nabla}(\beta(N)) + \beta(N)\bar{Z}, X \rangle, \\
\langle [X, \beta], \beta \rangle &= b X(b) - \langle \bar{\nabla}_\beta X, \beta \rangle = (b \bar{\nabla} b + \beta(N)^2\bar{Z} - \beta(N)\bar{A}(\beta^T), X), \\
\langle [X, \beta], n \rangle &= c_1 \langle [X, \beta], N \rangle - \gamma_1 \langle [X, \beta], \beta \rangle \\
&= \langle (c_1 \beta(N) - \gamma_1 \beta(N)^2)\bar{Z} - \gamma_1 b \bar{\nabla} b + \gamma_1 \beta(N)\bar{A}(\beta^T), X \rangle.
\end{align*}
\]
By (4.11), $g(Z, X) = \langle Z, X \rangle$. With the help of Lemma 2.2 we complete the proof. □

5 The Reeb type integral formula

In this section we apply results in Sections 1–4 to prove a new integral formula for a closed Riemannian manifold with a set of linearly independent 1-forms and a codimension one distribution, which generalizes the Reeb’s integral formula (0.1).

**Theorem 5.1.** Let $g$ be a new Riemannian metric determined by $D = \ker \omega$, 1-forms $\beta_i (1 \leq i \leq p)$ on a closed Riemannian manifold $(M, a)$ and a function $\phi(s)$, where $s = (s_1, \ldots, s_p)$, with conditions (2.4), (2.10). Then
\[
\int_M \mu_g(n)(\rho \phi)^{-1} \left\{ \rho c_1 \sigma_1(\bar{A}) - (m/2) n(\rho) + \rho \gamma_1 \langle \beta_i(\bar{Z}) - N(\beta_i(N)) \rangle \right\} \, d\text{vol}_a = 0.
\]
(5.1)

**Proof.** For the metric $g$ the Reeb’s integral formula (0.1) reads
\[
\int_M H_\beta \, d\text{vol}_g = 0.
\]
(5.2)

By (5.2), we have
\[
\int_M \mu_g(n) \sigma_1(A^T) \, d\text{vol}_a = 0.
\]

Corollary 4.2 and using $f^i \text{div} \beta_i^T = \text{div} (f^i \beta_i^T) - \beta_i^T(f^i)$ with $f^i = \mu_g(n) \gamma_1 / \phi$, yield (5.1).

The integral formula (5.1) holds when all 1-forms are defined outside a closed submanifold of codimension $\geq 2$ under convergence of some integrals, see discussion in [7, 16]. The singular case is important since many manifolds admit no codimension-one distributions or foliations, while all of them admit non-vanishing 1-forms outside some “set of singularities”.

**Corollary 5.2.** In conditions of Theorem 5.1 for $p = 1$, let $b$ and $\beta(N)$ be constant. Then
\[
\int_M \langle q_1 \bar{A}(\beta^T) + q_2 \bar{Z}, \beta^T \rangle \, d\text{vol}_a = 0,
\]
(5.3)
where the constants $q_1$ and $q_2$ are given by

$$q_1 = -\rho(\rho + (b^2 - \beta(N)^2)\gamma_2)^{-1}(c_1\rho_1\gamma_1(1 + s\gamma_1) + \gamma_2(c_1 - \beta(N)\gamma_1)),$$
$$q_2 = \gamma_1\rho - c_1\rho_1\rho(\rho + (b^2 - \beta(N)^2)\gamma_2)^{-1}(1 + s\gamma_1)(c_1 - \beta(N)\gamma_1).$$

**Proof.** If $b$ and $\beta(N)$ are constant, that is $\beta^\flat$ and its $D^\perp$-component have constant lengths, then $s, \rho, \rho_1, \gamma_1, c_1$ and $\phi(s), \mu_p(n)$ are also constant. In this case, (5.1) yields (5.3). \hfill \Box

There are topological obstructions to the existence of codimension one totally geodesic and Riemannian foliations on a closed Riemannian manifold, see [4, 6]. For such foliations we get

