Existence of free boundaries using the mean curvature
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Abstract. This paper deals with a free boundary problem for both Laplacean and p-Laplacian operators. We begin by proving the existence of solution (which is of class $C^2$) for the associated shape optimization problem. Then, after performing the shape derivative we will present two approaches in order to get sufficient conditions of existence of the free boundaries. The first one needs the use of some maximum principle. The second one uses the monotonicity of the mean curvature and can be applied for general divergence operators.

Key words: Dirichlet problem; free boundaries; Laplacian; p-Laplacian; mean curvature; minimal surface; shape derivative; shape optimization.

1 Introduction

Let $\mu$ be a positive measure with compact support $K_\mu$ (with a nonempty interior) and let $k > 0$ be a parameter. We look for an open and bounded set $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) such that

1. $\Omega$ strictly contains $K_\mu$ and
2. there exists a function $u_\Omega$, satisfying the following overdetermined problem

$$(FB) \left\{ \begin{array}{ll} - \text{div}(A(|\nabla u_\Omega|)\nabla u_\Omega) = \mu & \text{in } \Omega, \\
 u_\Omega = 0 & \text{on } \partial \Omega, \\
 |\nabla u_\Omega| = k & \text{on } \partial \Omega \quad \text{(overdetermined condition).} \\
 \end{array} \right.$$ 

Imposing boundary conditions for both $u_\Omega$ and $|\nabla u_\Omega|$ on $\partial \Omega$ makes problem (FB) overdetermined, so that in general without any assumptions on data this problem has no solution. Notice that since $u_\Omega = 0$ on $\partial \Omega$ then $|\nabla u_\Omega| = -\frac{\partial u_\Omega}{\partial \nu}$, where $\nu$ is the outward normal vector to $\partial \Omega$. 

In the linear case, when $A = 1$ and the equation becomes $-\Delta u = \mu$, (FB) is called the quadrature surfaces free boundary problem and arises in many areas of physics (free streamlines, jets, Hele-show flows, electromagnetic shaping, gravitational problems etc.) It has been intensively studied from different points of view, by several authors. For more details about the methods used for solving this problem see the introduction in [12]. In [2] using the maximum principle together with the compatibility condition of the Neumann problem, the authors gave sufficient condition of existence for problem (FB). When $A(t) = t^{p-2}$ the equation becomes $-\Delta p \mu u = \mu$. As far as the authors know this problem still open. In [4] using essentially the Hopf’s comparison principle (see Lemma 2.5 below), the author gave a sufficient condition of existence for this problem. The purpose of the present paper is to put conditions on $\mu$ and $k$ in order to satisfy 1 and 2. Our approach here consists on solving the shape optimization problem associated to (FB). Then performing the shape derivative, we will get the overdetermined condition but not in the entire boundary of $\Omega$. To conclude, we will give two theorems. The proof of the first one needs the use of some maximum principle. For the second theorem, we will use the monotonicity of the mean curvature for the domains which are of class $C^2$. The outline of the paper is as follows. Section 2 contains some preliminary results. Section 4 is devoted to the shape optimization problems while some auxiliary results are stated and proved in Section 4. In Section 5, we state and prove the main theorems. Section 6 contains some concluding remarks.

2 Preliminaries

Let $D$ be an open ball of $\mathbb{R}^N$ ($N \geq 2$) which will contain all the sets we use in this paper.

**Definition 2.1.** Let $K_1$ and $K_2$ be two compact subsets of $D$. We call a Hausdorff distance of $K_1$ and $K_2$ (or briefly $d_H(K_1, K_2)$), the following positive number:

$$d_H(K_1, K_2) = \max [\rho(K_1, K_2), \rho(K_2, K_1)],$$

where $\rho(K_i, K_j) = \max_{x \in K_i} d(x, K_j) \quad i, j = 1, 2$ and $d(x,K_j) = \min_{y \in K_j} |x - y|$.

**Definition 2.2.** Let $\omega_n$ be a sequence of open subsets of $D$ and $\omega$ be an open subset of $D$. Let $K_n$ and $K$ be their complements in $\overline{D}$. We say that the sequence $\omega_n$ converges in the Hausdorff sense, to $\omega$ (or briefly $\omega_n \xrightarrow{H} \omega$) if

$$\lim_{n \to +\infty} d_H(K_n, K) = 0.$$

**Definition 2.3.** Let $\omega_n$ be a sequence of open subsets of $D$ and $\omega$ be an open subset of $D$. We say that the sequence $\omega_n$ converges in the compact sense, to $\omega$ (or briefly $\omega_n \xrightarrow{K} \omega$) if

- every compact subset of $\omega$ is included in $\omega_n$, for $n$ large enough, and
- every compact subset of $\overline{\omega}^c$ is included in $\overline{\omega}_n^c$, for $n$ large enough.
Definition 2.4. Let $\omega_n$ be a sequence of open subsets of $D$ and $\omega$ be an open subset of $D$. We say that the sequence $\omega_n$ converges in the sense of characteristic functions, to $\omega$ (or briefly $\omega_n \overset{L}{\rightarrow} \omega$) if $\chi_{\omega_n}$ converges to $\chi_\omega$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, $p \neq \infty$, ($\chi_\omega$ is the characteristic function of $\omega$).

