Integral formulae for codimension-one foliated Finsler manifolds

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Abstract. We study extrinsic geometry of a codimension-one foliation $\mathcal{F}$ of a Finsler space $(M,F)$, in particular, of a Randers space $(M,\alpha + \beta)$. Using a unit vector field $\nu$ orthogonal (in the Finsler sense) to the leaves of $\mathcal{F}$, we define a new Riemannian metric $g$ on $M$, which for Randers case depends nicely on $(\alpha, \beta)$. For that $g$ we derive several geometric invariants of $\mathcal{F}$ (e.g. the Riemann curvature and the shape operator) in terms of $F$; then under natural assumptions on $\beta$ which simplify derivations, we express them in terms of invariants arising from $\alpha$ and $\beta$. Using our approach of [13], we produce the integral formulae for $\mathcal{F}$ of closed $(M,F)$ and $(M,\alpha + \beta)$, which relate integrals of mean curvatures with those involving algebraic invariants obtained from the shape operator of $\mathcal{F}$ and the Riemann curvature in the direction $\nu$. They generalize formulae by Brito-Langevin-Rosenberg (that total mean curvatures of any order for a foliated closed Riemannian space of constant curvature don’t depend on a choice of $\mathcal{F}$).


Key words: Finsler space; Randers norm; foliation; Riemann curvature; integral formula; shape operator; Cartan torsion; variation formula.

1 Introduction

Two recent decades brought increasing interest in Finsler geometry (see [2, 4, 15] and the bibliographies therein), in particular, in extrinsic geometry of hypersurfaces of Finsler manifolds (see the items above and, for example, [14]). Among all the Finsler structures, Randers metrics (introduced in [9] and being the closest relatives of Riemannian ones) play an important role.

Extrinsic geometry of foliated Riemannian manifolds is also of definite interest since some time (see [11, 12] and, again, the bibliographies therein). Among other topics of interest, one can find a number of papers devoted to so called integral formulae (see surveys in [12, 1]), which provide obstructions for existence of foliations...
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(or compact leaves of them) with given geometric properties. A series of integral formulae has been provided in [13]. They include the formulae in [10] that the total mean curvature of the leaves is zero, and generalize the formulae in [3], which show that total mean curvatures (of arbitrary order \(k\)) for codimension-one foliations on a closed \((m+1)\)-dimensional manifold of constant sectional curvature \(K\) depend only on \(K\), \(k\), \(m\) and the volume of the manifold, not on a foliation. One of such formulae was used in [7] to prove that codimension-one foliations of a closed Riemannian manifold of negative Ricci curvature are far (in a sense defined there) from being umbilical.

In this paper we study extrinsic geometry of a codimension-one transversely oriented foliation \(F\) of a closed Finsler space \((M, F)\), in particular, of a Randers space \((M, a + \alpha)\), \(a\) being the norm of a Riemannian structure \(a\) and \(\alpha\) a 1-form of \(\alpha\)-norm smaller than 1 everywhere on \(M\). Using a unit normal (in the Finsler sense) to the leaves of \(F\) we define a new Riemannian structure \(g\) on \(M\), which in Randers case depends nicely on \(\alpha\) and \(\beta\). For that \(g\), we derive several geometric invariants of \(F\) (e.g. the Riemann curvature and the shape operator) in terms of \(F\); under natural assumptions on \(\beta\) which simplify derivations, we express them in terms of corresponding invariants arising from \(\alpha\) and some quantities related to \(\beta\). Then, using the approach of [13], we produce the integral formulae for \(F\) on \((M, F)\) and \((M, a + \beta)\); some of them generalize the formulae in [3].

Our formulae relate integrals of \(\sigma\)'s with those involving algebraic invariants (see Appendix) obtained from \(A_p\) \((p \in M)\) – the shape operator of a foliation \(F\), \(R_p\) – the Riemann curvature in the direction \(\nu\) normal to \(F\), and their products of the form \((R_p)\nu A_p, j = 1, 2, \ldots\) In fact, we get a bit more: we produce an infinite sequence of such formulae for a smooth unit vector field \(\nu\) on \(M\) involving these algebraic invariants. To simplify calculations, we work on locally symmetric \((\nabla R = 0\) with respect to \(g)\) Finsler manifolds, where our approach can be applied with the full force (Section 3). We show that our formulae reduce to these in [3] in the case of constant curvature and to those in [13] in the Riemannian case. Using Finsler geometry of Randers spaces we produce also (Section 4) integral formulae on codimension-one foliated Riemannian manifolds which involve not only \(A_p\)'s and \(R_p\)'s but also an auxiliary 1-form \(\beta\).

We discuss a number of particular cases and provide consequences of our new formulae.

\section{Preliminaries}

Recall Euler’s Theorem: If a function \(f\) on \(\mathbb{R}^{m+1}\) is smooth away from the origin of \(\mathbb{R}^{m+1}\) then the following two statements are equivalent:

- \(f\) is positively homogeneous of degree \(r\), that is \(f(\lambda y) = \lambda^r f(y)\) for all \(\lambda > 0\);
- the radial derivative of \(f\) is \(r\) times \(f\), namely, \(f_{\nu}(y) y_{\nu} = r f(y)\).

The obvious consequence of Euler’s Theorem helps us to represent several formulae in what follows:

\textbf{Corollary 2.1.} If a smooth function \(f\) on \(\mathbb{R}^{m+1} \setminus \{0\}\) obeys the 2-homogeneity condition \(f(\lambda y) = \lambda^2 f(y)\) for \(\lambda > 0\) then \(f(y) = \frac{1}{2} f_{\nu}(y) y_{\nu}y_{\nu}\) for smooth functions \(f_{\nu}(y)\) on \(\mathbb{R}^{m+1} \setminus \{0\}\).
Proof. By Euler’s Theorem, \( f_{y'}(y) y' = 2f(y) \). Since \( f_{y'}(\lambda y) = \lambda f_{y'}(y) \), by Euler’s Theorem, we have \( f_{y'}(y) = f_{y'y'}(y)y' \).

\[ \square \]

2.1 The Minkowski and Randers norms

**Definition 2.1** (see [15]). A Minkowski norm on a vector space \( \mathbb{R}^{m+1} \) is a function \( F : \mathbb{R}^{m+1} \to [0, \infty) \) with the following properties (of regularity, positive 1-homogeneity and strong convexity):

\( M_1 : F \in C^\infty(\mathbb{R}^{m+1} \setminus \{0\}) \), \( M_2 : F(\lambda y) = \lambda F(y) \) for all \( \lambda > 0 \) and \( y \in \mathbb{R}^{m+1} \),

\( M_3 : \) For any \( y \in \mathbb{R}^{m+1} \setminus \{0\} \), the following symmetric bilinear form is positive definite on \( \mathbb{R}^{m+1} \):

\[
g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]_{s=t=0}.
\]

By \( M_2 \), \( g_{\lambda y} = g_y \) for all \( \lambda > 0 \). By \( M_3 \), \( \{y \in \mathbb{R}^{m+1} : F(y) \leq 1\} \) is a strictly convex set. Note that

\[
g_y(y, y) = F^2(y).
\]

One can check that \( F(u + v) \leq F(u) + F(v) \) (the triangle inequality) and \( F_y'(y) u' \leq F(u) \) (the fundamental inequality) for all \( y \in \mathbb{R}^{m+1} \setminus \{0\} \) and \( u, v \in \mathbb{R}^{m+1} \). By Corollary 2.1, we have \( F^2(y) = g_{ij}(y) y^i y^j \), where \( g_{ij} = \frac{1}{2} [F^2]_{y^i y^j} = F_{y^i} F_{y^j} \) are smooth functions in \( \mathbb{R}^{m+1} \setminus \{0\} \) which, in general, cannot be extended continuously to all of \( \mathbb{R}^{m+1} \). The following symmetric trilinear form \( C \) for Minkowski norms is called the Cartan torsion:

\[
C_y(u, v, w) = \frac{1}{2} \frac{\partial}{\partial t} [g_u + tw(u, v)]_{t=0} \quad \text{where} \quad y \in \mathbb{R}^{m+1} \setminus \{0\}, u, v, w \in \mathbb{R}^{m+1}.
\]

The homogeneity of \( F \) implies the following:

\[
C_y(u, v, w) = \frac{1}{4} \frac{\partial^3}{\partial s \partial t \partial t} [F^2(y + ru + sv + tw)]_{s=t=0} \quad C_{\lambda y} = \lambda^{-1}C_y \quad (\lambda > 0).
\]

We have \( C_y(y, \cdot, \cdot) = 0 \). The mean Cartan torsion is given by \( I_y(u) := \text{Tr} C_y(\cdot, \cdot, u) \).

Observe that

\[
C_{ijk} := C(\partial_i', \partial_j', \partial_k') = \frac{1}{2} \frac{\partial}{\partial y^k} g_{ij} = \frac{1}{4} [F^2]_{y^i y^j y^k}, \quad I_k = g^{ij} C_{ijk}.
\]

Let \((b_i)\) be a basis for \( \mathbb{R}^{m+1} \) and \((\theta^i)\) the dual basis in \( (\mathbb{R}^{m+1})^* \). The Busemann-Hausdorff volume form is defined by \( dV_F = \sigma_F(x) \theta^0 \wedge \cdots \wedge \theta^{m+1} \), where \( \sigma_F = \frac{\text{vol } B^{m+1}}{\text{vol } B^{m+1}} \).

Here \( \mathbb{R}^{m+1} := \{ y \in \mathbb{R}^{m+1} : \|y\| < 1 \} \) is a Euclidean unit ball, and \( \text{vol } B^{m+1} \) is the Euclidean volume of a strongly convex subset \( B^{m+1} := \{ y \in \mathbb{R}^{m+1} : F(y^ib_i) < 1 \} \) (so that for the unit cubic \( U = [0, 1]^{m+1}, \text{vol } U = 1 \)).

The distortion of \( F \) is defined by \( \tau(y) = \log(\sqrt{\det g_{ij}(y)}/\sigma_F) \). It has the 0-homogeneity property: \( \tau(\lambda y) = \tau(y) \) \((\lambda > 0)\), and \( \tau = 0 \) for Riemannian spaces.

The angular form is defined by \( h_y(u, v) = g_y(u, v) - F(y)^{-2} g_y(y, u) g_y(y, v) \). Observe that \( h_y(u, u) \geq g_y(u, u) - F(y)^{-2} g_y(y, y) g_y(u, u) = 0 \) and equality holds if and only if \( u \parallel y \).
A vector \( n \in \mathbb{R}^{m+1} \) is called a normal to a hyperplane \( W \subset \mathbb{R}^{m+1} \) if \( g_n(n, w) = 0 \) \((w \in W)\). There are exactly two normal directions to \( W \), see [15], which are opposite when \( F \) is reversible (i.e., \( F(-y) = F(y) \) for all \( y \in \mathbb{R}^{m+1} \)).

**Definition 2.2.** Let \( \langle \cdot, \cdot \rangle \) be a scalar product and \( \alpha(y) = \|y\|_\alpha = \sqrt{\langle y, y \rangle} \) for \( y \in \mathbb{R}^{m+1} \) the corresponding Euclidean norm on \( \mathbb{R}^{m+1} \). If \( \beta \) is a linear form on \( \mathbb{R}^{m+1} \) with \( \|\beta\|_\alpha < 1 \) then the following function \( F \) is called the Randers norm:

\[
F(y) = \alpha(y) + \beta(y) = \sqrt{\langle y, y \rangle} + \beta(y).
\]

For Randers norm (2.4) on \( \mathbb{R}^{m+1} \), the bilinear form \( g_y \) obeys, see [15],

\[
g_y(u, v) = \alpha^{-2}(y)(1 + \beta(y)) \langle u, v \rangle + \beta(v) \beta(u)
- \alpha^{-2}(y) \beta(y) \langle y, u \rangle \langle y, v \rangle + \alpha^{-1}(y) \beta(u) \langle y, v \rangle + \beta(v) \langle y, u \rangle,
\]

\[
\det g_y = (F(y)/\alpha(y))^{m+2} \det a.
\]

Let \( N \in \mathbb{R}^{m+1} \) be a unit normal to a hyperplane \( W \) in \( \mathbb{R}^{m+1} \) with respect to \( \langle \cdot, \cdot \rangle \), i.e.,

\[
\langle N, w \rangle = 0 \quad (w \in W), \quad \alpha(N) = \|N\|_\alpha = \sqrt{\langle N, N \rangle} = 1.
\]

Let \( n \) be a vector \( F \)-normal to \( W \), lying in the same half-space with \( N \) and such that \( \|n\|_\alpha = 1 \). Set

\[
g(u, v) := g_n(u, v), \quad u, v \in \mathbb{R}^{m+1}.
\]

Then \( g(n, n) = F^2(n) \), see (2.2), and \( F(n) = 1 + \beta(n) \).

The 'musical isomorphisms' \( ^* \) and \( ^\flat \) will be used for rank one tensors and symmetric rank \( 2 \) tensors on \( (\mathbb{R}^{m+1}, a) \) and Riemannian manifolds. For example, if \( \beta \) is a 1-form on \( \mathbb{R}^{m+1} \) and \( v \in \mathbb{R}^{m+1} \) then \( \langle \beta^\flat, u \rangle = \beta(u) \) and \( v^\flat(u) = \langle v, u \rangle \) for any \( u \in \mathbb{R}^{m+1} \).