**Corollary 5.3.** In conditions of Theorem 5.1 for $p = 1$, let $b$ and $\beta(N)$ be constant. (i) If $A = 0$ and $q_2 \neq 0$ then either $\beta(Z) \equiv 0$ or $\beta(Z)_x \cdot \beta(Z)_{x'} < 0$ for some points $x \neq x'$. (ii) If $\dot{Z} = 0$ and $q_1 \neq 0$ then either $\langle A(\beta^\flat\top), \beta^\flat \rangle \equiv 0$ or $\langle A(\beta^\flat\top), \beta^\flat \rangle_x \cdot \langle A(\beta^\flat\top), \beta^\flat \rangle_{x'} < 0$ for some points $x \neq x'$.

**Example 5.1.** (i) For Randers metric ($p = 1$), by (5.1) we get, see [13],

$$\int_M (c_1)^m+1c^{-1}((c_1)\sigma_1(\dot{A}) - \frac{m+2}{2}(N + c_1^{-1}\beta^\flat)(c_1) + c_1N(c))$$

$$- (c_1 - c)[N(c) + (c_1^{-2}(\beta^\flat\top) + \dot{Z}, \beta^\flat))] d\text{vol}_a = 0,$$

which is the Reeb formula when $\beta = 0$. If $\beta(N) = 0$ then (5.4) reads

$$\int_M c^{2m+1}(c_1\sigma_1(\dot{A}) - (m+1)cN(c) - (m+2)\beta^\flat(c)) d\text{vol}_a = 0.$$ 

If $b$ and $\beta(N) \neq 0$ are constant then (5.4) reads $\int_M \langle \dot{A}(\beta^\flat\top) + c\dot{Z}, \beta^\flat \rangle d\text{vol}_a = 0$, see also (5.3) with $q_1 = c_1^{-1}(c - c_1)$ and $q_2 = c_1(c - c_1)$.

(ii) For Kropina metric, if $\beta(N) = 0$ then $\mu_p(n) = (2/b)^{2m+2}$, and

$$\gamma_1 = -\sqrt{2}/(2b), \quad \gamma_2 = 0, \quad c_1 = 1/\sqrt{2},$$
$$s = b/\sqrt{2}, \quad \rho = 4/b^2, \quad \rho_0 = 12/b^4, \quad \rho_1 = -8\sqrt{2}/b^3.$$ 

Hence, by Proposition 4.1 for $p = 1$, $\sigma_1(A^p) = \frac{b}{2} \sigma_1(\dot{A}) - \frac{1}{2} \text{div} \beta^\flat + \frac{m}{\sqrt{2}} n(b) + \frac{1}{2+b} \beta^\flat(b),$ and, we get integral formula

$$\int_M \left( \frac{2}{b} \right)^{2m+2} \left\{ b\sigma_1(\dot{A}) + \sqrt{2} mn(b) - \frac{2m+1}{b} \beta^\flat(b) \right\} d\text{vol}_a = 0,$$

which for $b = \text{const}$ reduces to (0.1) for metric $a$.

(iii) The following application of (5.3) (when $b$ and $\beta(N)$ are constant) seems to be interesting. Let $\dot{Z} = 0, q_1 \neq 0$ and $\alpha$-unit vector field $X \in \mathfrak{X}_M$ be an eigenvector of $\dot{A}$ with an eigenvalue $\lambda : M \setminus \Sigma \to \mathbb{R}$. Then $\beta^\flat = \varepsilon'X + \varepsilon N$, where $\varepsilon = \text{const} \in (0, \delta_0)$ and $\varepsilon' = \text{const} \in (0, \sqrt{1 - \varepsilon^2})$, obeys (5.3). Thus, $\int_M \lambda d\text{vol}_a = 0$. Consequently, either $\lambda \equiv 0$ on $M$ or $\lambda(x)X(x') < 0$ for some points $x \neq x'$. Furthermore, this implies Reeb formula (0.1) for $\langle \cdot , \cdot \rangle$:

$$\int_M \sigma_1(\dot{A}) d\text{vol}_a = \sum_i \int_M \lambda_i d\text{vol}_a = 0.$$
6 The counterpart of Reeb integral formula

In this section we assume for simplicity that $\mathcal{D}$ is integrable and $p = 1$, and use $(\alpha, \beta)$-metrics.