Lemma 2.1. ([8], [18]) If $\omega_n$ is a sequence of open subsets of $D$, there exists a subsequence (still denoted by $\omega_n$) which converges, in the Hausdorff sense, to some open subset of $D$.

Definition 2.5. [3] Let $C$ be a compact convex set, the bounded domain $\omega$ satisfies C-GNP if

1. $\omega \supset \text{int}(C)$,
2. $\partial \omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray $\Delta_c$ such that $\Delta_c \cap \omega$ is connected, and
4. for every $x \in \partial \omega \setminus C$ the inward normal ray to $\omega$ (if exists) meets $C$.

Remark 2.6. If $\Omega$ satisfies the C-GNP and $C$ has a nonempty interior, then $\Omega$ is connected.

Theorem 2.2. If $\omega_n \in \mathcal{O}_C$, then there exists an open subset $\omega \subset D$ and a subsequence (again labeled $\omega_n$) such that (i) $\omega_n \overset{H}{\rightarrow} \omega$, (ii) $\omega_n \overset{K}{\rightarrow} \omega$, (iii) $\chi_{\omega_n}$ converges to $\chi_\omega$ in $L^1(D)$ and (iv) $\omega \in \mathcal{O}_C$.

For the proof of this theorem, see Theorem 3.1 in [3].

Proposition 2.3. Let $\{\omega_n, \omega\} \subset \mathcal{O}_C$ such that $\omega_n \overset{H}{\rightarrow} \omega$. Let $u_n$ and $u_\omega$ be respectively the solutions of $P(\omega_n, \mu)$ and $P(\omega, \mu)$. Then $u_n$ converges strongly in $H^1_0(D)$ to $u_\omega$ ($u_n$ and $u_\omega$ are extended by zero in $D$).

This proposition is proven for $N = 2$ or 3 (see Theorem 4.3 in [3]).

Definition 2.7. Let $C$ be a convex set. We say that an open subset $\omega$ has the C-SP, if

1. $\omega \supset \text{int}(C)$,
2. $\partial \omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray $\Delta_c$ such that $\Delta_c \cap \omega$ is connected, and
4. $\forall x \in \partial \omega \setminus C$ $K_x \cap \omega = \emptyset$, where $K_x$ is the closed cone defined by

$$\{ y \in \mathbb{R}^N : (y - x) \cdot (z - x) \leq 0, \forall \ z \in C \}.$$ 

Remark 2.8. $K_x$ is the normal cone to the convex hull of $C$ and $\{x\}$.

Proposition 2.4. $\omega$ has the C-GNP if and only if $\omega$ satisfies the C-SP.
For the proof of this proposition see Proposition 2.3 in [3].

**Lemma 2.5. (Hopf’s Comparison principle).** Let \( U \subset \mathbb{R}^N \) be open and bounded, and \( v_1, v_2 \in C^1(U) \), with \( \Delta_p v_1 \leq \Delta_p v_2 \). Then the following hold.

1. If \( v_1 \geq v_2 \) on \( \partial U \), then \( v_1 \geq v_2 \) in \( U \).

2. Suppose \( v_1 > v_2 \) in \( U \), \( v_1(x) = v_2(x) \) for some \( x \in \partial U \), \( \vert \nabla v_2 \vert \geq \gamma \) in \( U \) (for some \( \gamma > 0 \)), and \( U \) satisfies the interior sphere condition. Then \( \frac{\partial v_1}{\partial \nu}(x) > \frac{\partial v_2}{\partial \nu}(x) \), where \( \nu \) is the unit outward normal vector on \( \partial U \), at \( x \).

3. If \( v_1 \geq v_2 \) and \( v_1 \neq v_2 \) in \( U \), \( \vert \nabla v_2 \vert \geq \gamma \) in \( U \) (for some \( \gamma > 0 \)), then \( v_1 > v_2 \) in \( U \).

