On the Brill-Noether theory of curves in a weighted projective plane

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Abstract. We study the gonality and the existence of low degree pencils on curves with a model on a weighted projective plane, when their singularities are only ordinary nodes or ordinary cusps and they are general in the weighted projective plane.

Key words: weighted projective plane; gonality; ordinary singularity.

1 Introduction

In this paper we consider the first steps of the Brill-Noether theory of curves on a weighted projective plane ([7], [8], [1]) (a very classical topic, but as far as we know the results of this note are new). See [2], [3], [4], [5], [6], [9] for smooth and singular plane curves.

Fix positive integers $a, b, c$ and let $\mathbb{P} := \mathbb{P}(a, b, c)$ denote the weighted projective space with weights $a, b, c$. Up to isomorphisms of the ambient weighted projective plane we may assume that any 2 of the integer $a, b, c$ are coprimes ([1, Proposition 3C.5], [7, Proposition 1.3]). We may assume $a \leq b \leq c$. Since $(a, b) = (b, c) = (a, b) = 1$, we are in one of the following cases:

1. $a = b = c = 1$;
2. $a = b = 1$, $c > 1$;
3. $a < b < c$, $(a, b) = 1$, $(a, c) = 1$, $(b, c) = 1$.

In the first case we have $\mathbb{P} \cong \mathbb{P}^2$. In the second case $\mathbb{P}$ is embedded as a cone over a rational normal curve of $\mathbb{P}^c$ and the blowing up of the vertex of the cone gives the Hirzebruch surface $F_c$ ([1, page 124], [8, 1.2.3]). In this case it seems easier to work directly on $F_c$ (the case $b = 1$ of Theorem 1.2 is true by [10]). Hence from now on we assume $a < b < c$ and $(a, b) = (a, c) = (b, c) = 1$.

We fix variables $x_1, x_2, x_3$ and give weight $a$ to $x_1$, $b$ to $x_2$ and $c$ to $x_3$. For all integers $t \geq 0$ let $K[x_1, x_2, x_3]_{a, b, c, t}$ be the linear subspace of $K[x_1, x_2, x_3]$ generated...
by the monomials \( x_1^{a_1}x_2^{a_2}x_3^{a_3} \) with \( a_i \geq 0 \) for all \( i \) and \( aa_1 + ba_2 + ca_3 = t \), i.e. the monomials with weight \( t \). We recall that \( \mathbb{P} \) has only quotient singularities (if \( a = 1 < b \), \( \text{Sing}(\mathbb{P}) = \{(0 : 1 : 0), (0 : 0 : 1)\} \), if \( a > 1 \), then \( \text{Sing}(\mathbb{P}) = \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\} \), that the set of all rational equivalence classes of Weil divisors is a free abelian group of rank 1 ([1, Corollary 5.8]), that \( \mathcal{O}_\mathbb{P}(t) \), \( t \in \mathbb{Z} \), is the set of all rank one reflexive sheaves on \( \mathbb{P} \), that \( h^1(\mathcal{O}_\mathbb{P}(t)) = 0 \) for all \( t \in \mathbb{Z} \), \( h^0(\mathcal{O}_\mathbb{P}(t)) = K[x_1, x_2, x_3]_{a,b,c,t} \) for all \( t \geq 0 \), that \( \mathcal{O}_\mathbb{P}(t) \) is locally free if and only if \( t \equiv 0 \mod abc \). The line bundle \( \mathcal{O}_\mathbb{P}(abc) \) is very ample ([1, Remark 3]). Hence for all \( t > 0 \) a general element of \( |\mathcal{O}_\mathbb{P}(abc)| \) is a smooth and connected curve. Fix a positive integer \( d \) and take \( C \in |\mathcal{O}_\mathbb{P}(dbc)| \) such that \( C \) is smooth. Since \( C \) is a Cartier divisor of \( \mathbb{P} \) and \( C \) is smooth, we have \( C \cap \text{Sing}(\mathbb{P}) = \emptyset \). Hence each \( \mathcal{O}_C(t) \), \( t \in \mathbb{Z} \), is a line bundle. We have \( \mathcal{O}_\mathbb{P}(1) \cdot \mathcal{O}_\mathbb{P}(1) = \frac{1}{abcd} \) in the rational Chow ring of \( \mathbb{P} \) (use [11, Corollary A.2] or that the covering map \( \mathbb{P}^2 \to \mathbb{P} \) is the quotient by the group \( \mu_a \times \mu_b \times \mu_c \) and hence it has degree \( abc \)). Since \( \omega_{\mathbb{P}} \cong \mathcal{O}_\mathbb{P}(-a - b - c) \) ([1, Corollary 6B.8], [7, Theorem 5.2], [8, 3.3.4 and 3.5.2]), the adjunction formula gives \( \omega_C \cong \mathcal{O}_C(dabc - a - b - c) \) ([1, Corollary 6B.9], [8, 3.5.2]) Hence \( C \) has genus \( 1 + d(dabc - a - b - c)/2 \). Since \( h^1(\mathcal{O}_\mathbb{P}(t)) = 0 \) for all \( t \), for each integer \( w \geq 0 \) the restriction map \( \rho_w : H^0(\mathcal{O}_\mathbb{P}(w)) \to H^0(\mathcal{O}_C(w)) \) is surjective. Hence \( h^0(\mathcal{O}_C(t)) = \dim(K[x_1, x_2, x_3]_{a,b,c,t}) \) for all \( t < dabc \). In particular we have \( h^0(\mathcal{O}_C(ab)) = 2 \). Hence \( C \) has gonality at most \( \text{deg}(\mathcal{O}_C(ab)) = dab \) (use again that \( \mathcal{O}_\mathbb{P}(1) = \frac{1}{abcd} \)). The line bundle \( \mathcal{O}_C(ab) \) is spanned, because \( (0 : 0 : 1) \) is the only base point of \( |\mathcal{O}_\mathbb{P}(1)| \) and \( (0 : 0 : 1) \notin C \).

