Theorems on conformal mappings of complete Riemannian manifolds and their applications

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Abstract. We prove several Liouville-type non-existence theorems for conformal mappings of complete Riemannian manifolds. As well, we provide applications of these results to General Relativity and to the theory of conharmonic transformations.


Key words: complete Riemannian manifold; conformal diffeomorphism; conharmonic transformation; non-existence theorem.

1 Subharmonic and superharmonic functions

Let \((M, g)\) be an \(n\)-dimensional \((n \geq 2)\) Riemannian manifold. We recall that \(f \in C^2 \text{M}\) is subharmonic (resp. superharmonic or harmonic) if \(\Delta f \geq 0\) (resp. \(\Delta f \leq 0\) or \(\Delta f = 0\)) for the Laplace-Beltrami operator \(\Delta f = \text{div} (\text{grad} f)\). In particular, if \((M, g)\) is compact, then every harmonic (subharmonic or superharmonic) functions is constant by Hopf’s theorem [1].

We prove the following Lemma on superharmonic functions, which consists of two statements that are analogous to two Yau propositions on subharmonic functions (see [2]). Yau has stated in [2, p. 660] that on a complete Riemannian manifold \((M, g)\), each subharmonic function \(u \in C^2 \text{M}\), whose gradient has integrable norm on \((M, g)\), must be harmonic. Secondly, he has shown in [7, p. 663] that on a complete Riemannian manifold, each non-negative subharmonic function \(u \in C^2 \text{M}\) such that \(\int_M u^p dVol < \infty\) for some \(1 < p < \infty\), must be constant. In particular, if the volume of \((M, g)\) is infinite, then \(u = 0\).

Lemma 1.1. If \((M, g)\) is a connected complete Riemannian manifold (without boundary), then any superharmonic function \(\varphi \in C^2 \text{M}\) with \(\|\text{grad} \varphi\| \in L^1 (M, g)\) is harmonic and each non-positive superharmonic function \(\varphi \in C^2 \text{M}\) such that \(\varphi \in L^p (M, g)\) for some \(1 < p < \infty\) must be constant. In particular, if the volume of \((M, g)\) is infinite, then \(\varphi = 0\).
2 Conformal diffeomorphisms of complete Riemannian manifolds

Let \((M, g)\) and \((\bar{M}, \bar{g})\) be pseudo-Riemannian or Riemannian manifolds such that \(\dim M = \dim \bar{M} = n\) for any \(n \geq 3\). Then a diffeomorphism \(f : (M, g) \to (\bar{M}, \bar{g})\) is called conformal if it preserves angles between any pair curves. In this case, \(\bar{g} = e^{2\sigma} g\) for some scalar function \(\sigma\) (see [2, p. 663]). If the function \(\sigma\) is a constant then \(f\) is a homothetic mapping. In particular, if \(\sigma = 0\), \(f\) is an isometric mapping.

If \(\sigma \in C^2 M\) then for each pair of corresponding points \(x \in M\) and \(\bar{x} = f(x) \in \bar{M}\) we have the equation (see [3, p. 90])

\[
e^{2\sigma} \bar{s} = s - 2(n - 1) \Delta \sigma - (n - 1)(n - 2) \|\text{grad} \, \sigma\|^2,
\]

where \(s\) and \(\bar{s}\) denote the scalar curvatures of \((M, g)\) and \((\bar{M}, \bar{g})\), respectively. In the case when \((M, g)\) and \((\bar{M}, \bar{g})\) are Riemannian manifolds we can formulate the following Liouville-type non-existence theorem.

**Theorem 2.1.** Let \((M, g)\) be an \(n\)-dimensional \((n \geq 3)\) complete Riemannian manifold and \(f : (M, g) \to (\bar{M}, \bar{g})\) be a conformal diffeomorphism onto another Riemannian manifold \((\bar{M}, \bar{g})\) such that \(\bar{g} = e^{2\sigma} g\) and \(\bar{s} \geq e^{-2\sigma} s\) for some function \(\sigma \in C^2 M\) and the scalar curvatures \(s\) and \(\bar{s}\) of \((M, g)\) and \((\bar{M}, \bar{g})\), respectively. Then the following propositions are true.

1. If \(\|\text{grad} \, \sigma\| \in L^1(M, g)\), then \(f\) is a homothetic mapping.

2. If \(\sigma\) is non-positive function and \(\sigma \in L^p(M, g)\) for some \(1 < p < \infty\) then \(f\) is a homothetic mapping. In particular, if the volume of \((M, g)\) is infinite, then \(f\) is an isometric mapping.