This lemma is proven in ([23], Lemma 3.2, Proposition 3.4.1, 3.4.2)

As in the linear case, to obtain a continuity result for the Dirichlet problem in the non linear case, we can use the compact convergence and the \( p \)-stability of the limit domain (we say that an open set \( \Omega \) is \( p \)-stable if for any \( u \in H^{1,p}(\mathbb{R}^N) \) such that \( u = 0 \) a.e. in \( \text{int}(\Omega^c) \), we get \( u|_{\Omega} \in H^{1,p}_0(\Omega) \)). Here, we will use the theorem (see below) obtained by Bucur and Trebeschi where they generalize the Sverak’s result [21].

In [7], the authors gave a compactness-continuity result for the solution of a non linear Dirichlet problems (in particular with the \( p \)-Laplacian operator) when the domain varies.

**Definition 2.9. (\( \gamma_p \)-convergence)*** We say that a sequence \( \Omega_n \) of open subsets of \( D \) \( \gamma_p \)-converges to \( \Omega \) if and only if for any \( u \in H^{-1,q}(D) \) \((\frac{1}{p} + \frac{1}{q} = 1) \) the solutions \( u_n \) of the Dirichlet problems \( P(\Omega_n, \mu) \) converges strongly in \( H^{1,p}(D) \), as \( n \to +\infty \), to the solution \( u_\Omega \) of \( P(\Omega, \mu) \), \( (u_n \) and \( u_\Omega \) are extended by zero to \( D \)).

Set
\[
\mathcal{O}_l(D) = \{ \omega \subseteq D \mid \sharp\omega^c \leq l \}
\]
where \( \sharp\omega^c \) denotes the number of connected components of the complement of \( \omega \).

**Theorem 2.6.** [7] Let \( N \geq p > N - 1 \). Consider \( \Omega_n \in \mathcal{O}_l(D) \) and assume \( \Omega_n \xrightarrow{H} \Omega \), then \( \Omega \in \mathcal{O}_l(D) \) and \( \Omega_n \, \gamma_p \)-converges to \( \Omega \).

**Remark 2.10.** If \( p > N \), any sequence of open sets which converge in the Hausdorff sense is \( \gamma_p \)-convergent.

**Corollary 2.7.** Assume that the convex \( C \) has a nonempty interior. If \( \Omega_n \in \mathcal{O}_C \) and \( \Omega_n \xrightarrow{H} \Omega \), then \( \Omega_n \, \gamma_p \)-converges to \( \Omega \).

**Proof.** If the interior of \( C \) is nonempty and \( \Omega_n \in \mathcal{O}_C \), according to Remark 2.6, \( \Omega_n \) is connected. Therefore \( \Omega_n \in \mathcal{O}_l(D) \). Now, if \( \Omega_n \xrightarrow{H} \Omega \), by the previous theorem \( \Omega_n \, \gamma_p \)-converges to \( \Omega \). \( \square \)
Theorem 2.8. Let $L$ be a compact subset of $\mathbb{R}^N$. Let $f_n$ be a sequence a function defined on $L$. We assume that the $f_n$ are of class $C^3$ and
\[
\frac{\partial f_n}{\partial x_i} \leq M, \quad \frac{\partial^2 f_n}{\partial x_i \partial x_j} \leq M, \quad \frac{\partial^3 f_n}{\partial x_i \partial x_j \partial x_k} \leq M,
\]
where $M$ is a strictly positive constant and is independent of $n$.

Define a sequence $\Omega_n$, by $\Omega_n = \{x \in L : f_n(x) > 0\}$ and suppose there exists $\alpha > 0$ such that $|f_n(x)| + |\nabla f_n(x)| \geq \alpha$ for all $x \in L$. If the $\Omega_n$ have the C-GNP, then there exists $\Omega$ of class $C^2$ and a subsequence (still denoted by $\Omega_n$) such that $\Omega_n$ converges in the compact sense, to $\Omega$.

3 Shape optimization problems

Up to now, $\mu = f$ where $f \in L^2(D)$ for $p = 2$ or $f \in L^\infty(D)$ for $p \neq 2$.

In [12],[20] (for $p = 2$) or in [14] (for $p \neq 2$) by using the moving plane method [11], the authors showed that if the problem (FB) admits a solution $(\Omega, u_\Omega)$ such that $\Omega$ is of class $C^2$ and $u_\Omega \in C^2(\Omega \setminus K_\mu) \cap C^1(\Omega)$, then all the inward normals at the boundary $\partial \Omega$ of $\Omega$ meet $C$ (the convex hull of $K_\mu$). Since we relate the existence of a solution for Problem (FB) to the existence of a minimum of some shape optimization problem, it is natural to solve this one in a class of domains with this geometric normal property.