Our first result is non-trivial only if \( c \gg ab \).

**Theorem 1.1.** Let \( C \in |\mathcal{O}_\mathbb{P}(dbc)| \) be a smooth curve. Assume \( dab - a - b - c > 0 \) and \( (a,b,d) \neq (1,2,1) \). Let \( w : C \to \mathbb{P}^1 \) be the morphism induced by \( |\mathcal{O}_C(ab)| \). Let \( z \) be any positive integer such that \( (z - 2)ab < dab - a - b - c \). Then there is no degree \( z \) morphism \( u : C \to \mathbb{P}^1 \) such that \( u \) is not partially composed with \( w \), i.e. such that the morphism \( (w, u) : C \to \mathbb{P}^1 \times \mathbb{P}^1 \) is birational onto its image.

The condition “ \( dab - a - b - c > 0 \)” is equivalent to assuming that \( C \) has genus \( \geq 2 \). The result is sharp, in the sense that it fails (just by 1) in the omitted case \( (a,b,d) = (1,2,1) \) (see Remark 2.1).

In the case \( a = 1 \), we prove the following result.

**Theorem 1.2.** Assume \( a = 1 < b \). Let \( C \in |\mathcal{O}_\mathbb{P}(dac)| \) be a smooth curve. Then \( C \) has gonality \( db \) and \( \mathcal{O}_C(b) \) is the unique line bundle \( L \) on \( C \) such that \( h^0(L) \geq 2 \) and \( \text{deg}(L) \leq db \).

In section 3 we consider the case of singular curves. We consider both the spanned line bundles of minimal degree on the singular curve and the case of the normalization of an integral curve.