**Lemma 2.2.** If the Randers norm obeys \( \beta(N) = 0 \) (i.e., \( \beta^\flat \in W \)) then

\[
n = cN - \beta^\flat,
\]

\[
g(u, v) = c^2(\langle u, v \rangle - \beta(u) \beta(v)), \quad u, v \in W,
\]

\[
g(n, n) = c^4, \quad g(n, v) = 0,
\]

where \( c := (1 - \|\beta\|_\alpha^2)^{1/2} > 0 \). The vector \( \nu = c^{-2}n \) is an \( F \)-unit normal to \( W \).

**Proof.** For arbitrary \( \beta \) and \( y = n \) and \( \alpha(n) = 1 \), the formula (2.5) reads

\[
g(u, v) = (1 + \beta(n))\langle u, v \rangle + \beta(u) \beta(v) - \beta(n) \langle n, u \rangle \langle n, v \rangle + \beta(u) \langle n, v \rangle + \beta(v) \langle n, u \rangle.
\]

Assuming \( u = n \), from (2.10) we find

\[
g(n, v) = (1 + \beta(n)) \langle n + \beta^\flat, v \rangle.
\]

Note that \( |\beta(n)| = |\langle \beta^\flat, n \rangle| \leq \alpha(\beta^\flat) \alpha(n) < 1 \); hence, \( 1 + \beta(n) > 0 \). We find from (2.11) with \( v \in W \) that \( n + \beta^\flat = \hat{c}N \) for some \( \hat{c} > 0 \). Using \( 1 = \langle n, n \rangle = \hat{c}^2 - 2\hat{c} \beta(N) + \|\beta\|_\alpha^2 \), we get two values

\[
\hat{c} = \beta(N) \pm (\beta(N)^2 + c^2)^{1/2}.
\]
By condition $\beta(N) = 0$ we have $\beta^2 \in W$, this yields $\dot{c} = c$ and (2.7). Thus,

$$\beta(n) = \beta(cN - \beta^1) = -\|\beta\|_n^2, \quad 1 + \beta(n) = c^2.$$  

Finally, (2.8) follows from (2.10). \hfill \square

**Lemma 2.3.** Let the Randers norm obeys $\beta(N) = 0$ (i.e., $\beta^2 \in W$). If $u, U \in W$ and

$$g(u, v) = \langle U, v \rangle \quad \text{for all} \quad v \in W$$  

then $\beta(u) = c^{-4}\beta(U)$ and

$$c^2 u = U + c^{-2}\beta(U) \beta^1.$$  

**Proof.** By (2.8), we have

$$g(u, v) = c^2\langle u - \beta(u)\beta^1, v \rangle.$$  

Then from (2.12), since $u, U$ and $\beta^1$ belong to $W$, we obtain

$$u - \beta(u)\beta^1 = c^{-2}U.$$  

Applying $\beta$ we get $\beta(u) - \beta(u)\|\beta\|_n^2 = c^{-2}\beta(U)$, $\beta(u) = c^{-4}\beta(U)$ and then (2.13). \hfill \square

### 2.2 Finsler spaces

Let $M^{m+1}$ be a connected smooth manifold and $TM$ its tangent bundle. The natural projection $\pi : TM_0 \to M$, where $TM_0 := TM \setminus \{0\}$ is called the slit tangent bundle. A Finsler structure on $M$ is a Minkowski norm $F$ in tangent spaces $T_pM$, which smoothly depends on a point $p \in M$. Note that $\pi_*$ maps the double tangent bundle $T^2M$ into $TM$ itself.

A spray on a manifold $M$ is a smooth vector field $G$ on $TM_0$ such that

$$\pi_*(G_v) = v, \quad G_{\lambda v} = \lambda (h_\lambda)_*(G_v) \quad (v \in TM_0, \, \lambda > 0),$$  

where $h_\lambda : v \mapsto \lambda v$ is the homothety of $TM$. The meaning of (2.14)$_1$ is that $G$ is a second-order vector field over $M$, and (2.14)$_2$ is the homogeneous quadratic condition. In local coordinates $(x^j)$, $G$ is expressed as $G(y) = y^i \partial_{x^i} - 2G^i \partial_{y^i}$, where $G^i(\lambda y) = \lambda^2 G^i(y) (\lambda > 0)$.

Using $G$ we define the following notions: covariant derivative, parallel translation (and parallel vectors) along a curve, geodesics and curvature. A curve $\gamma(t)$ in $TM_0$ satisfying $\dot{\gamma} = G_\gamma$ is an integral curve of $G$; it is equal to the canonical lift of $c := \pi \circ \gamma$. The covariant derivative of a vector field $u(t)$ along a curve $c(t)$ in $M$ is given by $D_c u = \{\dot{u}^i + \Gamma^i_{kj}(c) \dot{c}^k w^j\} \partial_{x^i|c}$. Here $G^i = \frac{1}{2} \Gamma^i_{kj} y^k y^j$ for smooth functions $\Gamma^i_{kj} = (G^i)_{y^k y^j}$ on $TM_0$, see Corollary 2.1. The following properties are obvious:

$$D_c (u + v) = D_c u + D_c v, \quad D_c (fu) = \dot{c}(f) u + f D_c u, \quad D_{\lambda c} u = \lambda D_c u$$  

for any $f \in C^\infty(M)$ and $\lambda > 0$, see [15]. A vector field $u(t)$ along $c$ is parallel if $D_c u(t) \equiv 0$, i.e.,

$$\dot{u}^i + \Gamma^i_{kj}(c) \dot{c}^k w^j = 0 \quad (i \geq 1).$$
A curve \(c(t)\) in \(M\) is called a geodesic of \(G\) if it is a projection of an integral curve of \(\mathcal{G}\); hence, \(\dot{c} = \mathcal{G}_c\). A curve \(c(t)\) is a geodesic if and only if the tangent vector \(\dot{u} = \dot{c}\) is parallel along itself: \(D_t \dot{c} = 0\). For a geodesic \(c(t)\) we have the following quasilinear system of second order ODEs

\[
\ddot{c}^i + 2G^i(\dot{c}) = 0, \quad i = 1, \ldots, m + 1.
\]

A Finsler metric \(F\) on \(M\) induces a Finsler spray \(\mathcal{G}\) on \(TM_0\), whose geodesics are locally shortest paths connecting endpoints and have constant speed. Its geodesic coefficients are given by

\[
G^i = \frac{1}{4} g^{il} (\lbrack F^2 \rbrack_{x^l y^l} y^k) - \lbrack F^2 \rbrack_{x^i} = \frac{1}{4} g^{il} (2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}) y^j y^k,
\]

see [15]. Here \(g_{ij}(y) = \frac{1}{2} \lbrack F^2 \rbrack_{y^i y^j}(y)\), compare (2.1). Then \(\Gamma^i_{jk}(y) = \frac{1}{2} y^i (\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j})\) are homogeneous of 0-degree functions on \(TM_0\).

**Remark 2.3.** A Finsler metric on a manifold \(M\) is called a Berwald metric if in any local coordinate system \((x, y)\) in \(TM_0\), the Christoffel symbols \(\Gamma^i_{jk}(x)\) are functions on \(M\) only, in which case the geodesic coefficients \(G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k\) are quadratic in \(y = y^l \partial_{x^l}\). On a Berwald space, the parallel translation along any geodesic preserves the Minkowski functionals; thus, such spaces can be viewed as Finsler spaces modeled on a single Minkowski space. Berwald metrics are characterized among Randers ones, \(F = \alpha + \beta\), by the following criterion: \(\beta\) is parallel with respect to \(\alpha\), see [15, Theorem 2.4.1]. If \(\beta\) is a closed 1-form, then Finslerian geodesics are the same (as sets) as the geodesics of the metric \(\alpha\).

A Finsler manifold is positively (resp. negatively) complete if every geodesic \(c(t)\) on \((0, t_0)\) can be extended for \((0, \infty)\) (resp. \(\langle -\infty, 0 \rangle\)), and \(F\) is complete if it is both positively and negatively complete. This property is satisfied by all closed Finsler manifolds. Let \((M, F)\) be positively complete; hence, for any \(p, q \in M\) there exists a globally minimizing geodesic from \(p\) to \(q\), see also Hopf-Rinov theorem [15, p. 178].

Let \(c_y\) be a geodesic with \(c_y(0) = p\) and \(c_y(0) = y \in T_p M\). The exponential map is defined by \(\exp_p(y) = c_y(1)\). By homogeneity of \(G\) one has \(c_y(t) = c_{\lambda y}(t)\) for \(t > 0\); hence, \(\exp_p(y) = c_p(1)\). Recall [14] that \(\exp_p\) is smooth on \(TM_0\) and \(C^1\) at the origin with \(d(\exp_p)_0 = \text{id}_{T_p M}\).

Consider a geodesic \(c(t)\), \(0 \leq t \leq 1\). A \(C^\infty\) map \(\mathcal{H} : (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow M\) is called a geodesic variation of \(c\) if \(\mathcal{H}(0, t) = c(t)\) and for each \(s \in (-\varepsilon, \varepsilon)\), the curve \(c_s(t) := \mathcal{H}(s, t)\) is a geodesic. For a geodesic variation \(\mathcal{H}\) of \(c\), the variation field \(Y(t) := \frac{\partial \mathcal{H}}{\partial s}(0, t)\) along \(c\) satisfies the Jacobi equation:

\[
(D_c D_c Y) + R_c(Y) = 0
\]

for some \((y \in TM)\)-dependent \((1,1)\)-tensor \(R_y\). Jacobi equation (2.15) serves as the definition of curvature. A vector field \(Y(t)\) satisfying (2.15) along a geodesic \(c(t)\) is called Jacobi field. We have \(g_{ij}(Y(t), \dot{c}(t)) = \lambda^2(a + bt)\) and \(g_{ij}(D_c Y(t), \dot{c}(t)) = \lambda^2 b\) for some constants \(a\) and \(b\) and \(\lambda = F(\dot{c})\). The orthogonal component \(Y^\perp(t) = Y(t) - (a + bt)c(t)\) of the Jacobi field \(Y(t)\) along \(c(t)\) is also a Jacobi field such that \(Y^\perp(t)\) and \(D_c Y^\perp(t)\) are \(g_c\)-orthogonal to \(\dot{c}(t)\). Define \(R_c^{(1)}(t) : T_{c(t)} M \rightarrow T_{c(t)} M\) by \(R_c^{(1)}(u(t)) = \) 

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Given a transversally oriented codimension-one foliation $\mathcal{F}$ of a Finsler manifold $(M^{m+1}, F)$, there exists a globally defined $F$-normal (to the leaves) smooth vector field $n$ which defines a Riemannian metric $g := g_n$ with the Levi-Civita connection $\nabla$. We have $g(n, u) = 0$ ($u \in TF$) and $g(n, n) = F^2(n)$, see (2.9). Then $\nu = n/F(n)$ is an $F$-unit normal.

3.1 The Riemann curvature and the shape operator

In this section we apply the variational approach to find a relationship between the Riemann curvature of $F$ and $g$. It generalizes the following.
**Proposition 3.1** (see [15]). Let \( Y \) be a geodesic field on an open subset \( \mathcal{U} \) in a Finsler space \((M, F)\) and \( \hat{g} := gy \) the induced metric on \( \mathcal{U} \). Then the Riemann curvature of \( \hat{F} \) and \( \hat{F} := \sqrt{g} \) obey \( R_Y = \hat{R}_Y \). Moreover, \( Y \) is a geodesic field of \( \hat{F} \) and for the Levi-Civita connection we have \( D_Y X = \hat{D}_Y X \).

For a codimension-one Riemannian foliation, a unit normal \( \nu \) is a geodesic vector field; hence, by Proposition 3.1, transformations \( R_\nu \) defined for \( F \) by (2.15) coincide with the Jacobi operator \( R(\cdot, \nu)\nu \) of the metric \( g \). Recall that the second differential is defined by \( \nabla^2_{u,v} = \nabla_u \nabla_v - \nabla_{u,v} \) for any \( u, v \).