The counterpart of the Reeb integral formula for the second mean curvature reads

$$\int_M (2\sigma_2(\bar{A}) - \overline{\text{Ric}}_{N,N}) \, d\text{vol}_a = 0.$$  

(6.1)

Here $\overline{\text{Ric}}_{N,N} = \text{Tr}_a(u \rightarrow \bar{R}_{N,u} N)$ is the Ricci curvature of $a$ in the $N$-direction. The proof of (6.1), see e.g. [11], is based on the Divergence theorem applied to

$$\overline{\text{div}}(\sigma_1(\bar{A}) N + \bar{Z}) = \overline{\text{Ric}}_{N,N} - 2\sigma_2(\bar{A}).$$

We will generalize (6.1) for codimension one foliations with general $(\alpha, \beta)$-metrics on $M$. In this case, the volume form of $g$ with $\mu_g$ given in (3.6) obeys

$$d\text{vol}_g = \mu_g(n) \, d\text{vol}_a.$$  

(6.2)

Let $\text{Ric}^0_{\nu,\nu} = \text{Tr}_g(u \rightarrow R^0_{\nu,u} \nu)$ be the Ricci curvature of $g$ in the $\nu$-direction, where $R^0_{\nu,u} = [\nabla_u, \nabla_u] - \nabla_{[u,u]}$ is the curvature tensor derived using the Levi-Civita connection of $g$. The *Chern connection* $D^\nu$ is torsion-free and almost metric, it is determined by

$$g(D^\nu_{\nu} u, v) - g(\nabla_u v, w) = C_{\nu}(D^\nu_{\nu} u, v, w) - C_{\nu}(D^\nu_{\nu} u, v, w) - C_{\nu}(D^\nu_{\nu} u, v, w),$$

(6.3)

see [14], for any vector fields $u, v, w$, where $g(\nabla_u v, w)$ is given in (4.6).

The difference $T = D^\nu - \nabla$ is called the *contorsion tensor*. It is a symmetric tensor because both connections, $\nabla$ and $D^\nu$, are torsion-free. By (6.3), $D^\nu_{\nu} = \nabla_{\nu,\nu}$ holds; hence, $\nabla_{\nu,\nu} = 0$ (thus, $\nu$ is geodesic for $F$ if and only if it is geodesic for $g$).

Comparing the curvature $R^0_{\nu,u} = [D^\nu_{\nu} - D^\nu_{\nu}]$ of $D^\nu$ with $R^0_{\nu,u}$, we find

$$R^D_{\nu,u} - R^D_{\nu,u} = (\nabla_u T)_{\nu} - (\nabla_{\nu T})u - [T_{\nu}, T_u], \quad u \in TM.$$  

(6.4)

In [5], the Ricci curvature $\text{Ric}^D_y = \text{Tr}_g(u \rightarrow R^D_{\nu,u} y)$ of $(\alpha, \beta)$-metric is expressed through $\text{Ric}_y$ of $\alpha$; in particular, $\nabla_{\nu} = 0$ provides $\text{Ric}^D_y = \text{Ric}_y (y \neq 0)$.

Let $C^D_{\nu}$ be a $(1,1)$-tensor $g$-dual to the symmetric bilinear form $C_{\nu}(Z, \cdot, \cdot)$:

$$g(C^D_{\nu}(u), v) = C_{\nu}(Z, u, v), \quad u, v \in TM.$$  

Note that $A^g + C^D_{\nu}$ is the shape operator of the leaves with respect to $D^\nu$, see [13]. By (6.3), we get

$$T_{\nu} = -C^D_{\nu}, \quad \text{Tr} T_{\nu} = -\sigma_1(C^D_{\nu}) = -I_{\nu}(Z).$$  

(6.5)

Unlike Theorem 5.1, the following theorem contains non-Riemannian quantities.