**Proof.** If \(f : (M, g) \to (\bar{M}, \bar{g})\) is a conformal diffeomorphism a connected complete Riemannian manifold \((M, g)\) onto another Riemannian manifold \((\bar{M}, \bar{g})\) such that \(\bar{g} = e^{2\sigma} g\) for some function \(\sigma \in C^2 M\), then from (2.1) we obtain

\[
2(n - 1) \Delta \sigma = s - e^{2\sigma} \bar{s} - (n - 1)(n - 2) \|\text{grad} \, \sigma\|^2.
\]

Let \(s \leq e^{2\sigma} \bar{s}\) then (2) shows \(\Delta \sigma \leq 0\). It means that \(\sigma\) is a superharmonic function. By the condition of our theorem, the gradient of \(\sigma\) has integrable norm on \((M, g)\) and
we obtain from (2.2) that $\Delta \sigma = 0$ (see our Lemma). In this case, $\sigma$ is a harmonic function. Since $n \geq 3$, we see from (2.2) that $\sigma$ is constant. In the other hand, if $\sigma$ is a non-positive function such that $s \leq e^{2s} \bar{s}$ and $\sigma \in L^p(M, g)$ for some $1 < p < \infty$ then using the Lemma we can conclude that $\sigma$ is a constant function. It is obvious that if the volume of $(M, g)$ is infinite, then $\sigma = 0$ (see our Lemma). The proof of the theorem is complete. \hfill $\square$

In particular, if we assume that $s \geq 0$ and $\bar{s} \leq 0$ in the condition of our theorem, then the inequality $s \geq \lambda^2 \bar{s}$ must be satisfied. Then, as a result the proofs of the theorem, we can conclude that $s = \bar{s} = 0$. Therefore we have

**Corollary 2.2.** Let $(M, g)$ be an $n$-dimensional ($n \geq 0$) complete Riemannian manifold and $f : (M, g) \to (M, \bar{g})$ be a conformal diffeomorphism onto another Riemannian manifold $(\bar{M}, \bar{g})$ such that $\bar{g} = e^{2\sigma} g$ for some function $\sigma \in C^2 M$, $s \geq 0$ and $\bar{s} \leq 0$ for the scalar curvatures $s$ and $\bar{s}$ of $(M, g)$ and $(\bar{M}, \bar{g})$, respectively. If the one of the following conditions holds:

1. $\|\text{grad} \sigma\| \in L^1(M, g)$,
2. $\sigma \in L^p(M, g)$ for some $1 < p < \infty$ and $\sigma \leq 0$,

then $f$ is a homothetic mapping and $s = \bar{s} = 0$. If in the second case the volume of $(M, g)$ is infinite, then $f$ is an isometric mapping.

Let $\sigma = \log \lambda$ for some positive scalar function $\lambda \in C^2 M$ then

$$\Delta \sigma = \lambda^{-1} \Delta \lambda - \lambda^{-2} \|\text{grad} \lambda\|^2, \quad \|\text{grad} \sigma\|^2 = \lambda^{-2} \|\text{grad} \lambda\|^2.$$  

In this case, (2.2) can be rewritten in the following equivalent form

$$(2.3) \quad 2(n-1)\lambda \Delta \lambda = \lambda^2 (s - \lambda^2 \bar{s}) - (n-1)(n-4) \|\text{grad} \lambda\|^2.$$  

If $s \geq \lambda^2 \bar{s}$ for $n \leq 4$ then from (2.3) we obtain that $\lambda \Delta \lambda \geq 0$. On the other hand, Yau has proved in [2, p. 664] that if a smooth function $\lambda \in C^2 M$ on a complete Riemannian manifold $(M, g)$ such that $\lambda \Delta \lambda \geq 0$, then either $\int_M |\lambda|^p dV_g = \infty$ for all $p \neq 1$ or $\lambda \equiv \text{constant}$. Therefore, in the case when $(M, g)$ and $(\bar{M}, \bar{g})$ are Riemannian manifolds we have

**Theorem 2.3.** Let $(M, g)$ be an $n$-dimensional ($n = 3, 4$) complete Riemannian manifold and $f : (M, g) \to (M, \bar{g})$ be a conformal diffeomorphism onto another Riemannian manifold $(\bar{M}, \bar{g})$ such that $\bar{g} = \lambda^2 g$ and $s \geq \lambda^2 \bar{s}$ for some positive function $\lambda \in C^2 M$ and for the scalar curvatures $s$ and $\bar{s}$ of $(M, g)$ and $(\bar{M}, \bar{g})$, respectively. If $\lambda \in L^p(M, g)$ for some $p \neq 1$, then $f$ is a homothetic mapping.