Let \( Y_t \) (\( |t| \leq \varepsilon \)) be a smooth family of \( F \)-unit vector fields on an open subset \( \mathcal{U} \) in \((M, F)\). Put \( Y_t = \partial_t Y_t \) and \( g_t = g_{Y_t} \), where \( g_t := g_{Y_t} \) is a family of metrics on \( \mathcal{U} \). By definition (2.3) of the Cartan torsion, we have

\[
(3.1) \quad \hat{g}_t = 2C_{Y_t}(\cdot, \cdot, \dot{Y}_t).
\]

Note that 
\[
\hat{g}_t(Y_t, \cdot) = 2C_{Y_t}(Y_t, \cdot, \dot{Y}_t) = 0.
\]

**Proposition 3.2.** Let \( Y_t \) (\( |t| \leq \varepsilon \)) doesn’t depend on \( t \) at a point \( p \in \mathcal{U} \) and \( u, v \in T_p M \). Then

\[
-\partial_t R_t(u, Y_t, Y_t, v) = C_Y(u, \nabla^i_t Y_t, \nabla^i_t \dot{Y}_t) + C_Y(\nabla^i_t Y_t, v, \nabla^i_t \dot{Y}_t)
+ C_Y(\nabla^i_t Y_t, v, \nabla^i_v \dot{Y}_t) + C_Y(u, \nabla^i_t Y_t, \nabla^i_v \dot{Y}_t)
+ C_Y(u, v, \nabla^i_t \dot{Y}_t) + 2\nabla^i_t C_Y(u, v, \nabla^i_t \dot{Y}_t).
\]

The shape operators \( A_t \) (when \( Y_p = \nu_p \)) of \( F \) with respect to \( g_t \) and the volume forms \( dV_t \) at \( p \) obey

\[
(3.3) \quad g_t(\partial_t A_t(u), v) = -C_{\nu}(u, v, \nabla^i_t \dot{Y}_t), \quad \partial_t(dV_t) = 0.
\]

**Proof.** Put \( \Pi(u, v) = \partial_t \nabla^i_t v \) for \( t \)-independent vector fields \( u, v \). Then, see [16],

\[
(3.4) \quad 2g_t(\Pi(u, v), w) = (\nabla^i_t \hat{g}_t)(u, w) + (\nabla^i_t \hat{g}_t)(v, w) - (\nabla^i_t \hat{g}_t)(u, v),
\]

and for arbitrary \( t \)-dependent vector fields \( X_t \) and \( Z_t \) we obtain

\[
\partial_t \nabla^i_t X_t = \Pi(X_t, Z_t) + \nabla^i_t(\partial_t Z_t) + \nabla^i_t \partial_t X_t Z_t.
\]

By definition,

\[
R_t(u, Z_t) Y_t = \nabla^i_t(\nabla^i_{Z_t} Y_t) - \nabla^i_{Z_t}(\nabla^i_t Y_t) - \nabla^i_{[u, Z_t]} Y_t.
\]

So,

\[
\partial_t R_t(u, Z_t) Y_t = \partial_t(\nabla^i_t(\nabla^i_{Z_t} Y_t)) - \partial_t(\nabla^i_{Z_t}(\nabla^i_t Y_t)) - \partial_t(\nabla^i_{[u, Z_t]} Y_t).
\]

Deriving the terms of the above,

\[
\partial_t(\nabla^i_{Z_t}(\nabla^i_t Y_t)) = \Pi(Z_t, \nabla^i_t Y_t) + \nabla^i_{Z_t}(\Pi(u, Y_t)) + \nabla^i_{Z_t}(\partial_t(\nabla^i_t Y_t)) + \nabla^i_{Z_t}(\nabla^i_t Y_t),
\]

\[
\partial_t(\nabla^i_t(\nabla^i_{Z_t} Y_t)) = \Pi(u, \nabla^i_t Z_t Y_t) + \nabla^i_t(\Pi(Z_t, Y_t)) + \nabla^i_t(\partial_t(\nabla^i_{Z_t} Y_t)) + \nabla^i_t(\nabla^i_{Z_t} Y_t),
\]

\[
\partial_t(\nabla^i_{[u, Z_t]} Y_t) = \Pi([u, Z_t], Y_t) + \nabla^i_{[u, Z_t]} \dot{Y}_t + \nabla^i_{[u, \dot{Z}_t]} Y_t.
\]
with $\dot{Z}_t = \partial_t Z_t$, we obtain a ‘time-dependent’ version of [16, Proposition 2.3.4],

$$\partial_s R_t(u, Z_t) Y_t = (\nabla_u^t \Pi)(Z_t, Y_t) - (\nabla_{Z_t}^t \Pi)(u, Y_t) + \partial_t (u, Z_t) \dot{Y}_t + \partial_t (u, \dot{Z}_t) Y_t.$$ 

We shall compute $\partial_s R_t(u, Y_t, Y_0, v) := \partial_s g_t(R_t(u, Y_t)Y_t, v)$ at $p$; thus, terms with $\dot{Y}$ will be canceled at the final stage. Assume at a ‘time’ $t$ of our choice, $\nabla = \nabla^t$ and $\nabla u = \nabla v = 0$ at $p$. Then perform the following preparatory calculations at $p$:

$$\frac{1}{2} Y((\nabla_u^t \dot{g}_t)(Y_t, v)) = Y(u (C_{Y_t}(Y_t, v, \dot{Y}_t)) - C_{Y_t}(\nabla_u^t Y_t, v, \dot{Y}_t))$$

$$= -C_{Y_t}(\nabla_u Y_t, v, \nabla Y \dot{Y}_t),$$

$$\frac{1}{2} Y((\nabla_{Y_t}^t \dot{g}_t)(u, v)) = Y(Y_t (C_{Y_t}(u, v, \dot{Y}_t))) - Y(C_{Y_t}(\nabla_{Y_t}^t u, v, \dot{Y}_t))$$

$$- Y(C_{Y_t}(u, \nabla_{Y_t}^t v, \dot{Y}_t)) = C_{Y_t}(u, v, \nabla_{Y_t} \dot{Y}_t) + 2(\nabla_{Y_t} C_{Y_t})(u, v, \nabla Y \dot{Y}_t),$$

$$\frac{1}{2} Y((\nabla_{u}^t \dot{g}_t)(u, Y_t)) = Y(v (C_{Y_t}(u, Y_t, \dot{Y}_t))) - C_{Y_t}(u, \nabla_{v} Y_t, \dot{Y}_t))$$

$$= -C_{Y_t}(u, \nabla_{v} Y_t, \nabla Y \dot{Y}_t),$$

$$\langle \nabla_{Y_t} \Pi \rangle(u, Y_t) = \langle \nabla_{Y_t} (\Pi(u, Y_t)) - \Pi(u, \nabla_{Y_t} Y_t), v \rangle$$

$$= Y(\Pi(u, Y_t), v) - \langle \Pi(u, \nabla_{Y_t} Y_t), v \rangle$$

$$= \frac{1}{2} Y\left[ (\nabla_{Y_t}^t \dot{g}_t)(Y_t, v) + (\nabla_{Y_t}^t \dot{g}_t)(u, v) - (\nabla_{u}^t \dot{g}_t)(u, Y_t) \right]$$

$$- \frac{1}{2} \left[ \langle \nabla_{\nabla_{Y_t} Y_t} \dot{g}_t \rangle(u, v) + \langle \nabla_{u} \dot{g}_t \rangle(\nabla_{Y_t} Y_t, v) - \langle \nabla_{v} \dot{g}_t \rangle(u, \nabla_{Y_t} Y_t) \right]$$

$$\quad = C_{Y_t}(u, \nabla_{v} Y_t, \nabla_{Y_t} \dot{Y}_t) - C_{Y_t}(\nabla_{u} Y_t, v, \nabla Y \dot{Y}_t)$$

$$\quad + 2(\nabla_{Y_t} C_{Y_t})(u, v, \nabla_{Y_t} \dot{Y}_t) + C_{Y_t}(u, v, \nabla_{Y_t} \nabla_{Y_t}^t \dot{Y}_t) - C_{Y_t}(u, v, \nabla_{Y_t} \nabla_{Y_t}^t \dot{Y}_t)$$

$$\quad - C_{Y_t}(\nabla_{Y_t} Y_t, v, \nabla_{u} \dot{Y}_t) + C_{Y_t}(u, \nabla_{Y_t} Y_t, \nabla_{u} \dot{Y}_t).$$

Here the terms with $C_{Y_t}(Y_t, \cdot, \cdot)$ were canceled on $\mathcal{U}$, and the identity $[Y_t, v]^\top = - (\nabla_u^t Y_t)^\top$ at $p$ (where $^\top$ is the orthogonal to $Y$ at $p$ component of a vector) was applied. Similarly, we use at $p$

$$u[\nabla_{Y_t} \dot{g}_t](Y_t, v) = -2C_{Y_t}(\nabla_{Y_t} Y_t, v, \nabla_{u} \dot{Y}_t),$$

$$\langle \nabla_{\nabla_{Y_t} Y_t} \dot{g}_t \rangle(Y, v) = 0,$$

$$\langle \nabla_{u} \dot{g}_t \rangle(Y, \nabla_{u} Y_t) = 0,$$

$$\langle \nabla_{Y_t} \dot{g}_t \rangle(\nabla_{u} Y_t, v) = 2C_{Y_t}(\nabla_{u} Y_t, v, \nabla_{Y_t} Y_t)$$
to find
\[
(\langle \nabla_u \Pi \rangle(Y_t, Y_t), v) = \langle \nabla_u (\Pi(Y_t, Y_t)) - 2 \Pi(Y_t, \nabla_u Y_t), v \rangle
= u(\Pi(Y_t, Y_t), v) - 2 \langle \Pi(Y_t, \nabla_u Y_t), v \rangle
= u[(\nabla_u \hat{g})(Y_t, v) - \frac{1}{2} (\nabla_u \hat{g})(Y_t, Y_t)]
- (\nabla_{\nabla_u Y_t} \hat{g})(Y_t, v) - (\nabla_Y \hat{g})(\nabla_u Y_t, v) + (\nabla_v \hat{g})(Y_t, \nabla_u Y_t)
= -2C_Y(\nabla_Y Y_t, v, \nabla_u Y_t) - 2C_Y(\nabla_u Y_t, v, \nabla_Y Y_t).
\]

Since \( \dot{Y} = 0 \) at \( p \), we have
\[
\partial_t b_t(u, v) = g(\partial_t \nabla_u v, Y) + g(\partial_t \nabla_u v, \partial_t Y)
= \frac{1}{2} \left( (\nabla_u \hat{g})(v, Y) + (\nabla_v \hat{g})(u, Y) - (\nabla_Y \hat{g})(u, v) \right) + g(\nabla_u v, \dot{Y})
= -\nabla_Y (C_Y(u, v, \dot{Y})) = -C_Y(u, v, \nabla_Y \dot{Y}).
\]

From this, using \( b_t(u, v) = g_t(A_t(u), v) \), we get (3.3):
\[
g_t(A_t(u), v) = \partial_t b_t(u, v) - \hat{g}(A(u), v) = -C_Y(u, v, \nabla_Y \dot{Y}).
\]

By the formula for the volume form of a \( t \)-dependent metric, \( \partial_t (dV_t) = \frac{1}{2} (\text{Tr} \hat{g}) dV_t \), see [16], and definition of the mean Cartan torsion, we get
\[
(3.5) \quad \partial_t (dV_t) = I_{Y_t}(\dot{Y}_t) dV_t.
\]

Next, (3.3) follows from (3.5) and \( \dot{Y}(p) = 0 \).

Let \( L \) be a leaf through a point \( p \in M \), and \( \rho \) the local distance function to \( L \) in a neighborhood of \( p \). Denote by \( \nabla \) the Levi-Civita connection of the (local again) Riemannian metric \( \hat{g} := g_{\nabla \rho} \). Note that \( \nabla \rho = \nu \) on \( L \). The shape operator \( A : T \mathcal{F} \to T \mathcal{F} \) (self-adjoint for \( g \)) is defined at \( p \in M \) by (compare [15] with the opposite sign)
\[
A(u) = -\nabla_u \nu \quad (u \in T_p \mathcal{F}).
\]
The shape operator \( A^g : T \mathcal{F} \to T \mathcal{F} \) with respect to the metric \( g \) is defined at \( p \in M \) by
\[
A^g(u) = -\nabla_u \nu \quad (u \in T_p \mathcal{F}).
\]
Note that \( 2g(\nabla_u \nu, \nu) = u(g(\nu, \nu)) = 0 \) (\( u \in T \mathcal{F} \)); hence, \( \nabla_u \nu \in T \mathcal{F} \). The mean curvature function (of the leaves with respect to \( g \)) is defined by \( H^g = \text{Tr} A^g \). Recall that \( \mathcal{F} \) is \( g \)-totally umbilical if \( A^g = H^g I_m \), and is \( g \)-totally geodesic if \( A^g \equiv 0 \).
Corollary 3.3. Let $L$ be a hypersurface in an open set $\mathcal{U} \subset M$. If an $F$-unit vector field $Y_t$ ($0 \leq t \leq \varepsilon$) is given in $\mathcal{U}$ and orthogonal to $L$ then for the metric $g_t := g_{Y_t}$ for all $u, v \in T_p L$ ($p \in L$) we have

$$\partial_t R_t(u, Y_t, Y_t, v) = C_Y(A_t(u), v, \nabla^1_{Y_t} Y_t) + C_Y(u, A_t(v), \nabla^1_{Y_t} Y_t)$$

(3.6)

$$g(\partial_t A_t(u), v) = -C_Y(u, v, (\nabla^1_{Y_t} Y_t)^2) - 2(\nabla^1_{Y_t} C_Y)(u, v, \nabla^1_{Y_t} Y_t),$$

(3.7)

Proof. This follows from $\dot{Y}_t = 0$ on $L$, the definition of $A_t$ (for $g_t$) and (3.2)-(3.3).

Definition 3.1. A vector field $\dot{Y}$ defined in some neighborhood $\mathcal{U} \subset M$ of a point $p \in \mathcal{U}$ is called a geodesic extension of a vector $Y_p \in T_p M$ if $\dot{Y}(p) = Y_p$ and the integral curves of $\dot{Y}$ are geodesics of the Finsler metric. Similarly, we define a geodesic extension of a (e.g. normal) vector field along a hypersurface $L \subset \mathcal{U}$. In both cases, $\overline{g} := g_{\dot{Y}}$ is called the osculating Riemannian metric of $F$ on $\mathcal{U}$.

We will use osculating metric (given locally) to express the Riemannian curvature of $g = g_{\overline{Y}}$ (for an unit $F$-normal $\nu$ to $F$) in terms of Riemannian curvature and the Cartan torsion of $F$.