## 2 Proof of Theorems 1.1 and 1.2

**Remark 2.1.** Let \( C \in |\mathcal{O}_\mathbb{P}(dbc)| \) be a smooth curve of genus \( g \geq 2 \). Assume \( (a,b,c) = (1,2,1) \) (the case excluded in the statement of Theorem 1.1). Since \( b = 2 \) and \( (b,c) = 1, c \) is odd. We have \( g = 1 + (c - 3)/2 \). The spanned line bundle \( \mathcal{O}_C(2) \) has degree 2 and hence \( C \) is hyperelliptic. There is a degree \( z \) spanned line bundle
whose associated morphism is not composed with the hyperelliptic involution if and only if \( z \geq g + 1 = 2 + (c - 3)/2 \).

**Proof of Theorem 1.1:** Assume the existence of such a morphism and take \( z \) minimal for which it exists. Set \( R := u^*(O_{P^3}(1)) \). \( R \) is a spanned line bundle of degree \( z \) and in particular \( h^0(R) \geq 2 \). Let \( g = 1 + d(dabc - a - b - c)/2 \) be the genus of \( C \).

First assume \( z > g \), i.e. \( z - 2 \geq g - 1 \). We get \( d(dabc - a - b - c)/2 \leq dabc - a - b - c \). Since \( dabc - a - b - c > 0 \), we get \( d = 1 \) and \( ab = 2 \), i.e. \( d = 1, a = 1, b = 2 \). We excluded this case in the proof of Theorem 1.1.

Now assume \( z \leq g \) and hence \( h^1(R) > 0 \). Fix a general fiber of \( u \). Since \( h^1(R) > 0 \) and \( \omega_C \cong O_C(dabc - a - b - c) \), we have \( h^1(I_Z(dabc - a - b - c)) > 0 \). Assume for the moment that \( Z \) is reduced (this is always the case in characteristic zero). Fix an ordering \( P_1, \ldots, P_2 \) of the points of the support of \( Z \). Since \( R \) is spanned and \( h^1(R) > 0 \), we have \( h^1(O_C(Z')) = h^1(O_C(Z)) \) for each \( Z' \subset Z \) with \( \text{deg}(Z') = z - 1 \).

To get \( Z' \) we have \( \dim(I_Z(dabc - a - b - c)) = 1 + dabc - a - b - c \) by \( \text{dim}(\text{Sing}(P)) \). Fix a general \( P \) and \( dabc - a - b - c > 0 \), we get \( d = 1 \) and \( ab = 2 \), i.e. \( d = 1, a = 1, b = 2 \). We excluded this case in the proof of Theorem 1.1.

Now assume \( Z \) is not reduced, i.e. that \( u \) is not separable. We get that the base field has characteristic \( p > 0 \). Since the base field is algebraically closed, we also get that it is composed with a Frobenius of \( \mathbb{P}^3 \), contradicting the minimality of \( z \). \( \square \)

**Proof of Theorem 1.2:** We have \( \text{Sing}(P) = \{(0 : 1 : 0), (0 : 0 : 1)\} \). Since \( C \in |O_{\mathbb{P}^3}(dabc)| \), it is a Cartier divisor of \( \mathbb{P}^3 \). Since \( C \) is smooth, then \((0 : 0 : 1) \notin C \).

Hence \( O_C(b) \) is a spanned line bundle of degree \( db \). Since \( h^1(O_{\mathbb{P}^3}(b - dabc)) = 0 \), we have \( h^0(O_C(b)) = 2 \). Take a line bundle \( L \) with minimal degree \( z \leq db \) with \( h^0(L) \geq 2 \) and assume \( L \notin O_C(b) \). Fix a general \( Z \in |L| \). As in last part of the proof of Theorem 1.1 we reduce to the case in which \( Z \) is reduced. Since \( L \) is spanned, we may assume \( Z \cap \{ z_0 = 0 \} = \emptyset \). We fix an ordering \( P_1, \ldots, P_2 \) of the points of \( Z \) and set \( Z' := \{P_1, \ldots, P_{n-1}\} \).

As in the proof of Theorem 1.1 to get a contradiction it is sufficient to prove that \( h^1(I_{Z'}(dabc - a - b - c)) = 0 \).