Using the shape derivative, the problem (FB) can be seen as the Euler equation of the following problem of minimization, e.g. [22] and [17]:

\((OP)\) Find $\Omega \in \mathcal{O}_C$ such that $J(\Omega) = \min_{\omega \in \mathcal{O}_C} J(\omega)$,

where
\[
\mathcal{O}_C = \{\omega \subset D : \omega \text{ satisfies C-GNP}\}
\]
and
\[
J(\omega) = \int_\omega \left( \frac{1}{p} |\nabla u_\omega|^p - f u_\omega + \frac{k^p}{p} \right) dx
\]
with $u_\omega$ the solution of the Dirichlet problem.
\[
P(\omega, f) \left\{ \begin{array}{ll} -\Delta_p u_\omega = f & \text{in } \omega, \\ u_\omega = 0 & \text{on } \partial \omega. \end{array} \right.
\]

3.1 Existence of the minima

Theorem 3.1. There exists $\Omega \in \mathcal{O}_C$ which minimizes the functional $J$ on $\mathcal{O}_C$. $\Omega$ is of class $C^2$.

We will give the proof in the case where $p \neq 2$. For $p = 2$, just replace in the proof the Hopf’s comparison principle by the maximum principle. The continuity result thanks to Proposition 2.3 from above.
Proof. Using the variational formulation of the Dirichlet problem $P(\omega, f)$, we get
\[ \int_\omega |\nabla u_\omega(x)|^p dx = \int_\omega f u_\omega. \]
If $u_D$ denotes the solution of the Dirichlet problem $P(D, f)$, by the Hopf's comparison principle (see Lemma 2.5 part 1.), $0 \leq u_\omega \leq u_D$ so
\[ J(\omega) = -\frac{p-1}{p} \int_\omega f u_\omega + \frac{k^p}{p} \int_\omega dx \geq -\frac{p-1}{p} \int_D f u_D \]
and $\inf J$ exists. Let $\Omega_n$ be a minimizing sequence in $O_C$. (one can choose it as in Theorem 2.8 from above). Since $int(C) \subset \Omega_n \subset D$, according to (i) of the Theorem and the continuity of the inclusion for the Hausdorff topology, there exist an open set $\Omega$ and a subsequence of $\Omega_n$ (still denoted by $\Omega_n$) such that $\Omega_n \rightarrow^H \Omega$ and $int(C) \subset \Omega \subset D$. (ii) of Theorem 2.8 together with Theorem 2.8 imply that $\Omega$ is of class $C^2$. Now by (iii) of Theorem 2.2 $\int_{\Omega_n} dx$ converges to $\int_\Omega dx$, and by Corollary 2.7, $\int_D f u_n \chi_{\Omega_n}$ converges to $\int_D f u_\Omega \chi_\Omega = \int_\Omega |\nabla u_\Omega(x)|^p dx$. Hence $J(\Omega) \leq \liminf_{n \rightarrow +\infty} J(\Omega_n)$. According to (iv) of Theorem 2.2, $\Omega \in O_C$, therefore $J(\Omega) = \min_{\omega \in O_C} J(\omega)$. The regularity $C^2$ of $\Omega$ thanks to Theorem 2.8.

Put
\[ O_\Omega = \{ \omega \subset \Omega : \omega \text{ satisfies } C\text{-GNP} \} \]
and
\[ j(\omega) = k|\partial \omega| + \int_{\partial \omega} \frac{\partial u_\omega}{\partial \nu} dx \]
where $\nu$ is the exterior normal vector to $\partial \omega$, $|\partial \omega|$ denotes the perimeter of $\omega$ and $u_\omega$ the solution of the Dirichlet problem $P(\omega, f)$. By Green formula, $j$ becomes
\[ j(\omega) = k|\partial \omega| - \int_\omega f(x) dx \]

Theorem 3.2. There exists $\Omega^* \in O_\Omega$ which minimizes the functional $j$ on $O_\Omega$. $\Omega^*$ is of class $C^2$.

For the proof of this theorem, we use (iii) and (iv) of Theorem 2.2. Once again, the $C^2$ regularity of $\Omega^*$ thanks to Theorem 2.8.