Given a vector field $Y$, let $C^g_Y$ be a $(1, 1)$-tensor $g_Y$-dual to the symmetric bilinear form $C_Y(\cdot, \cdot, \nabla_Y Y)$. Note that $C_n(\cdot, \cdot, \nabla n) = C^g(\cdot, \cdot, c^g \nabla \nu \nu) = c^g C(\cdot, \cdot, \nabla \nu \nu)$.

Theorem 3.4. Let $\nu$ be a unit normal to a codimension-one foliation of a Finsler space $(M^{m+1}, F)$. The Riemann curvatures (in the $\nu$-direction) of $F$ and $g = g_{\overline{Y}}$ are related by

$$g((R_\nu - R^g_\nu)(u), v) = -C_\nu(A^g(u) + \frac{1}{2} C^g_{\nu}(u), v, \nabla_\nu \nu)$$

(3.8)

$$-C_\nu(u, A^g(v) + \frac{1}{2} C^g_{\nu}(v), \nabla_\nu \nu) + C_\nu(u, v, \nabla^2_{\nu, \nu} \nu - C^g_{\nu}(\nabla_\nu \nu)) + 2(\nabla_\nu C_\nu)(u, v, \nabla_\nu \nu) \quad (u, v \in T_p L).$$

The shape operators and volume forms are related by

$$A - A^g = C^g, \quad dV_g = e^{\tau(\nu)} dV_F.$$  

(3.9)

In particular, the traces are related by

$$\text{Ric}_\nu - \text{Ric}^g_\nu = I_\nu(\nabla^2_{\nu, \nu} \nu - C^g_{\nu}(\nabla_\nu \nu)) + 2(\nabla_\nu I_\nu)(\nabla_\nu \nu)$$

$$-\text{Tr}(C^g_{\nu}(C^g_{\nu} + 2 A^g)),$$

(3.10)

$$\text{Tr} A - \text{Tr} A^g = I_\nu(\nabla_\nu \nu).$$

Proof. Let $U$ be a “small” neighborhood of $p \in L$ such that any two geodesics starting from $L \cap U$ in the $\nu$-direction do not intersect in $U$. Then for any $q \in U$ there is a unique geodesic $\gamma$ starting from $L$ in the $\nu$-direction such that $\gamma(s) = q$ for some $s \geq 0$, in other words, $q = \exp_{\gamma(0)}(s \dot{\gamma}(0)).$ Thus, $\dot{\gamma} : q \rightarrow \dot{\gamma}(s) (q \in U)$ is an $F$-unit geodesic vector field ($\nabla_\gamma \dot{Y} = 0$) - a geodesic extension of $\nu|_L$.

Consider a family of vector fields $Y_t = t \dot{Y} + (1 - t) \nu$ ($0 \leq t \leq 1$) on $U$, define the Riemannian metrics $g_t := g_{Y_t}$, $g_1$ being osculating, and denote by $R_t$ their Riemann...
curvatures. Since \( \dot{Y}_t = \dot{Y} - \nu \) and \( Y_{t|L} = \nu_{t|L} = \dot{Y}_{|L} \) for all \( t \), we have \( \dot{Y}_{t|L} = 0 \) and \( g_{t|L} = g_{L} \). By (3.1) and (3.4), we get \( \Pi_{(\nu, \nu)} = \Pi_{(\nu, \dot{Y})} = 0 \) on \( L \); hence, \( \nabla^t_b \nu \) and \( \nabla^t_b \dot{Y} \) restricted on \( L \) don’t depend on \( t \). Next, we find

\[
\begin{aligned}
g(\Pi(\nu, \nu), v) &= C_\nu(u, v, \nabla_\nu (\dot{Y} - \nu)) = -C_\nu(u, v, \nabla_\nu \nu), \quad u, v \in TM_{|L},
\end{aligned}
\]
i.e., \( \Pi(\nu, \nu) = -C^t_\nu(u) \). We calculate on \( L \):

\[
\begin{aligned}
g(\partial_t(\nabla^t_\nu u), v) &= \nabla^t_\nu (C_Y(u, v, \dot{Y} - \nu)) + \nabla^t_\nu (C_Y(u, v, \dot{Y} - \nu)) - \nabla^t_\nu (C_Y(u, v, \dot{Y} - \nu)) \\
&= (\nabla^t_\nu (C_Y(u, v, \dot{Y} - \nu)) + \nabla^t_\nu (C_Y(u, v, \dot{Y} - \nu)) \\
&= -C_\nu(u, v, \nabla_\nu (\dot{Y} - \nu)) = -C_\nu(u, v, \nabla_\nu \nu).
\end{aligned}
\]

Since, \( \partial_t(g(\nabla^t_\nu u, v)) = g(\partial_t(\nabla^t_\nu u, v)) \) and \( \partial_t(g(\nabla^t_\nu \nu, v)) = g(\partial_t(\nabla^t_\nu \nu, v)) \) on \( L \), we obtain

\[
\begin{aligned}
g(\nabla^t_\nu u, v) &= g(\nabla_\nu u, v) - t C_\nu(u, v, \nabla_\nu \nu), \\
g(\nabla^t_\nu \nu, v) &= g(\nabla_\nu \nu, v) - t C_\nu(u, v, \nabla_\nu \nu).
\end{aligned}
\]

Recall that \( \nabla^2_{\nu, \nu} \) is tensorial in \( u, v \). We show that \( (\nabla^t)^2_{\nu, \nu} \dot{Y} \) is \( t \)-independent on \( L \):

\[
\begin{aligned}
(\nabla^t)^2_{\nu, \nu} \dot{Y} &= \nabla^t_{\nu} (\nabla^t_{\nu} \dot{Y}) = \nabla^t_{\nu} (\nabla^t_{\nu} \dot{Y}) - t C^t_\nu (\nabla^t_{\nu} \dot{Y}) \\
&= \nabla^t_{\nu} (\nabla^t_{\nu} \dot{Y}) - t (\nabla^t_{\nu} \nabla^t_\nu \nu) (\dot{Y}) - t C^2_\nu (\nabla^t_{\nu} \dot{Y}) = \nabla^2_{\nu, \nu} \dot{Y}.
\end{aligned}
\]

Thus, \( (\nabla^t)^2_{\nu, \nu} \dot{Y})_{|L} = (\nabla^t)^2_{\nu, \nu} \dot{Y})_{|L} = 0 \). Using this and \( (\nabla^t_{\nu} \dot{Y})_{|L} = 0 \), we find on \( L \):

\[
\begin{aligned}
\nabla^t_{\nu} \dot{Y} &= -\nabla_\nu \nu, \\
(\nabla^t)^2_{\nu, \nu} \dot{Y} &= (\nabla^t)^2_{\nu, \nu} \dot{Y} - t C^t_\nu (\nabla^t_{\nu} \dot{Y}) = \nabla^2_{\nu, \nu} (\nabla^t_{\nu} \dot{Y}) = \nabla^2_{\nu, \nu} (\nabla^t_{\nu} \dot{Y}) = -2 t C^2_\nu (\nabla^t_{\nu} \dot{Y}) = -2 t C^2_\nu (\nabla^t_{\nu} \dot{Y})
\end{aligned}
\]

Then we obtain on \( L \):

\[
\begin{aligned}
C_{Y_1}(\cdots, \nabla_{Y_1} \dot{Y}) &= C_\nu(\cdots, \nabla_\nu \dot{Y}) = -C_\nu(\cdots, \nabla_\nu \nu), \\
C_{Y_1}(\cdots, \nabla_{Y_1} \dot{Y}) &= C_\nu(\cdots, \nabla^2_{\nu, \nu} \dot{Y}) = -C_\nu(\cdots, \nabla^2_{\nu, \nu} \nu).
\end{aligned}
\]

Next, we calculate on \( L \), using \( C_{Z}(Z, \cdots, Z) = 0 \) for \( Z = \nabla_\nu \nu, \)

\[
(\nabla_{Y_1} C_{Y_1})(\cdots, \nabla_{Y_1} \dot{Y}) = (\nabla_\nu C_{\nu C_{Y_1}+_{(1-t)} \nu})(\cdots, -\nabla_\nu \nu) = -(\nabla_\nu C_{\nu}) (\cdots, -\nabla_\nu \nu).
\]

By the above and (3.3), we obtain (3.9). By Corollary 3.3, for all \( t \in [0, 1] \), and using \( A_t = A^0 + t C^t_\nu \), see (3.9), and \( (\nabla^t)^2_{\nu, \nu} \nu = -2 t C^2_\nu (\nabla^t_{\nu} \nu) \), we obtain

\[
\begin{aligned}
\partial_t R_k(u, \nu, \nu, v) &= -C_\nu(A_t(u), v, \nabla_\nu \nu) - C_\nu(u, A_t(v), \nabla_\nu \nu) \\
&= +C_\nu(u, v, (\nabla^t)^2_{\nu, \nu} \nu) + 2 (\nabla_\nu C_\nu)(u, v, \nabla_\nu \nu) \\
&= -C_\nu(A^0(u) + t C^t_\nu(u), v, \nabla_\nu \nu) - C_\nu(u, A^0(u) + t C^t_\nu(v), \nabla_\nu \nu) \\
&= +C_\nu(u, v, -\nabla^2_{\nu, \nu} \nu + 2 t C^2_\nu (\nabla^t_{\nu} \nu) + 2 (\nabla_\nu C_\nu)(u, v, \nabla_\nu \nu)
\end{aligned}
\]
for \( u, v \in T_p L \), where the right hand side becomes linear in \( t \). Integrating this by \( t \in [0, 1] \) yields (3.8). Finally, using the equality for volume forms, \( \mathrm{d} \tilde{V} = \mathrm{d} V_g \), and definition of \( \tau \) (see Section 2.1), we get (3.9).  

Since any geodesic vector field \( Y \) satisfies conditions

\[
C_Y(u, v, \nabla_Y Y) = 0, \quad C_Y(u, v, \nabla^2_{Y,Y} Y) = 0 \quad (\forall u, v),
\]

the following corollary generalizes Proposition 3.1.

Corollary 3.5. If \( Y \) is a unit vector field on a Finsler space \( (M, F) \) and \( g := g_Y \) a Riemannian metric on \( M \) with the Levi-Civita connection \( \nabla \) and conditions (3.11), then \( R_Y = R^g_Y \).

Proof. By (3.11), we have \( C^2_Y = 0 \) and

\[
(\nabla_Y C_Y)(u, v, \nabla_Y Y) = \nabla_Y (C_Y(u, v, \nabla_Y Y)) - C_Y(u, v, \nabla^2_{Y,Y} Y) = 0.
\]

If a vector field \( \tilde{Y} \) is a local geodesic extension of \( Y(p) \) then \( R^g_Y = \tilde{R}_Y \) (and \( A^g = A \)) at \( p \), see (3.8) and (3.9). Thus, the claim follows from Proposition 3.1.

3.2 Integral formulae

Let \( \mathcal{F} \) is a codimension-one foliation of a closed Finsler space \( (M^{m+1}, F) \) with the Busemann-Hausdorff volume form \( \mathrm{d}V_F \). Define a family of diffeomorphisms \( \{ \phi_t : M \to M, \ 0 \leq t < \varepsilon \} \) (\( \varepsilon > 0 \) being small enough) by

\[
\phi_t(p) = \exp_p(t \nu), \quad \text{where} \quad \nu \in T_p M \quad \text{is an } F\text{-unit normal to } \mathcal{F} \text{ at } p \in M.
\]

Let \( c(t) \ (t \geq 0) \) be an \( F \)-geodesic with \( c(0) = p \) and \( \dot{c}(0) = \nu(p) \). Any geodesic variation built of \( \phi_t \)-trajectories determines an \( F \)-Jacobi field \( Y(t) \) on \( c \), and \( A_p(\mathcal{Y}(0)) = -[D_{c(t)} Y(t)]_{|t=0}, \) see [15, p. 225]. Recall that if vectors \( u(t) \) and \( v(t) \) are \( D \)-parallel along \( c(t) \) then \( g_{c(t)}(u(t), v(t)) \) is constant. Choose a positively oriented \( g_{c(t)}\)-orthonormal frame \( (e^1, \ldots, e^m) \) of \( T_p \mathcal{F} \) and extend it by parallel translation to the frame \( (E_1^u, \ldots, E^m_u) \) of vector fields \( g_{c(t)}\)-orthogonal to \( \dot{c}(t) \) along \( c(t) \). Denote also by \( E_u^{m+1} = \dot{c}(t) \) the tangent vector field along \( c(t) \). Denote by \( Y^i(t) \) \((i \leq m)\) the Jacobi field along \( c(t) \) satisfying \( Y^i(0) = e^i \) and \( D_c Y^i(0) = A_p(e^i) \). Let \( R(t) \) be the matrix with entries \( g_{c(t)}(R_c(E_i^u), E^j_u) \). Denote by \( Y(t) \) the \( m \times m \) matrix consisting of the scalar products \( g_{c(t)}(Y^i(t), E^j_u) \) \("F-Jacobi tensor\”). Then \( \mathcal{Y}(0) = I_m \) and \( \mathcal{Y}'(0) = A_p \). It is known (see, for instance, [15, Sections 2.1 and 2.2]) that

\[
|d \phi_t(p)| = \det Y(t),
\]

where \( |d \phi_t(p)| \) is the Jacobian of \( \phi_t \) at \( p \). Assume that \( R^{(1)}_{c(t)} \equiv 0 \) for any \( F \)-geodesic \( c(t) \) \((t \geq 0)\) (e.g. \( (M, F) \) is locally symmetric with respect to \( F)\). For \( t = 0 \), we have \( R^{(2)}_{c(0)} \equiv R^{(1)}_{c(t)} (t \equiv 0) \equiv 0 \). For short, write \( R_p := R(0) \). Note that \( \text{Tr } R_p = \text{Ric}(\nu(p)) \).