Since \( z \leq db \), we have \((z - 2)c \leq (db - 2)c \leq dabc - 1 - b - c \) and so it is sufficient to find \( D_i \in |O_{\mathbb{P}^3}(c)| \), \( 1 \leq i \leq z - 2 \), such that \( P_i \notin D_i \) and \( P_{i+1} \notin D_i \). Fix \( i \in \{1, \ldots, z - 2\} \). If there is \( T \in |O_{\mathbb{P}^3}(b)| \) with \( P_i \in T \) and \( P_{i+1} \notin T \), say \( T \) with equation \( u(z_0, z_1) \in K[z_0, z_1, z_2] \), then we take as \( D_i \), the divisor with \( z_0^{e} - u(z_0, z_1) \) as its equations. Now assume that \( D_{i+1} \) is contained in every element of \( |I_{P_i}(b)| \) and fix \( T \in |I_{P_i}(b)| \). Since \( P_i \notin \{(0 : 1 : 0), (0 : 0 : 1)\}, T \) is the only element of \( |O_{\mathbb{P}^3}(b)| \) containing \( P_i \). Let \( M \) be a general element of \( |I_{P_i}(c)| \). Set \( e := [c/b] \).

We have \( \dim(K[x_0, x_1, x_2]_{1,b,c,c+e-b}) = e \) and \( \dim(K[x_0, x_1, x_2]_{1,b,c,c+e}) = e + 2 \) and so \( h^0(O_{\mathbb{P}^3}(c - b)) \leq h^0(O_{\mathbb{P}^3}(c - 2)) \). Hence \( T \) is not a component of \( M \). We have \( P_i \notin T \cap M \).

Since \( O_{\mathbb{P}^3}(b) \cdot O_{\mathbb{P}^3}(c) = 1 \), \( P_i \) is a smooth point of \( P \) and \( P_i \in T \cap M \), \( P_i \) is the only element of \( P \setminus \text{Sing}(P) \) contained in \( M \cap T \). Hence \( P_{i+1} \notin M \). Take \( D_i := M \). \( \square \)

To check the key assumption of Theorem 1.1 the following well-known result may be useful.
Lemma 2.1. Take a smooth and connected curve $C \subset \mathbb{P}$ such that $(0 : 0 : 1) \notin C$ and assume the existence of $D \in |\mathcal{O}_P(ab)|$, $D \neq C$, such that the scheme $C \cap D$ has 1 connected component with multiplicity 2 and deg($w$) = 2 connected components with multiplicity 1. Let $w: C \rightarrow \mathbb{P}^1$ be the morphism induced by $|\mathcal{O}_C(ab)|$. Then $w$ is not composed with an involution, i.e. there are no triple $(X, w_1, w_2)$ with $X$ a connected smooth curve, $w_1: C \rightarrow X$, $w_2: X \rightarrow \mathbb{P}^1$, $w = w_2 \circ w_1$, deg($w_1$) $\geq 2$ and deg($w_2$) $\geq 2$.

Proof. If $ab = 2$ (i.e. if $(a, b) = (1, 2)$), then $w$ is not composed. In the general case we use that the monodromy group of $w$ is the full symmetric group (see [12, Proposition 2.1] for a characteristic free proof, but remember that the monodromy group is 1-transitive just because $C$ is an integral curve).

3 Singular curves

We only look at integral curves $T$, which are contained in the smooth locus of $\mathbb{P}$ and hence that are Cartier divisors of $\mathbb{P}$. Let $T$ be any such curve. There are many different Brill-Noether theories for integral singular curves. If we only look at spanned line bundles, then the proofs of Theorems 1.1 and 1.2 only require minimal modifications.