In particular, if we assume that $s \geq 0$ and $\bar{s} \leq 0$ in the condition of Theorem 2.3, then one can verify that in this case $f$ is a homothetic mapping and $s = \bar{s} = 0$. Therefore, we have

**Corollary 2.4.** Let $(M, g)$ be an $n$-dimensional ($n = 3, 4$) complete Riemannian manifold and $f : (M, g) \to (M, \bar{g})$ be a conformal diffeomorphism onto another Riemannian manifold $(\bar{M}, \bar{g})$ such that $\bar{g} = \lambda^2 g$ for some positive function $\lambda \in C^2 M$ and $\lambda \in L^p(M, g)$ for some $p \neq 1$. If $s \geq 0$ and $\bar{s} \leq 0$ for the scalar curvatures $s$ and $\bar{s}$ of $(M, g)$ and $(\bar{M}, \bar{g})$, respectively, then $f$ is a homothetic mapping and $s = \bar{s} = 0$. 
If we assume that \( \lambda = u^{\frac{2}{n-2}} \), then (2.3) immediately gives
\[
(2.4) \quad \frac{4(n-1)}{n-2} \Delta u = s u - s u^{\frac{n+2}{n-2}}.
\]
In the case of the Riemannian manifolds \((M, g)\) and \((\tilde{M}, \tilde{g})\), the equation (2.4) is the classical Yamabe equation (see [5, p. 39]). The equation (2.4) can be written in the form
\[
(2.5) \quad \frac{4(n-1)}{n-2} \Delta u = u (s - \lambda^2 s).
\]
Then for \( s \geq \lambda^2 \bar{s} \), from (2.4) we obtain that \( \Delta u \geq 0 \). On the other hand, Yau has shown in [2, p. 663] that if \( u \) is a non-negative subharmonic function defined on a complete Riemannian manifold \((M, g)\), then \( \int_M u^p dV_g = \infty \) for all \( p > 1 \), unless \( u = \text{constant} \). Therefore, in the case when \((M, g)\) and \((\tilde{M}, \tilde{g})\) are Riemannian manifolds, we have the following Liouville-type non-existence theorem.

\textbf{Theorem 2.5.} Let \((M, g)\) be a \( n \)-dimensional (\( n \geq 3 \)) complete Riemannian manifold and \( f : (M, g) \to (\tilde{M}, \tilde{g}) \) be a conformal diffeomorphism onto another Riemannian manifold \((\tilde{M}, \tilde{g})\) such that \( \tilde{g} = \lambda^2 g \) and \( \lambda^{(n-2)/2} \in L^p(M, g) \) for some positive function \( \lambda \in C^2 M \) and for some \( p \neq 1 \). If \( s \geq \lambda^2 \bar{s} \) for the scalar curvatures \( s \) and \( \bar{s} \) of \((M, g)\) and \((\tilde{M}, \tilde{g})\), respectively, then \( f \) is a homothetic mapping.

In particular, if we assume that \( s \geq 0 \) and \( \bar{s} \leq 0 \) in the condition of Theorem 2.5, then we can prove that \( f \) is a homothetic mapping and \( s = \bar{s} = 0 \). Therefore we have

\textbf{Corollary 2.6.} Let \((M, g)\) be a \( n \)-dimensional (\( n \geq 3 \)) complete Riemannian manifold and \( f : (M, g) \to (\tilde{M}, \tilde{g}) \) be a conformal diffeomorphism onto another Riemannian manifold \((\tilde{M}, \tilde{g})\) such that \( \tilde{g} = \lambda^2 g \) and \( \lambda^{(n-2)/2} \in L^p(M, g) \) for some positive function \( \lambda \in C^2 M \) for some \( p \neq 1 \). If \( s \geq 0 \) and \( \bar{s} \leq 0 \) for the scalar curvatures \( s \) and \( \bar{s} \) of \((M, g)\) and \((\tilde{M}, \tilde{g})\), respectively, then \( f \) is a homothetic mapping and \( s = \bar{s} = 0 \).

\section{An application to the theory of conharmonic transformations}

A mapping \( f : (M, g) \to (\tilde{M}, \tilde{g}) \) is called \textit{conharmonic transformation} (Ishi, [4]) if it is a conformal transformation, i.e., \( \tilde{g} = e^{2\sigma} g \) for some scalar function \( \sigma \in C^2 M \) satisfying the equation
\[
(3.1) \quad \Delta \sigma = -\frac{n-2}{2} ||\text{grad} \ \sigma||^2
\]
for any \( n \geq 3 \). The conharmonic transformations introduced by Ishi are a subgroup of the group of conformal transformations which preserve the harmonicity of certain class of smooth functions (see [5]). From (3.1) we conclude that \( \sigma \) is a superharmonic function. Then the following Corollary is obvious from Theorem 2.1.