The \( F \)-Jacobi equation \( \mathcal{Y}' = -R(t)\mathcal{Y} \) implies that

\[
\mathcal{Y}^{(2k)}(0) = (-R_p)^k, \quad \mathcal{Y}^{(2k+1)}(0) = (-R_p)^k A_p, \quad k = 0, 1, 2, \ldots
\]
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Hence, our Jacobi tensor has the form

\[ Y(t) = \sum_{k=0}^{\infty} Y^{(k)}(0) \frac{t^k}{k!} = I_m + tA_p - \frac{t^2}{2!} R_p - \frac{t^3}{3!} R_p A_p + \frac{t^4}{4!} R_p^2 + \ldots. \]

Certainly, the radius of convergence of the series is uniformly bounded from below on \( M \) (by \( 1/\|R\|_F > 0 \)). The volume of \( M \) is defined by \( \text{Vol}_F(M) = \int_M dV_F \). Therefore – by Dominated Convergence Theorem – its integration together with Change of Variables Theorem yield the equality for any \( t \geq 0 \) small enough

\[
(3.12) \quad \text{Vol}_F(M) = \int_M \det (I_m + tA_p - \frac{t^2}{2!} R_p - \frac{t^3}{3!} R_p A_p + \frac{t^4}{4!} R_p^2 + \ldots) \, dV_F,
\]

where \( dV_F \) is the volume form of \( F \). Formula (3.12) together with Lemma 5.2 of Appendix imply our main result (which generalizes that of [13] valid for the Riemannian case). Note that the invariants \( \sigma_\lambda(A_1, \ldots, A_k) \) of a set of real \( m \times m \) matrices \( A_i \) are defined and discussed in Appendix.

**Theorem 3.6.** If \( F \) is a codimension-one foliation on a closed Finsler manifold \((M^{m+1}, F)\), which is \( F \)-locally symmetric, then for any \( 0 \leq k \leq m \) one has

\[
(3.13) \quad \int_M \sum_{\|\lambda\| = k} \sigma_\lambda(B_1(p), \ldots, B_k(p)) \, dV_F = 0,
\]

where \( B_{2k}(p) = \frac{(-1)^k}{(2k)!} (R_p)^k \), \( B_{2k+1}(p) = \frac{(-1)^k}{(2k+1)!} (R_p)^k A_p \) for \( p \in M \).

The formulae (3.13) for few initial values of \( k, k = 1, \ldots, 3 \), read as follows:

\[
(3.14) \quad \int_M \sigma_1(A_p) \, dV_F = 0, \\
(3.15) \quad \int_M (\sigma_2(A_p) - \frac{1}{2} \text{Tr}(R_p)) \, dV_F = 0, \\
(3.16) \quad \int_M (\sigma_3(A_p) - \frac{1}{2} \text{Tr}(A_p) \text{Tr}(R_p) - \frac{1}{3} \text{Tr}(R_p A_p)) \, dV_F = 0.
\]

The formulae (3.14) and (3.15) are well known for arbitrary foliated Riemannian manifolds, see the Introduction. For \( m = 1 \), (3.15) reduces to the integral of flag (Gauss) curvature, \( \int_M K \, dV_F = 0 \).

**Remark 3.2.** 1. The compactness of \( M \) in Theorem 3.6 can be replaced by weaker conditions: \( M \) is positively complete of \( F \)-volume, and has ‘bounded geometry’ in the following sense:

\[
(3.17) \quad \sup_{p \in M} \|R_p\|_F < \infty, \quad \sup_{p \in M} \|A_p\|_F < \infty.
\]

2. Similar formulae exist for codimension-one foliations of on arbitrary (non-locally symmetric with respect to \( F \)) Finsler manifolds. They are more complicated since they contain terms which depend on covariant derivatives of \( R_p \). More precisely, they contain just terms of the form \( R_p^{(k)} \), where \( R_p^{(1)} = D_{\nu(p)} R_p \), \( R_p^{(2)} = D_{\nu(p)} D_{\nu(p)} R_p \) and so on. For the \( F \)-Jacobi tensor \( Y(t) \) we get

\[
Y(t) = I_m + tA_p - \frac{t^2}{2!} R_p - \frac{t^3}{3!} (R_p A_p + R_p^{(1)}) + \frac{t^4}{4!} (R_p^2 - R_p^{(2)}) - 2R_p^{(1)} A_p + \ldots
\]
The $t^3$ term of (3.12) becomes, compare (3.16),
\[ \int_M \left( \sigma_3(A_p) - \frac{1}{6} \operatorname{Tr}(R_p) \operatorname{Tr}(A_p) + \frac{1}{3} \operatorname{Tr}(R_p A_p) - \operatorname{Tr} R_p^{(1)} \right) \, dV_F = 0. \]

In general, the $t^k$ term in (3.12) contains $R_p^{(j)}$'s with $j \leq k - 2$.

**Corollary 3.7.** Let $\mathcal{F}$ be a codimension-one foliation on a $F$-locally symmetric complete Finsler manifold $(M, F)$ of finite $F$-volume and bounded (in the sense of (3.17)) geometry. If $\operatorname{rank}(A_p) \leq 1$ for all $p \in M$ (for example, $\mathcal{F}$ is $F$-totally geodesic) then the Riemannian curvature $R_p$ vanishes identically provided that $M$ has everywhere non-negative (or, non-positive) Ricci curvature $\operatorname{Ric}_p = \operatorname{Tr} R_p$.

**Proof.** Since in this case $\sigma_2(A_p) = 0$, integral formula (3.15) implies the claim. □

Given a unit normal $\nu$ to $\mathcal{F}$, denote by $Q_R$ the symmetric $(0,2)$-tensor in the rhs of (3.8). Then, see (3.10),
\[ \operatorname{Tr} Q_R = I_{\nu}(\nabla^2_{\nu\nu} + C^2_{\nu}(\nabla_{\nu}\nu)) + 2(\nabla_{\nu}I_{\nu})(\nabla_{\nu}\nu) - \operatorname{Tr} (C^2_{\nu}(C^2_{\nu} + 2A^g)). \]

Define the 1-form $\theta_g$ by the equality
\[ \theta_g(X) = g([X,\nu],\nu) \quad (X \in TM). \]

Note that $\nabla_{\nu}\nu = \theta^g_{\nu}$ is the mean curvature of $\nu$-curves with respect to $g$. Comparing (3.13) for $F$ and $g$, we obtain a series of integral formulas, the first two of which are given in the following.

**Theorem 3.8.** Let $\tau(\nu) = \text{const}$ on a codimension-one foliated Finsler space $(M, F)$. Then

**Proof.** By (3.9), $A = A^g + C^g_{\nu}$, where $A = A_p$. Thus, (3.18) follows from (3.14), using (3.9) and Theorem 3.4. Note that by (5.4) with $k = 1$ and (5.6) (of Appendix), and by (3.10), we have
\[ \sigma_2(A_p) = \sigma_2(A^g) + \operatorname{Tr}(A^g) \operatorname{Tr} C^g_{\nu} - \operatorname{Tr}(A^g C^g_{\nu}), \]
\[ \operatorname{Ric}_\nu = \operatorname{Tr} R_p = \operatorname{Ric}^g_{\nu} + \operatorname{Tr} Q_R. \]

Thus, (3.19) follows from (3.15), using (3.9) and (5.6) with $k = 2$ (of Appendix). □

### 3.3 Examples

**Finsler manifolds of constant flag curvature.** We provide examples, these of $(M, F)$ with constant flag curvature $K(\nu, P)$ on $M$, i.e., such that $R_p = K I_m$ for some $K \in \mathbb{R}$.

a) For $(M, F)$ with zero flag curvature, $R_p = 0$, and we obtain the Jacobi tensor of a simple form, linear in $t$: $Y(t) = I_m + tA_p \ (t \geq 0)$. Then (3.12) reduces to
\[
\text{Vol}_F(M) = \int_M \det(I_m + t A_p) \, dV_F. \]
From this we obtain the Finsler generalization of the case \(K = 0\) of [3, Theorem 1.1], i.e.,

\[
(3.20) \quad \int_M \sigma_k(A_p) \, dV_F = 0, \quad k > 0.
\]

b) Assume now that the flag curvature \(K(\nu, P)\) of \((M, F)\) is constant and positive, say \(K = 1\). Then \(R_p = I_m\) and one can rewrite the Taylor series for \(Y(t) (t \geq 0)\) in the form \(Y(t) = \cos t (I_m + (\tan t)A_p)\). Change of Variables Theorem for integration implies that the equality

\[
\text{Vol}_F(M) = (\cos t)^m \int_M \det(I_m + (\tan t)A_p) \, dV_F
\]
holds for arbitrary \(t \geq 0\) small enough. One can use the substitution \(\tan t \to \tilde{t}\) and the identity \(\cos^2 t = (1 + \tilde{t}^2)^{-1}\) for further derivations.

c) The case of negative constant flag curvature \(K(\nu, P)\) of \(M\), say \(K = -1\), is similar to the case (b). One can use the substitution \(\tanh(t) \to \tilde{t}\) and the identity \(\cosh^2 t = (1 - \tilde{t}^2)^{-1}\) for derivations.

The above yields the following extension of Theorem 1.1 in [3].

**Corollary 3.9.** Let \(F\) be a transversally oriented codimension-one foliation on a Finsler manifold \((M^{m+1}, F)\) of finite \(F\)-volume and \(\sup_{p \in M} \|A_p\|_F < \infty\) (e.g., closed) with a unit normal \(\nu\) and condition \(R_p = K I_m\). Then, for any \(0 \leq k \leq m\),

\[
(3.21) \quad \int_M \sigma_k(A_p) \, dV_F = \begin{cases} 
K^{k/2} (m/2) \text{Vol}_F(M), & m, k \text{ even}, \\
0, & m \text{ or } k \text{ odd}.
\end{cases}
\]

**Remark 3.3.** By Theorem 8.2.4 in [8], if a Finsler manifold \(M\) is closed and has constant negative curvature then it is Randers.

If \((M, F)\) is \(F\)-locally symmetric and the leaves of \(F\) are \(F\)-totally geodesic (i.e., \(A_p = 0\)) then

\[
Y^{(2k+1)}(0) = 0, \quad Y^{(2k)}(0) = (-R_p)^k.
\]

Finally we get the \(F\)-Jacobi tensor \(Y(t) = I_m - \frac{t^2}{2!} R_p + \frac{t^4}{4!} R_p^2 - \frac{t^6}{6!} R_p^3 + \ldots\), and (3.13) reduces to

\[
\int M \sum_{|\lambda| = k} \sigma_\lambda (-\frac{1}{2!} R_p, \frac{1}{4!} R_p^2, \ldots, \frac{(-1)^k}{(2k)!} R_p^k) \, dV_F = 0.
\]

For codimension-one \(F\)-totally geodesic foliations on arbitrary positively complete (or closed) Finsler manifolds of finite \(F\)-volume, we get

\[
(3.22) \quad \int_M \text{Tr } R_p \, dV_F = 0, \quad \int_M \text{Tr } R_p^{(1)} \, dV_F = 0,
\]

and so on. Equalities (3.22) imply directly the following statement (see also Corollary 3.7).
Corollary 3.10. Let $\mathcal{F}$ be a codimension-one $F$-totally geodesic foliation on a $F$-locally symmetric positively complete Finsler manifold $(M, F)$ of finite $F$-volume and with condition (3.17)$_1$. Then $R_p$ vanishes identically provided that either $M$ has everywhere non-negative (or, non-positive) Ricci curvature $\text{Ric}$, or $\sigma_2(R_p)$ is non-negative.

It has been observed in [7] that codimension-one foliations of compact negatively-Ricci curved Riemannian spaces are far (in a sense) from being totally umbilical. In the case of an $F$-totally umbilical foliation, $A_p = H I_m$, therefore on a locally symmetric Finsler space $(M, F)$ the following can be derived from (3.15) – (3.16) etc. with the use of Lemma 5.1 of Appendix:

\begin{align}
\int_M (m-1)(m-2)H^2 - \text{Tr} R_p \, dV_F &= 0, \\
\int_M H \left( \frac{m(m-1)(m-2)}{3m-2} H^2 - \text{Tr} R_p \right) \, dV_F &= 0.
\end{align}

These integrals for $k$ even ((3.23), (3.24), etc.) contain polynomials depending on $H^2$ only. If all the coefficients of such polynomials are positive, then the polynomials are positive for all values of $H$ and one may easily get obstructions for existence of totally umbilical foliations on some Finsler manifolds.