**Theorem 6.1.** Let $g$ be a new metric determined by a codimension-one foliation $\mathcal{F}$ ($\mathcal{F} = \mathcal{D}$), a 1-form $\beta$ on $(M, a)$, and a function $\phi(s)$ with the conditions (2.4),
\( (2.10) \) and \( \nabla \beta^\sharp = 0 \). Then

\[
\int_M \left\{ \left( (c_1 \rho)^2 \left( 2 \sigma_2 (\bar{A}) - \text{Ric}_{N,N} \right) + \frac{1}{4} m (m - 1) n (\rho)^2 - (m - 1) c_1 \rho n (\rho) \sigma_1 (\bar{A}) + \frac{1}{2} \beta (U) (2 \gamma_3 \mathcal{A} (\beta^T) + U, \beta^T) - \frac{1}{2} (b^2 - \beta (N))^2 (2 \gamma_3 \mathcal{A} (\beta^T) + U, U) - 2 c_1 \sigma_1 (\bar{A}) - (m - 1) n (\rho) \right) (2 \gamma_3 \mathcal{A} (\beta^T) + U, \beta^T) + 2 \rho c_1 \sigma_1 (\bar{A}) \right) \right\} (\rho \phi (s))^{-2} \mu_g (n) d \text{vol}_a = 0,
\]

where \( A^p, \mathcal{A} \) and \( U \) are given in Proposition 4.1, \( Z \) is given in Proposition 4.4 and \( \mu_g (n) \) is given in (3.6) with \( y = n \) and \( s = \beta (n) \).

**Proof.** We will use the adjoint (1,2)-tensor \( T^* \) defined by

\[
g (T^*_u, v, w) = g (T_u w, v)
\]

for \( u, v, w \in TM \). Note that \( T^*_u \nu = 0 \) and define \( \text{Tr}_g T^* = \sum_i T^*_b b_i \) – the trace of \( T^* \) with respect to \( g \). Assuming \( (\nabla_u b_i)^\top = 0 \) and \( (\nabla_b, \nu)^\perp = 0 \) at a point \( x \in M \), calculate at \( x \):

\[
\begin{align*}
\sum_i g ((\nabla_i T)_\nu \nu, b_i) &= 2 \sum_i g (T^*_{b_i} b_i, A^p (b_i)) = 2 \sigma_1 (C^p_\nu A^p), \\
\sum_i g ((\nabla_b T)_i \nu, b_i) &= \text{div}_g (\text{Tr}_g T^*), \quad \sum_i g ([T_i, T_u] \nu, b_i) = - \sigma_1 ((C^p_\nu)^2),
\end{align*}
\]

using the symmetry \( T_i \nu = T_u b_i \). Then, applying (6.4) we get

\[
\text{Ric}_{\nu, \nu}^D - \text{Ric}_{\nu, \nu}^g = \sum_i \left[ g ((\nabla_i T)_\nu \nu, b_i) - g ((\nabla_b T)_i \nu, b_i) + g ([T_i, T_u] \nu, b_i) \right] = 2 \sigma_1 (C^p_\nu A^p) - \sigma_1 ((C^p_\nu)^2) - \text{div}_g^\perp (\text{Tr}_g^T T^*).
\]

From (6.7) and

\[
\text{div}_g^\perp (\text{Tr}_g T^*) = \text{div}_g ((\text{Tr}_g T^*)^\perp) - g (\text{Tr}_g T^*, \sigma_1 (A^p) \nu - Z)
\]

we obtain

\[
\text{div}_g ((\text{Tr}_g T^*)^\perp) = \text{Ric}_{\nu, \nu}^g - \text{Ric}_{\nu, \nu}^D
\]