Theorem 3.1. Let $C \in |\mathcal{O}_P(dabc)|$ be an integral curve. Assume $dabc - a - b - c > 0$, i.e. assume that $C$ has arithmetic genus $\geq 2$, and $(a, b, d) \neq (1, 2, 1)$. Let $w: C \rightarrow \mathbb{P}^1$ be the morphism induced by $|\mathcal{O}_C(ab)|$. Fix a positive integer $z$ such that $(z - 2)ab \leq dabc - a - b - c$ and there is a degree $z$ spanned line bundle $R$ on $C$. Let $u: C \rightarrow \mathbb{P}^y$, $y := h^0(R) - 1$, be the morphism induced by $H^0(R)$. In positive characteristic assume that either $u$ is separable or that the algebraic group $\text{Pic}^0(C)$ has no unipotent part. Then the morphism $(w, u): C \rightarrow \mathbb{P}^y \times \mathbb{P}^1$ is not birational onto its image.

Theorem 3.2. Assume $a = 1 < b$. Let $C \in |\mathcal{O}_P(dac)|$ be a integral curve such that $C \cap \text{Sing}(\mathbb{P}) = \emptyset$. In positive characteristic assume that either $u$ is separable or that the algebraic group $\text{Pic}^0(C)$ has no unipotent part. Then $\mathcal{O}_C(b)$ is the unique line bundle $R$ on $C$ such that $h^0(R) \geq 2$, $R$ is spanned and $\text{deg}(R) \leq db$.

Proofs of Theorems 3.1 and 3.2: Take any spanned line bundle $R$ on $C$ with $h^0(R) \geq 2$ and call $Z$ the zero-locus of a general section of $R$. Set $z := \text{deg}(Z)$. Since $R$ is spanned, we have $Z \cap \text{Sing}(C) = \emptyset$. In characteristic zero $Z$ is reduced and we may continue the proofs of Theorems 1.1 and 1.2. Now assume $p := \text{char}(K) > 0$ and that $Z$ is not reduced. Set $B := Z_{\text{red}}$. Let $u: C \rightarrow \mathbb{P}^y$, $y := h^0(R) - 1$, be the morphism induced by $H^0(R)$. Since $Z$ is general, it is not reduced if and only if $u$ is not separable and, if $p^e > 0$, is the inseparable degree of $u$, then each connected component of $Z$ has degree $p^e$ and $Z = p^eB$ (this equality is non-ambiguous, because $B \subset C_{\text{reg}}$). Varying $Z$ in $|L|$ we get infinitely many effective divisors $B$ which, multiplied by $p^e$, are linearly equivalent. By assumption the $p^e$-torsion of $\text{Pic}^0(C)$ is finite. Hence $C$ has a line bundle $A$ of degree $z/p^e$ with $h^0(A) \geq 2$, a contradiction. □

Let $Y \subset \mathbb{P}$ be an integral curve with $Y \cap \text{Sing}(\mathbb{P}) = \emptyset$ and only ordinary nodes and ordinary cusps as its singularities. Set $S := \text{Sing}(Y)$ and $s := \sharp(S)$. Since $Y \cap \text{Sing}(\mathbb{P}) = \emptyset$, $Y$ is a Cartier divisor of $\mathbb{P}$ and hence there is an integer $d > 0$ such that $Y \in |\mathcal{O}_P(dabc)|$. The adjunction formula, gives $\omega_Y \cong \mathcal{O}_Y(dabc - a - b - c)$. Since
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Proofs of Theorems 3.3 and 3.4: 

Theorem 3.3. Assume $(z - 2)ab \leq dab - a - b - c$, $s + z \leq 2 + d(dabc - a - b - c)/2$ and that $S \subset \mathbb{P}$ is a general subset with cardinality $s$. Then there is no degree $z$ morphism $u: C \to \mathbb{P}^1$ such that the morphism $(w, u): C \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image.

Theorem 3.4. Assume $a = 1 < b$, $s + db \leq 2 + d(dabc - a - b - c)/2$ and that $S$ is general in $\mathbb{P}$. Then $\mathcal{O}_C(b)$ is the only line bundle $L$ on $C$ with $\deg(L) \leq db$ and $h^0(L) \geq 2$.

Acknowledgements. The author was partially supported by MIUR and GN-SAGA of INdAM (Italy).

References


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