4 Codimension-one foliated Randers spaces

Let $\mathcal{F}$ be a transversally oriented codimension-one foliation of $M^{m+1}$ equipped with a Randers metric

$$F(y) = \sqrt{a(y,y) + \beta(y)}, \quad \|\beta\|_\alpha < 1, \quad \beta^g \in \Gamma(T\mathcal{F}).$$

As before, let us write $a(\cdot, \cdot) = \langle \cdot, \cdot \rangle$. Let $N$ be a unit $a$-normal vector field to $\mathcal{F}$, i.e., $\langle N, N \rangle = 1$ and $\langle N, v \rangle = 0$ for $v \in T\mathcal{F}$, and $n$ an $F$-normal vector field to $\mathcal{F}$ with the property $\langle n, n \rangle = 1$. Denote by $\nabla$ the Levi-Civita connection of the Riemannian metric $a$ and by $\nabla$ the Levi-Civita connection of the Riemannian metric $g = g_n$ on $M$. According to [4, (1.15) and (1.19)] we have

\begin{align}
\tau(y) &= (m+2) \log \sqrt{(1 + \beta(y)/a(y)) c^{-2}}, \\
I_n(v) &= \frac{m+2}{2F(y)} \left( \beta(v) - \frac{\langle v, y \rangle \beta(y)}{a^2(y)} \right).
\end{align}

In particular, $\tau(n) = 0$ and $I_n(v) = \frac{m+2}{2c^2} (\beta^g - (c^2-1) n, v)$. Remark that for Randers spaces

$$C_n(u, v, w) = \frac{1}{m+2} \left( I_n(u) h_n(v, w) + I_n(v) h_n(u, w) + I_n(w) h_n(u, v) \right),$$

where the angular form $h_n$ is given by

\begin{align}
\langle h_n(u, v) \rangle = c^2 \left( \langle u, v \rangle - \langle u, n \rangle \langle v, n \rangle \right),
\end{align}
see [4, (1.11) and (1.20)]. Since $\sigma_F = e^{n+2}\sqrt{\det a_{ij}}$, see [4, p. 6], and $\sqrt{\det a_{ij}(n)} = c^{n+2}\sqrt{\det a_{ij}}$, see (2.6), the volume form of $F$ and canonical volume forms of Riemannian metrics $g$ and $a$ obey

\begin{equation}
(4.4) \quad dV_F = e^{n+2}dV_a, \quad dV_g = e^{n+2}dV_a, \quad dV_F = dV_g.
\end{equation}

Let $Z = \nabla_\nu \nu$ (which is dual of $\theta_g$ in Sect. 3.2) and $\tilde{Z} = \nabla_N N$ be the curvature vectors of $\nu$-curves and $N$-curves for Riemannian metrics $g$ and $a$, respectively.

### 4.1 The shape operators of $g$ and $a$

The shape operators of $F$ with respect to the metrics $a$ and $g$ are defined as follows:

$$\tilde{A}(u) = -\nabla_u N, \quad A^g(u) = -\nabla_u \nu,$$

where $u \in TF$ and $\nu = e^{-n}c^{1-N} (N - c^{-1}\beta^2)$ with $c = \sqrt{1 - \|\beta\|^2_\alpha} > 0$.

The derivative $\nabla u : TM \to TM$ is defined by $\langle \nabla u \rangle (v) = \nabla_v u = \nabla_u v$, where $v \in TM$. The conjugate derivative $\langle (\nabla u)^T \rangle : TM \to TM$ is defined by $\langle (\nabla u)^T \rangle (v) = \langle v, (\nabla u)(w) \rangle$ for all $v, w \in TM$. The deformation tensor $\text{Def}$,

$$2\text{Def}_u = \nabla u + (\nabla u)^T,$$

measures the degree to which the flow of a vector field $u \in \Gamma(TM)$ distorts the metric $a$. The same notation $\text{Def}_u$ will be used for its dual (with respect to $a$) $(1, 1)$-tensor. Set $\text{Def}_u^T(v) = (\text{Def}_u(v))^\top$. For $\beta \neq 0$, let

$$\tilde{A}(\beta^2)^{1,\beta} = \tilde{A}(\beta^2) - \langle \tilde{A}(\beta^2), \beta^T \rangle \beta \cdot \|\beta\|^{-2}_\alpha$$

be the projection of $\tilde{A}(\beta^2)$ on $(\beta^2)^\top$. Note that $\lim_{\beta \to 0} \tilde{A}(\beta^2)^{1,\beta} = 0$.

**Proposition 4.1.** Let $\beta(N) = 0$ on $M$. Then on $TF$ we have

\begin{equation}
(4.5) \quad c A^g = \tilde{A} - e^{-2}(c N - \beta^2)(c)I_m + e^{-1}(\text{Def}_\beta)^\top u_1 + u_2 \otimes \beta,
\end{equation}

where

$$U_1 = -\frac{1}{2} e^{-2}(c N - \beta^2)(c) \beta^T - 2e^{-1}(\text{Def}_\beta^\top \beta^2)^\top - \nabla_{N-c^{-1}\beta}^\top \beta^2 + c \tilde{Z} + c \beta(\tilde{Z}) \beta - \tilde{A}(\beta^2)^{1,\beta},$$

$$U_2 = \frac{1}{2} (\nabla_{N-c^{-1}\beta}^\top \beta^2 - c \tilde{Z} - \tilde{A}(\beta^2)^{1,\beta}).$$

**Proof.** By the well-known formula for Levi-Civita connection of $g$, using equalities $g(u, n) = 0 = g(v, n)$ and $g([u, v], n) = 0$, we have

\begin{equation}
(4.7) \quad 2g(\nabla_u n, v) = n(g(u, v)) + g([u, n], v) + g([v, n], u) \quad (u, v \in TF).
\end{equation}
One may assume $\nabla_X u = \nabla_X v = 0$ for all $X \in T_p M$ at a given point $p \in M$. Using (2.11) with $u = [u, n]$ and $v = v$, we obtain

$$n(g(u, v)) = n(c^2(\langle u, v \rangle - \beta(u) \beta(v)))$$
$$g([u, n], v) = c^2([\langle u, n \rangle, v] + \beta(v)[[u, n]], n))$$
$$g([v, n], u) = c^2([\langle v, n \rangle, u] + \beta(u)[[v, n]], n))$$

Substituting the above into (4.7), we find

$$2 g(\nabla u n, v) = n(c^2(\langle u, v \rangle - \beta(u) \beta(v))) - 2 c^3(\langle A(u), v \rangle - 2 c^2(\nabla_{\nabla \beta T}(u), v)$$
$$- c^2(\nabla_n \beta) \beta(v) - c^2 \beta(u)(\nabla_n \beta)(v) + c^3(A(\beta^3) + c\bar{Z}, u) \beta(v)$$

(4.8)
$$+ c^3 \beta(\beta^3)(A(\beta^3) + c\bar{Z}, v).$$

From (4.8), assuming $g(\nabla u n, v) = (\mathcal{D}(u), v)$ and using Lemma 2.3, we get

(4.9)
$$-2 c^4 A^g(u) = 2 \mathcal{D}(u) + c^{-2}(2 \mathcal{D}(u), \beta^2) \beta^2,$$

where $\mathcal{D} : T \mathcal{F} \to T \mathcal{F}$ is a linear operator, and

$$2 \mathcal{D}(u) = n(c^2(\langle u, v \rangle - \beta(u) \beta^v) - 2 c^3(\langle A(u), v \rangle - 2 c^2(\nabla_{\nabla \beta T}(u), v)$$
$$- c^2(\nabla_n \beta) \beta(v) - c^2 \beta(u)(\nabla_n \beta)(v) + c^3(A(\beta^3) + c\bar{Z}, u) \beta^v$$

(4.10)
$$+ c^3 \beta(\beta^3)(A(\beta^3) + c\bar{Z}).$$

From (4.10) we get

(4.9) - (4.11) we obtain

$$c^4 A^g = \bar{A} - c^{-1}(N - c^{-1} \beta^3) c_1^{-1} (\nabla_n \beta) \nabla_{\nabla \beta T}$$
$$- \frac{1}{2} c^{-2} ((N - \beta^3) c \beta^v - 2 c^{-1}(\nabla_{\nabla \beta T}(\beta^3)) - \nabla_{\nabla_{\nabla \beta T}} \beta^3 + c \bar{Z} + c (\bar{Z}, \beta^3) \beta^v$$

(4.12)
$$- \bar{A}(\beta^3) + \bar{A}(\beta^3, \beta^v) \beta^v \beta^v + \frac{1}{2} (\nabla_{\nabla_{\nabla \beta T}} \beta^3 - c \bar{Z} - \bar{A}(\beta^3)) \beta^v.$$

From the above the expected (4.5) - (4.6) follow. \hfill \Box

**Corollary 4.2.** Let $\beta(N) = 0$. If $\|\beta\|_a = \text{const}$ then on $T \mathcal{F}$ we have

$$c^4 A^g = \bar{A} - c^{-1} (\nabla_{\nabla \beta T}) + \frac{1}{2} (\nabla_{\nabla_{\nabla \beta T}} \beta^3 - c \bar{Z} - \bar{A}(\beta^3) \beta^v \beta^v + \frac{1}{2} (\nabla_{\nabla_{\nabla \beta T}} \beta^3 - c \bar{Z} + \bar{A}(\beta^3) \beta^v \beta^v.$$
If, in particular, $\nabla \beta = 0$ (i.e., $F$ is a Berwald structure) then

$$ cA^g = \bar{A} - \frac{1}{2} \left( \bar{A}(\beta^2)^{1/2} + cZ \right) \otimes \beta + \frac{1}{2} c^{-2} \left( \bar{A}(\beta^2)^{1/2} - cZ - c \langle Z, \beta^2 \rangle \beta^2 \right) \otimes \beta^2. $$

### 4.2 The Riemann curvature of $g$ and $a$

In this section we study relationship between Riemann curvature of two metrics, $g$ and $a$, on a Randers space.

**Proposition 4.3.** For a codimension-one foliation of $M$ with Riemannian metrics $g$ and $a$ we have

$$ Z = c^{-2} \bar{Z} - c^{-3} \nabla^\top c + c^{-4} \beta(\bar{Z} - c^{-1} \nabla^\top c) \beta^2, $$

$$ C_n^g = c^{-2} \bar{C} + c^{-4}(\beta \circ \bar{C}) \otimes \beta^2, $$

where

$$ 2 \bar{C} = \text{Sym}(\beta \otimes \bar{Z}) + c^{-3}(c \beta(\bar{Z}) - 2 \beta^2(c) - n(c))(I_m - \beta \otimes \beta^2) - c^{-1} \text{Sym}(\beta \otimes \nabla^\top c) + c^{-1}(\beta^2(c) + n(c))(I_m - 3 \beta \otimes \beta^2). $$

We also have

$$ \langle \nabla_u \bar{Z}, v \rangle = \langle \nabla_v \bar{Z}, u \rangle, \quad g(\nabla_u \bar{Z}, v) = g(\nabla_v \bar{Z}, u) \quad (u, v \in TF), $$

$$ R_N = (\text{Def}_Z)_{TF}^+ + \nabla_N A - A^2 - \bar{Z} \otimes \bar{Z}, \quad R_u^g = (\text{Def}_Z)_{TF}^+ + \nabla_u A - A^2 - Z^2 - Z^2 \otimes Z. $$

**Proof:** Extend $X \in T_p F$ at a point $p \in M$ onto a neighborhood of $p$ with the property $(\nabla_Y X)^\top = 0$ for any $Y \in T_p M$. By the well-known formula for the Levi-Civita connection, we obtain at $p$:

$$ g(Z, X) = g([X, \nu], \nu). $$

Then, using the equalities $\nu = c^{-1}N - c^{-2}\beta^2$ and $[X, fY] = X(f)Y + f[X, Y]$, we calculate

$$ g([X, \nu], \nu) = c^{-4}g(cN, \beta^2) - c^{-3}X(c)g(N, N) + c^{-2}g([X, N], N) - c^{-3}g([X, N], \beta^2). $$

Note that

$$ [X, N] = \nabla_X N - \nabla_N X = -\bar{A}(X) - \langle \nabla_N X, N \rangle N = -\bar{A}(X) + \langle Z, X \rangle N $$

and $N = cv + c^{-1}\beta^2$. Then, by Lemma 2.2 and the equalities

$$ g(\beta^4, \beta^2) = c^2((\beta^2, \beta^2) - (\beta^2)^2) = c^2(1 - c^2), $$

$$ g(N, \beta^2) = g(cv + c^{-1}\beta^2, \beta^2) = c^{-1}g(\beta^2, \beta^2) = c^3(1 - c^2), $$

$$ g(N, N) = g(cv + c^{-1}\beta^2, cv + c^{-1}\beta^2) = c^2 + c^{-2}g(\beta^2, \beta^2) = c^2(2 - c^2), $$

we obtain

$$ g([X, N], N) = -\langle \bar{A}(\beta^2), X \rangle + \langle \bar{Z}, X \rangle g(N, N) = c^2((2 - c^2)Z - c\bar{A}(\beta^2), X), $$

$$ g([X, N], \beta^2) = -\langle \bar{A}(\beta^2), X \rangle + \langle \bar{Z}, X \rangle g(N, \beta^2) = c^3((1 - c^2)Z - cA(\beta^2), X). $$
Hence,
\[ g(Z, X) = -c^{-1} X(c) + \langle \tilde{Z}, X \rangle = (\tilde{Z} - c^{-1} \nabla_c, X). \]

By Lemma 2.3, we get (4.14). From (4.2)–(4.3), (4.14) and a bit of help from Maple program we find
\[
2 C_n(u, v, Z) = \langle \tilde{Z}, u \rangle \beta(v) + \langle \tilde{Z}, v \rangle \beta(u) \\
+ c^{-3} (c \beta(\tilde{Z}) - 2 \beta^2(c) - n(c))(\langle u, v \rangle - \beta(u) \beta(v)) \\
- c^{-1} (u(c) \beta(v) + v(c) \beta(u)) + c^{-1} (\beta^2(c) + n(c))(\langle u, v \rangle - 3 \beta(u) \beta(v)).
\]

Using \( g(C_n^a(u, v) = \langle \tilde{C}(u), v \rangle \), where \( C_n^a \) is g-dual to \( C_n(\cdot, \cdot, \nabla n n) \), and
\[
2 \tilde{C}(u) = (\tilde{Z}, u) \beta^2 + \beta(u) \tilde{Z} + c^{-3} (c \beta(\tilde{Z}) - 2 \beta^2(c) - n(c))(u - \beta(u) \beta^2) \\
- c^{-1} (u(c) \beta^2 + \beta(u) \nabla^\top c) + c^{-1} (\beta^2(c) + n(c))(u - 3 \beta(u) \beta^2),
\]
we apply Lemma 2.3 to get (4.15).