\textbf{Corollary 3.1.} Let \( f : (M, g) \to (\tilde{M}, \tilde{g}) \) be a conharmonic transformation of an \( n \)-dimensional (\( n \geq 3 \)) complete Riemannian manifold \((M, g)\), i.e. \( \tilde{g} = e^{2\sigma} g \) for
some function \( \sigma \in C^2 \mathcal{M} \) which satisfies the equation (3.1). If \( \sigma \) has a gradient with integrable norm on \((\mathcal{M}, g)\), then the function \( \sigma \) is constant and \( f \) is a homothetic transformation.

Let \( \sigma = \log \lambda \) for some positive scalar function \( \lambda \in C^2 \mathcal{M} \) then (3.1) can be rewritten in the following equivalent form

\[
(3.2) \quad 2\lambda \Delta \lambda = (n - 4) \| \text{grad} \lambda \|^2.
\]

In this case, we can formulate a proposition that is an analogue of Theorem 2.5.

**Corollary 3.2.** Let \( f : (\mathcal{M}, g) \to (\mathcal{M}, \tilde{g}) \) be a conharmonic transformation of an \( n \)-dimensional \( (n \geq 4) \) complete Riemannian manifold \((\mathcal{M}, g)\), i.e. \( \tilde{g} = \lambda^2 g \) for some positive function \( \lambda \in C^2 \mathcal{M} \) which satisfies the equation (3.2). If \( \lambda \in L^p(\mathcal{M}, g) \) for some \( p \neq 1 \), then \( f \) is a homothetic mapping.

In particular, for \( n = 4 \) from (3.2) we obtain that \( \Delta \lambda = 0 \). Then \( \lambda \) is a positive harmonic function on a complete Riemannian manifold \((\mathcal{M}, g)\). We can easily state the following

**Theorem 3.3.** Let \( f : (\mathcal{M}, g) \to (\mathcal{M}, \tilde{g}) \) be a conharmonic transformation of an \( n \)-dimensional \( (n \geq 4) \) Riemannian manifold \((\mathcal{M}, g)\) such that \( \tilde{g} = \lambda^2 g \), then for the case \( n = 4 \) the function \( \lambda \) is harmonic.

**Remark 3.1.** Corollaries 3.1 and 3.2 generalize Proposition 4.7 from [6] on conharmonic transformations of compact manifolds.

### 4 An application to General Relativity

In this paragraph we give an application of our results to General Relativity using the classical Bochner technique for Lorentzian geometry (see, for example, [7]). Let \((\mathcal{M}, g)\) be a compact space-time, i.e. a four-dimensional compact Lorentzian manifold \((\mathcal{M}, g)\). For \( n = 4 \), the equation (2.3) can be rewritten in the form

\[
(4.1) \quad 6\Delta \lambda = \lambda \left( s - \lambda^2 \bar{s} \right).
\]

In this case, using Green's divergence theorem from (4.1), we obtain the integral formula

\[
(4.2) \quad \int_{\mathcal{M}} \lambda \left( s - \lambda^2 \bar{s} \right) dV_{\tilde{g}} = 0.
\]

It's obvious that the conditions \( s > \lambda^2 \bar{s} \), or \( s < \lambda^2 \bar{s} \) contrast with (4.1). Therefore, we can formulate the following non-existence theorem.

**Theorem 4.1.** Let \((\mathcal{M}, g)\) be a compact space-time. There does not exist any conformal transformation \( f : (\mathcal{M}, g) \to (\mathcal{M}, \tilde{g}) \) such that \( \tilde{g} = \lambda^2 g \) and \( s > \lambda^2 \bar{s} \) (or \( s < \lambda^2 \bar{s} \)) for some positive function \( \lambda \in C^2 \mathcal{M} \) and the scalar curvatures \( s \) and \( \bar{s} \) of \((\mathcal{M}, g)\) and \((\mathcal{M}, \tilde{g})\), respectively.

Moreover, we have the following
Corollary 4.2. Let \((M, g)\) be a compact space-time. There does not exist any conformal transformation \(f : (M, g) \to (M, \bar{g})\) such that \(\bar{g} = \lambda^2 g\), \(s > 0\) and \(\bar{s} < 0\) (or \(s > 0\) and \(\bar{s} < 0\)) for some positive function \(\lambda \in C^2 M\) and the scalar curvatures \(s\) and \(\bar{s}\) of \((M, g)\) and \((M, \bar{g})\), respectively.

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