(6.8) and

\[
g (\text{Tr}_g T^*, \nu) = - \sum_i C_\nu (D_\nu \nu, b_i, b_i) = - \sigma_1 (C^p_\nu) = - I_\nu (Z),
\]

\[
g (\text{Tr}_g T^*, u) = - \sum_i C_\nu (D_u \nu, b_i, b_i) = I_\nu ((A^p + C^p_\nu) u)
\]

for \( u \in D \). By the above we obtain

\[
g (\text{Tr}_g T^*, \sigma_1 (A^p) \nu - Z) = - I_\nu ((A^p + C^p_\nu + \sigma_1 (A^p) \text{id}) Z).
\]
By conditions, \( b = \text{const} \) and \( \bar{R}(X,Y)\beta^4 = 0 \) \((X,Y \in TM)\). Using
\[
\text{Ric}^{D}_{n,n} = \text{Ric}_{n,n} = c_1^2 \text{Ric}_{N,N} + \gamma_1^2 \text{Ric}_{\beta^4,\beta^4} - 2c_1\gamma_1 \langle \bar{R}(N,b_i)\beta^4, b_i \rangle
\]
and \( \text{Ric}^{D}_{\nu,\nu} = \phi^{-2} \text{Ric}^{D}_{n,n} \), we find
\[
\text{Ric}^{D}_{\nu,\nu} = (c_1/\phi)^2 \text{Ric}_{N,N}.
\]
By the above, (6.1) and (6.2) for \( g \), using (6.8) and Corollary 4.3, we find (6.6). \( \square \)

**Corollary 6.2.** In conditions of Theorem 6.1, let \( \beta(N) = \text{const} \), \( Z = 0 \) and \( q_3 \neq 0 \), where
\[
q_3 = \frac{q\rho(4\rho c_1 - (b^2 - \beta(N)^2)q) - 4\rho^2 c_1^2 \gamma_2}{4(\rho + (b^2 - \beta(N)^2)\gamma_2)},
\]
\[
q = \rho_1 c_1 \gamma_1 (1 + s \gamma_1) - (\rho_0 - \rho_1 \gamma_1) (c_1 - \beta(N)\gamma_1) - \gamma_1 \gamma_2 \beta(N).
\]
Then \( \bar{A}(\beta^T) = 0 \), hence \( \text{rank}(\bar{A}) < m \). If \( \mathcal{F} \) is totally umbilical then \( \mathcal{F} \) is totally geodesic.

**Proof.** By conditions, \( s, \rho, \rho_1, \gamma_i, c_1 \) are constant (since \( b \) and \( \beta(N) \) are constant) and \( \text{Ric}^{D}_{\nu,\nu} = \text{Ric}^{g}_{\nu,\nu} \). Hence, see (6.8),
\[
\int_M \{ g(\text{Tr} \, T^*, \sigma_1(A^\beta) - Z) - 2\sigma_1(A^\beta C^2_c) - \sigma_1((C^2_c)^2) \} \, d\text{vol}_g = 0.
\]
Thus, (6.6) and (6.1) yield
\[
\int_M \left\{ \frac{1}{4} \beta(U)\langle 2\gamma_3 A(\beta^T) + U, \beta^T \rangle - \frac{1}{4} (b^2 - \beta(N)^2)\langle 2\gamma_3 A(\beta^T) + U, \beta^T \rangle \right\} \, d\text{vol}_a = 0,
\]
where, in view of \( \nabla^T_n \beta^T = -\gamma_1 \beta(N) \bar{A}(\beta^T) \), we have
\[
U = q\bar{A}(\beta^T), \quad A = -c_1 \bar{A} + q \text{Sym}(\bar{A}(\beta^T) \otimes \beta^T).
\]
If \( \beta(N) = \text{const} \) then \( \beta(\bar{Z}) = 0 \) and \( \langle \bar{A}(\beta^T), \beta^T \rangle = 0 \):
\[
0 = \langle \nabla_{\beta^T} \beta^T, N \rangle = \langle \nabla_{\beta^T} (\beta^T + \beta(N)N), N \rangle = -\langle \beta^T, \bar{Z} \rangle,
\]
\[
0 = \langle \nabla_{\beta^T} \beta^T, N \rangle = \langle \nabla_{\beta^T} (\beta^T + \beta(N)N), N \rangle = -\langle \bar{A}(\beta^T), \beta^2 \rangle.
\]
By (6.9),
\[
\int_M q_3 \| \bar{A}(\beta^T) \|_a^2 \, d\text{vol}_a = 0,
\]
and \( q_3 \neq 0 \) yields \( \bar{A}(\beta^T) \equiv 0 \). If \( \mathcal{F} \) is totally umbilical then \( 0 = \langle \bar{A}(\beta^T), \beta^T \rangle = \| \beta^T \|^2_\alpha \sigma_1(\bar{A}) \), hence \( \sigma_1(\bar{A}) = 0 \). By the above, \( \bar{A} = 0 \) on \( M \). \( \square \)