We shall prove (4.16) and (4.17) for \( a \). It is sufficient to show that
\[
(4.18) \quad \langle \tilde{R}(u, N)N, v \rangle = \langle \nabla N \tilde{A} - \tilde{A}^2(u), v \rangle - \langle \tilde{Z}, u \rangle \langle \tilde{Z}, v \rangle + \langle \nabla_u \tilde{Z}, v \rangle, \quad u, v \in TM.
\]

Since the left hand side of (4.18) is symmetric, we obtain \( \langle \nabla_u \tilde{Z}, v \rangle = \langle \nabla_u \tilde{Z}, u \rangle \), see (4.17)1 and (4.16)1. Indeed,
\[
\langle \tilde{R}(u, N)N, v \rangle = \langle \nabla_u \nabla N N, v \rangle - \langle \nabla N \nabla_u N, v \rangle - \langle \nabla \nabla_u N - \nabla N u, N, v \rangle \\
= \langle \nabla_u \tilde{Z}, v \rangle + \langle \nabla N (\tilde{A}(u)), v \rangle - \langle \tilde{A}^2(u), v \rangle + \langle \nabla_u (\nabla N u, N), N, v \rangle - \langle \tilde{A}(\nabla_u N, u), v \rangle \\
= \langle \nabla N \tilde{A} - \tilde{A}^2(u), v \rangle - \langle \tilde{Z}, u \rangle \langle \tilde{Z}, v \rangle + \langle \nabla_u \tilde{Z}, v \rangle,
\]
that completes the proof of (4.18). The proof of (4.16)2 and (4.17)2 (for the metric \( g \)) is similar.

By (4.15), the equality \( C_n^a = 0 \) is independent of the condition \( \nabla \beta = 0 \). Moreover, we have the following.

**Corollary 4.4.** Let \( m > 3 \) and \( c = \text{const.} \) Then \( C_n^a = 0 \) if and only if \( \tilde{Z} = 0 \).

**Proof.** By our assumptions, \( \tilde{C} = \frac{1}{2} \text{Sym}(\beta \otimes \tilde{Z}) + \frac{1}{2} c^{-2} \beta(\tilde{Z})(I_m - \beta \otimes \beta^2) \). Hence, \( C_n^a = 0 \) reads
\[
\beta(\tilde{Z}) I_m = \beta(\tilde{Z}) \beta \otimes \beta^2 - c^2 \text{Sym}(\beta \otimes \tilde{Z}) - 2 (\beta \circ \tilde{C}) \circ \beta^2.
\]
Since the matrix \( \beta(\tilde{Z}) I_m \) is conformal, while the matrix in the right hand side of above equality has the form \( \omega \otimes \beta^2 - c^2 Z^\perp \beta \otimes \beta \) and rank \( \leq 3 \), for \( m > 3 \) we obtain
\[
\beta(\tilde{Z}) = 0, \quad \text{Sym}(\beta \otimes \tilde{Z}) + 2 c^{-2} (\beta \circ \tilde{C}) \circ \beta^2 = 0.
\]
By the first condition, \( Z \perp \beta^2 \); thus, the second condition yields \( \tilde{Z} = 0 \) (that is, \( F \) is a Riemannian foliation for the metric \( a \)) and \( \tilde{C} = 0 \). The converse claim follows directly from (4.15) and the definition of \( \tilde{C} \).

**Remark 4.1.** In [15] and [5] one may find coordinate presentations of \( R_y \) through \( \tilde{R}_y \) for all \( y \in TM \). For example, if \( \nabla \beta = 0 \) (i.e., \( F \) is a Berwald structure) then \( R_y(u) = \tilde{R}_y(u) \) for all \( u \). Alternative formulas with relationship between \( R_y \) and \( \tilde{R}_y \) follow from (4.17), where \( A^\flat \) and \( Z \) are expressed using \( A \) and \( \tilde{Z} \) given in Propositions 4.1 and 4.3.
4.3 Around the Reeb and Brito-Langevin-Rosenberg formula

Based on (3.13) and (3.21), one may produce a sequence of similar formulae for Randers spaces. We will discuss first two of them (i.e., $k = 1, 2$).

**Remark 4.2.** In [10], G. Reeb proved that the total mean curvature of the leaves of a codimension-one foliation on a closed Riemannian manifold equals zero. Note that $\text{Tr} \text{Def}_\beta = \int \nabla^\beta + \beta(Z)$, where $Z = \nabla_N N$ is the curvature vector of $N$-curves for the metric $a$.

Using notations of Appendix, we find from (4.6),

$$
\beta(U_1) = -\frac{2 - c^2}{2c} N(c) - \frac{1}{2} \beta^1(c) - \frac{2 - c^2}{2c} \beta(\mathcal{Z}), \quad \beta(U_2) = -\frac{1}{2} (c N - \beta^2)(c) - \frac{1}{2} c \beta(\mathcal{Z}).
$$

Hence,

$$
\beta(U_1) + \beta(U_2) = -c^{-1}(N(c) + \beta(\mathcal{Z})).
$$

Tracing (4.5), we get

$$
c \sigma_1(A^p) = \sigma_1(\mathcal{A}) - (m + 1) c^{-1} N(c) + m c^{-2} \beta^1(c) + c^{-1} \text{div} \beta^2.
$$

The volume forms of $g$ and $a$ obey $dV_g = e^{m+2} dV_a$, see (4.4). Using the Reeb formula for metric $g$,

$$
\int_M \sigma_1(A^p) dV_g = 0,
$$

the equality $\text{div}(e^m \beta^2) = e^m \text{div} \beta^2 + \beta^1(e^m)$ and the Divergence Theorem, we get

$$
\int_M (e^{m+1} \sigma_1(\mathcal{A}) - N(e^{m+1})) dV_a = 0,
$$

which for $\beta = 0$ is the Reeb formula for metric $a$. Remark that (4.19) is a particular case of a general formula for any $f \in C^2(M)$, see [12, Lemma 2.5]:

$$
\int_M (f \sigma_1(\mathcal{A}) - N(f)) dV_a = 0.
$$

The next results concern Brito-Langevin-Rosenberg type formulas for foliated Randers spaces.

The *Newton transformations* $T_k(A) (0 \leq k \leq m)$ of an $m \times m$ matrix $A$ (see [12]) are defined either inductively by $T_0(A) = I_m$, $T_k(A) = \sigma_k(A)I_m - AT_{k-1}(A)$ ($k \geq 1$) or explicitly as

$$
T_k(A) = \sigma_k(A)I_m - \sigma_{k-1}(A) A + \ldots + (-1)^k A^k, \quad 0 \leq k \leq m,
$$

and we have $T_k(\lambda A) = \lambda^k T_k(A)$ for $\lambda \neq 0$. Observe that if a rank-one matrix $A := U \otimes \beta$ (and similarly for $A := \omega \otimes \beta^2$) has zero trace, i.e., $\beta(U) = 0$, then

$$
A^2 = U(\beta^2)^t \cdot U(\beta^2)^t = U \beta(U) (\beta^2)^t = \beta(U) A = 0.
$$

Note that for $c = \text{const}$ we have, see (4.15), $C_n^4 = c^{-2} \mathcal{C} + c^{-4} (\beta \circ \mathcal{C}) \otimes \beta^2$, where $C_n^4 = c^2 C_n^4$ and

$$
2 \mathcal{C} = \text{Sym}(\beta \otimes \mathcal{Z}) + c^{-2} \beta(\mathcal{Z})(I_m - \beta \otimes \beta^2).
Theorem 4.5. Let \((M^{m+1}, \alpha + \beta)\) be a codimension-one foliated closed Randers space with constant sectional curvature \(K\) of \(a\). If a nonzero vector field \(\beta^2 \in \Gamma(TF)\) obeys \(\nabla \beta = 0\), then \(K=0\) and for \(1 \leq k \leq m\) we have
\[
\int_M \left( \sum_{j > 0} \sigma_{k-j,j}(\tilde{A}, c C^2_\nu) + (T_{k-1}(\tilde{A} + c C^2_\nu)(\beta^2), U_1) 
+ \langle T_{k-1}(\tilde{A} + c C^2_\nu + U_1^2 \otimes \beta^2)(U_2), \beta^2 \rangle \right) dV_a = 0,
\]
where \(U_1 = \frac{1}{2} e^{-2}(\tilde{A}(\beta^2) - c \tilde{Z}), U_2 = -\frac{1}{2}(\tilde{A}(\beta^2) + c \tilde{Z})\). Moreover, if \(m > 3\) and \(\tilde{Z} = 0\) then
\[
\int_M \left( (e^{-2}T_{k-1}(\tilde{A}) - T_{k-1}(\tilde{A} + \frac{1}{2} e^{-2} \tilde{A}(\beta^2) \otimes \beta^2)))(\tilde{A}(\beta^2)), \beta^2 \right) dV_a = 0.
\]

Proof. By our assumptions, \(c = \text{const}\) and \(\tilde{R}(x,y)z = \tilde{R}(\langle y, z \rangle x - \langle x, z \rangle y)\). Hence, on \(TF\)
\[
\tilde{R}_N = \tilde{K}I_m, \quad \tilde{R}_{\beta^2} = (1 - c^2)\tilde{K}I_m, \quad \tilde{R}(\cdot, N)\beta^2 = 0.
\]
If \(\nabla \beta = 0\) then \(\tilde{R}(U, \beta^2, \beta^2, U) = 0\) and \(\tilde{K}(U \wedge \beta^2) = 0\) for all \(U \perp \beta^2\); hence, in our case, \(\tilde{K} = 0\). By Remark 4.1, \(R_y = R_y\) for all \(y \in TM_0\); hence, \(R_y = 0\). Since \(\nabla \beta^2 = 0\), we obtain \(\beta(\tilde{Z}) = 0\) and \(\langle \tilde{A}(\beta^2), \beta^2 \rangle = 0\):
\[
\langle \beta^2, \tilde{Z} \rangle = \langle \beta^2, \nabla_N N \rangle = -\langle \nabla_N \beta^2, N \rangle = 0,
\]
\[
\langle \tilde{A}(\beta^2), \beta^2 \rangle = -\langle \beta^2, \nabla_{\beta^2} N \rangle = \langle \nabla_{\beta^2} \beta^2, N \rangle = 0.
\]
By (3.9) and Corollary 4.2,
\[
c \tilde{A} = c A^\# + c C^2_\nu = \tilde{A} + c C^2_\nu + A_1 + A_2,
\]
where \(A_1 = U_1^2 \otimes \beta^2\) and \(A_2 = U_2 \otimes \beta\) are rank \(\leq 1\) matrices (since \(\langle U_1, \beta^2 \rangle = 0\)). By Corollary 5.5 of Appendix, we have
\[
c^k \sigma_k(A) = \sigma_k(\tilde{A}) + \sum_{j > 0} \sigma_{k-j,j}(\tilde{A}, c C^2_\nu) + U_1(T_{k-1}(\tilde{A} + c C^2_\nu)(\beta^2)) 
+ \beta(T_{k-1}(\tilde{A} + c C^2_\nu + A_1)(U_2)).
\]
Recall that \(dV_F = e^{m+2} dV_\alpha\), see (4.4). Comparing (3.21) (when \(K = 0\)) with
\[
\int_M \sigma_k(\tilde{A}_p) dV_a = 0,
\]
we find (4.20). By Corollary 4.4, if \(m > 3\), \(\tilde{Z} = 0\) then \(C^2_\nu = 0\); hence, (4.20) yields (4.21).

Example 4.3. For \(k = 1\), (4.20) yields the Reeb type formula
\[
\int_M \sigma_1(C^2_\nu) dV_a = 0.
\]

Corollary 4.6. Let \((M^{m+1}, \alpha + \beta), m > 3\), be a codimension-one foliated closed Randers space with constant sectional curvature \(K\) of \(a\). If \(\tilde{Z} = 0\) and a nonzero vector field \(\beta^2 \in \Gamma(TF)\) obeys \(\nabla \beta = 0\) then \(K = 0\) and \(\tilde{A}(\beta^2) = 0\) at any point of \(M\). If, in addition, \(F\) is totally umbilical \((\tilde{A} = H \cdot I_m)\) then \(F\) is totally geodesic.
Proof. For $k = 2$, the integrand in (4.21) reduces to $\frac{c^2-1}{4c^2} \| \bar{A}(\beta') \|^2$. Thus, when $c \neq 1$, the claim follows.