**Example 6.1.** For Randers metric, we obtain \( q_3 = \frac{1}{4} c_1^2 \sqrt{c - \beta(N)^2} \) with \( c_1 = c + \beta(N) \) and \( c = \sqrt{1 - b^2 + \beta(N)^2} \). For Kropina metric, we have \( q_3 = -\frac{1}{16} \beta(N)(16c_1s^3 + b^2\beta(N) - \beta(N)^3) \) \( s^{-10} \) with \( s = \sqrt{b - \beta(N)} \).
Let \( k_1 \leq k_2 \leq \ldots \leq k_m \) be the eigenvalues of \( A^g \). One can consider the integral

\[
U_F = \int_M \sum_{i<j} (k_i - k_j)^2 \, d\text{vol}_g,
\]

which measures “how far from \( g \)-umbilicity” is a foliation \( F \), see [6] for Riemannian case. Put

\[
\mu_{\text{min}} = \min_{y \in TM \setminus \{0\}} \mu_{g_y}(y).
\]

**Theorem 6.3.** Let \( g \) be a new Riemannian metric determined by a codimension-one foliation \( F \), a 1-form \( \beta \) on \((M, a)\), and a function \( \phi \) with conditions (2.4), (2.10), \( \nabla \beta = 0 \), \( \beta(N) = \text{const} \) and \( \text{Ric}_{N,N} \leq -r < 0 \). Then

\[
(6.10) \quad U_F \geq m \, r \, (c_1/\phi(s))^2 \mu_{\text{min}} \text{Vol}_a(M).
\]

In particular, if \( c_1 \neq 0 \) then \( F \) is nowhere \( g \)-totally umbilical.

**Proof.** One may show that

\[
\sum_{i<j} (k_i - k_j)^2 = (m - 1) \sigma_1^2(A^g) - 2m \sigma_2(A^g).
\]

Hence, and by (6.1) for \( g \) we obtain

\[
U_F \geq -m \int_M 2 \sigma_2(A^g) \, d\text{vol}_g = -m \int_M \text{Ric}^g_{\nu,\nu} \, d\text{vol}_g.
\]

By conditions, \( \text{Ric}^g_{\nu,\nu} = (c_1/\phi(s))^2 \text{Ric}_{N,N} \), and \( s, \rho, \rho_i, \gamma_i, c_1, \phi(s), \mu_g(\nu) \) are constant. Thus,

\[
U_F \geq -m \, (c_1/\phi(s))^2 \mu_{\text{min}} \int_M \text{Ric}_{N,N} \, d\text{vol}_a,
\]

which reduces to (6.10) since our assumption \( \text{Ric}_{N,N} \leq -r < 0 \). \( \square \)

Following [3] for Riemannian case, define the **energy of a vector field** \( \nu \) by

\[
\mathcal{E}(\nu) = \frac{m+1}{2} \text{Vol}_g(M) + \frac{1}{2} \int_M \|D\nu\|_g^2 \, d\text{vol}_g.
\]

By (6.1) for \( g \) and the inequality \( \|D\nu\|_g^2 \geq \frac{2}{m} \sigma_2(A^g) \), see [3], we get the following.

**Theorem 6.4.** Let \( g \) be a new Riemannian metric determined by a codimension-one foliation \( F \), a 1-form \( \beta \) on \((M, a)\), and a function \( \phi \) with conditions (2.4), (2.10), \( \nabla \beta = 0 \) and \( \beta(N) = \text{const} \). Then for a unit \( g \)-normal \( \nu \),

\[
\mathcal{E}(\nu) \geq \mu_{\text{min}} \left( \frac{m+1}{2} \text{Vol}_a(M) + \frac{c_1^2}{2m} \phi^2 \int_M \text{Ric}_{N,N} \, d\text{vol}_a \right).
\]
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