Nevertheless, we will give alternative proof with use of integral formula (3.15). Our Randers space $(M, \alpha + \beta)$ is now Berwald. For the rank 1 matrices $A_1 = U_1^1 \otimes \beta^1$ and $A_2 = U_2 \otimes \beta$, where $U_1 = \frac{1}{2} c^{-2} \bar{A}(\beta^2)$ and $U_2 = -\frac{1}{2} \bar{A}(\beta^2)$ and $\langle \bar{A}(\beta^2), \beta^2 \rangle = 0$, see (4.13) with $\bar{Z} = 0$, we have

$$\begin{align*}
\text{Tr}(A_1A_2) &= \langle U_1, U_2 \rangle \beta(\beta^2) = \frac{c^2-1}{4c^2} \| \bar{A}(\beta^2) \|^2, \\
\text{Tr}(\bar{A}A_1) &= \langle U_1, \bar{A}(\beta^2) \rangle = \frac{1}{2c^2} \| \bar{A}(\beta^2) \|^2, \\
\text{Tr}(\bar{A}A_2) &= \langle U_2, \bar{A}(\beta^2) \rangle = -\frac{1}{2} \| \bar{A}(\beta^2) \|^2.
\end{align*}$$

Thus, $\text{Tr}(A_1A_2 + \bar{A}A_1 + \bar{A}A_2) = \frac{1-c^2}{4c^2} \| \bar{A}(\beta^2) \|^2$. By the identity for square matrices

$$\sigma_2(\sum_i A_i) = \frac{1}{2} \text{Tr}^2(\sum_i A_i) - \frac{1}{2} \text{Tr}(\sum_i A_i)^2 = \sum_i \sigma_2(A_i) + \sum_{i<j} ((\text{Tr} A_i)(\text{Tr} A_j) - \text{Tr}(A_iA_j)),$$

and $\sigma_2(A_1) = \sigma_2(A_2) = 0$, by the above and since $c A = c A^g = \bar{A} + A_1 + A_2$, we get

$$c^2 \sigma_2(A) = c^2 \sigma_2(A^g) = \sigma_2(\bar{A}) + \frac{1}{4} (c^{-2} - 1) \| \bar{A}(\beta^2) \|^2_a.$$

From the integral formulae, (3.20), for $F$ and for Riemannian metric $a$,

$$\int_M \sigma_2(\bar{A}) \, dV_a = 0, \quad \int_M \sigma_2(A) \, dV_F = 0,$$

where the volume forms are related by $dV_F = e^{m+2} dV_a$, see (2.6), we find that

$$(c^{-2} - 1) \int_M \| \bar{A}(\beta^2) \|^2_a \, dV_a = 0.$$

Since $c \neq 1$ (for $\beta \neq 0$), we obtain $\bar{A}(\beta^2) = 0$. \hspace{1cm} \square

Similar integral formulae exist for codimension one totally umbilical (i.e., $\bar{A} = H I_m$, where $H = \frac{1}{m+1} \text{Tr} A$) and totally geodesic foliations. Notice that non-flat closed Riemannian manifolds of constant curvature do not admit such foliations.

**Corollary 4.7.** Let $F$ be a codimension-one totally umbilical (for the metric $a$) foliation of a closed Randers space $(M^{n+1}, \alpha + \beta)$ with constant sectional curvature $K$ of $a$. If a nonzero vector field $\beta^2 \in \Gamma(TF)$ obeys $\nabla \beta^2 = 0$ then $K = 0$, $F$ is totally geodesic and for $1 \leq k \leq m$ (for $k = 1$, see also Example 4.3) we have

$$\begin{align*}
\int_M \left( c^k \sigma_k(C_a^k) - \frac{1}{2} c^{-1} \langle T_{k-1}(c C_a^k)(\beta^2), \bar{Z} \rangle \\
- \frac{c}{2} \langle T_{k-1}(c C_a^k) - \frac{1}{2} c^{-1} \bar{Z} \otimes \beta^2)(\bar{Z}, \beta^2) \rangle \right) \, dV_a = 0.
\end{align*}$$

(4.23)

Proof. Since $\langle \bar{A}(\beta^2), \beta^2 \rangle = 0$ (see the proof of Theorem 4.5), we obtain $\bar{H} = 0$. Thus, (4.23) follows from (4.20) with $\bar{A} = 0$ and $\beta(\bar{Z}) = 0$. \hspace{1cm} \square

**Remark 4.4.** In results of this section, a closed manifold can be replaced by a complete manifold of finite volume with bounded geometry, see conditions (3.17).
5 Appendix: Invariants of a set of matrices

Here, we collect the properties of the invariants \( \sigma(\lambda_1, \ldots, \lambda_k) \) of real matrices \( A_i \) that generalize the elementary symmetric functions of a single symmetric matrix \( A \). Let \( S_k \) be the group of all permutations of \( k \) elements. Given arbitrary quadratic \( m \times m \) real matrices \( A_1, \ldots, A_k \) and the unit matrix \( I_m \), one can consider the determinant \( \det(I_m + t_1 A_1 + \ldots + t_k A_k) \) and express it as a polynomial of real variables \( t = (t_1, \ldots, t_k) \). Given \( \lambda = (\lambda_1, \ldots, \lambda_k) \), a sequence of nonnegative integers with \( |\lambda| := \lambda_1 + \ldots + \lambda_k \leq m \), we shall denote by \( \sigma(\lambda) \) its coefficient at \( t^\lambda = t_1^{\lambda_1} \cdots t_k^{\lambda_k} \):

\[
\det(I_m + t_1 A_1 + \ldots + t_k A_k) = \sum_{|\lambda| \leq m} \sigma(\lambda) t^\lambda.
\]

Evidently, the quantities \( \sigma(\lambda) \) are invariants of conjugation by \( GL(m) \)-matrices:

\[
\sigma(\lambda) = \sigma(QA Q^{-1}, \ldots, QA_k Q^{-1})
\]

for all \( A_i \)'s, \( \lambda \)'s and nonsingular \( m \times m \) matrices \( Q \). Certainly, \( \sigma(\lambda) \) (for a single symmetric matrix \( A \)) coincides with the \( i \)-th elementary symmetric polynomial of the eigenvalues \( \{k_j\} \) of \( A \).

In the next lemma, we collect properties of these invariants.

**Lemma 5.1** (see [13]). For any \( \lambda = (\lambda_1, \ldots, \lambda_k) \) and any \( m \times m \) matrices \( A_i, A \) and \( B \) one has

\[
(I) \quad \sigma(\lambda_0, A_2, \ldots, A_k) = 0 \text{ if } \lambda_1 > 0 \text{ and } \sigma(\lambda_0, A_1, \ldots, A_k) = \sigma(\lambda) (A_2, \ldots, A_k) \quad \text{where } \lambda = (\lambda_2, \ldots, \lambda_k),
\]

\[
(II) \quad \sigma(\lambda_0, A_{s(1)}, \ldots, A_{s(k)}) = \sigma(\lambda_0 A_1, \ldots, A_k), \quad \text{where } s \in S_k \text{ and } \lambda \circ s = (\lambda_{s(1)}, \ldots, \lambda_{s(k)}),
\]

\[
(III) \quad \sigma(A_m, A_2, \ldots, A_k) = (m-|\lambda|) \sigma(\lambda_0) \sigma(\lambda),
\]

\[
(IV) \quad \sigma(A_1, A, A_3, \ldots, A_k) = (\lambda_1 + \lambda_2, \lambda_3, \ldots, \lambda_k) \quad \text{where } \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k),
\]

\[
(V) \quad \sigma(A_1, A, A_2, \ldots, A_k) = \sum_{i=1}^{k} \sigma(A_1, A_2, \ldots, A_k) + \sigma(A, A_1, A_2, \ldots, A_k)
\]

The invariants defined above can be used in calculation of the determinant of a matrix \( B(t) \) expressed as a power series \( B(t) = \sum_{i=0}^{\infty} t^i B_i \). Indeed, if one wants to express \( \det(B(t)) \) as a power series in \( t \), then the coefficient at \( t^l \) depends only on the part \( \sum_{i \leq l} t^i B_i \) of \( B(t) \).

**Lemma 5.2** ([13]). If \( B(t), t \in \mathbb{R} \), is the \( m \times m \) matrix given by \( B(t) = \sum_{i=0}^{\infty} t^i B_i \), \( B_0 = I_m \) then

\[
\det(B(t)) = 1 + \sum_{k=1}^{\infty} \left( \sum_{\lambda, \|\lambda\| = k} \sigma(\lambda_0, B_1, \ldots, B_k) \right) t^k,
\]

where \( \|\lambda\| = \lambda_1 + 2\lambda_2 + \ldots + k\lambda_k \) for \( \lambda = (\lambda_1, \ldots, \lambda_k) \).

Since \( \det : M(m) \to \mathbb{R} \), \( M(m) \approx \mathbb{R}^{m^2} \) being the space of all \( m \times m \)-matrices, is a polynomial function, the series in (5.3) is convergent for all \( t \in (-r_0, r_0) \), where

\[
r_0 = \frac{1}{2} \limsup_{k \to \infty} \|B_k\|^{1/k}
\]

is the radius of convergence of the series \( B(t) \).

By the First Fundamental Theorem of Matrix Invariants, see [6], all the invariants \( \sigma_\lambda \) can be expressed in terms of the traces of the matrices involved and their products.
Lemma 5.3 ([13]). For arbitrary matrices $B, C$ and $k, l > 0$ we have
\[
\sigma_{k,l}(B, C) = \sigma_k(B) \sigma_l(C) - \sum_{i=1}^{\min(k,l)} \sigma_{k-l+i,i}(B, C, BC).
\]
In particular, for $l = 1$, it follows that
\[
(5.4) \quad \sigma_{k,1}(B, C) = \sum_{i=0}^{k} (-1)^i \sigma_{k-i,1}(B) \text{Tr}(B^iC) = \text{Tr}(T_k(B)C).
\]

Lemma 5.4. Let $A, C$ be $m \times m$ matrices and rank $A = 1$. Then
\[
(5.5) \quad \sigma_k(C + A) = \sigma_k(C) + \text{Tr}(T_{k-1}(C)A).
\]

Proof. There exists a nonsingular matrix $Q$ such that $\tilde{A} = QAQ^{-1}$ has one nonzero element, $\tilde{a}_{ii} \neq 0$ for some $i$ (the simplest rank one matrix). By (5.2), $\sigma_{k,l}(\tilde{C}, \tilde{A}) = \sigma_{k,l}(C, A)$ where $\tilde{C} = QCQ^{-1}$. By Laplace’s formula (which expresses the determinant of a matrix in terms of its minors), det$(I_m + t\tilde{C} + s\tilde{A})$ is a linear function in $s \in \mathbb{R}$; hence, see (5.1), $\sigma_{k,l}(\tilde{C}, \tilde{A}) = 0$ for $l > 1$. By the above, $\sigma_{k,l}(C, A) = 0$ for $l > 1$ and all $k$. Using the identity, see [13],
\[
(5.6) \quad \sigma_k(C_1 + C_2) = \sum_{i=0}^{k} \sigma_{k-i,i}(C_1, C_2),
\]
we find that
\[
\sigma_k(C + A) = \sigma_k(C) + \sigma_{k-1,1}(C, A).
\]

By (5.4), $\sigma_{k-1,1}(C, A) = \text{Tr}(T_{k-1}(C)A)$ and (5.5) follows.

Corollary 5.5. Let $C, D, A_i$ be $m \times m$ matrices and rank $A_i = 1$ ($1 \leq i \leq s$). Then
\[
\sigma_k(C + D + A_1 + \ldots + A_s) = \sigma_k(C) + \sum_{j>0} \sigma_{k-j,j}(C, D) + \text{Tr}(T_{k-1}(C + D)A_1) + \ldots + \text{Tr}(T_{k-1}(C + D + A_1 + \ldots + A_{s-1})A_s).
\]

Proof. This follows from Lemma 5.4 and (5.4). For $s = 1$, we obtain
\[
\sigma_k(C + D + A_1) \overset{(5.5)}{=} \sigma_k(C + D) + \text{Tr}(T_{k-1}(C + D)A_1)
\overset{(5.6)}{=} \sigma_k(C) + \sum_{j>0} \sigma_{k-j,j}(C, D) + \text{Tr}(T_{k-1}(C + D)A_1).
\]

Then, by induction for $s$, (5.7) follows.

Let $C_i$ and $P_j$ be $m$-vectors (columns) and $I_m$ the identity $m$-matrix and $1 \leq i \leq j \leq m$. Note that $C_i P_j$ are $m \times m$-matrices of rank 1 with
\[
\sigma_1(C_i P_j) = C_i^t P_j, \quad \sigma_2(C_i P_j) = 0, \quad (I_m + C_i P_j)^{-1} = I_m - (1 + C_i P_j)^{-1} C_i P_j.
\]

Lemma 5.6. We have $\det(I_m + \sum_{i=1}^{k} C_i P_i) = 1 + \det(\{C_i P_j\}_{1 \leq i, j \leq k})$. For example,
\[
\det(I_m + C_1 P_1) = 1 + C_1^t P_1, \quad \det(I_m + C_1 P_1 + C_2 P_2) = 1 + C_1^t P_1 + C_2^t P_2 + C_1^t P_1 C_2^t P_2 - C_1^t P_2 C_1^t P_1, \quad \text{and so on.}
\]
References


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