3.2 The optimality conditions

In this paragraph, we are going to use the standard tool of the domain derivative to write down the optimality condition. Let us recall the definition of the domain derivative, see for instance [22] and [17]. Since the minimum $\Omega$ of the functional $J$ is of class $C^2$. Let us consider a deformation field $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ and set $\Omega_t = \{ x + tv(x), x \in \Omega \}$, $t > 0$. The application $Id + tv$ is a perturbation of the identity which is a Lipschitz diffeomorphism for $t$ small enough. By definition, the derivative of $J$ at $\Omega$ in the direction $V$ is
\[ dJ(\Omega, V) = \lim_{t \rightarrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}. \]
As the functional $J$ depends on the domain $\Omega$ through the solution of the Dirichlet problem $P(\Omega, f)$, we need to define also the domain derivative of $u_\Omega$. If $u'_\Omega$ denotes the domain derivative of $u_\Omega$, then

$$
u'_\Omega = \lim_{t \to 0} \frac{u_{\Omega_t} - u_\Omega}{t}.$$ 

Recall that the shape derivative of the volume is $\int_{\partial \Omega} V \nu \, d\sigma$

Now for $F(\Omega) = \int_{\Omega} h(u_\Omega) \, dx$, the Hadamard formula gives

$$dF(\Omega, V) = \int_{\Omega} h'(u_\Omega) u'_\Omega \, dx + \int_{\partial \Omega} h(u_\Omega) V \cdot \nu \, d\sigma.$$ 

Furthermore, we can prove ([22], [17]) that $u'_\Omega$ is a solution of some linear Dirichlet problem with

$$u'_\Omega = -\frac{\partial u_\Omega}{\partial \nu} V \cdot \nu \text{ on } \partial \Omega.$$ 

This, together with $u_\Omega = 0$ on $\partial \Omega$ implies

$$dF(\Omega, V) = \int_{\partial \Omega} h(u_\Omega) V \cdot \nu \, d\sigma.$$ 

Now by Green formula

$$J(\Omega) = -\frac{1}{p} \int_{\Omega} |\nabla u_\Omega|^p + \frac{1}{p} k^p \int_{\Omega} dx.$$ 

So if we put $h(u_\Omega) = |\nabla u_\Omega|^p$, according to what precedes we obtain

$$dJ(\Omega; V) = \frac{1}{p} \int_{\partial \Omega} (k^p - |\nabla u_\Omega(x)|^p) V \nu \, d\sigma.$$ 

where $\nu$ is the outward normal vector to $\partial \Omega$.

Now since $\Omega$ is the minimum for the functional $J$, $dJ(\Omega; V) \geq 0$ for every admissible direction $V$. Therefore

$$\int_{\partial \Omega} (k^p - |\nabla u_\Omega(x)|^p) V \nu \, d\sigma \geq 0 \text{ for every admissible direction } V.$$ 

We mean by admissible displacement the one which allows us to keep the C-GNP or the C-SP (according to Proposition 2.4 from above). Since $\Omega$ has the C-GNP, it satisfies the C-SP. Then

$$\forall x \in \partial \Omega \setminus C \quad K_x \cap \Omega = \emptyset.$$ 

For $t$ sufficiently small, let $\Omega_t = \Omega + tV(\Omega)$ be the deformation of $\Omega$ in the direction $V$. Let $x_t \in \partial \Omega_t$. There exists $x \in \partial \Omega$ s.t. $x_t = x + tV(x)$. Using the definition of $K_x$, and the equality above, it is obvious to get (for $t$ small enough and for every displacement $V$):

$$\forall x_t \in \partial \Omega_t \setminus C \quad K_{x_t} \cap \Omega_t = \emptyset.$$
which means that $\Omega_t$ satisfies the $C$-SP (and so the $C$-GNP) for every displacement $V$ when $t$ is sufficiently small. Then, using $V$ and $-V$, and the fact that the set of the functions $V \cdot \nu$ is dense in $L^2(\partial \Omega)$, we deduce

\begin{equation}
|\nabla u_\Omega(x)| = k \text{ on } \partial \Omega \setminus \partial C.
\end{equation}

On the other hand, the admissible directions $V$ on $\partial \Omega \cap \partial C$ must satisfy $V(x) \cdot \nu(x) \geq 0$, and one gets

\begin{equation}
|\nabla u_\Omega(x)| \leq k \text{ on } \partial \Omega \cap \partial C.
\end{equation}

Now, thanks to Hadamard formula, the shape derivative of $j$ on $\Omega$ is

$$dj(\Omega^*; V) = \int_{\partial \Omega^*} (NkH_{\partial \Omega^*} - f)V \cdot \nu \, d\sigma \geq 0$$

for every admissible direction $V$.

Arguing as above and using the fact that $int(C) \subset \Omega^*$, we get

\begin{equation}
\begin{cases}
H_{\partial \Omega^*} = 0 \text{ on } \partial \Omega^* \setminus \partial C \\
H_{\partial \Omega^*} \geq \frac{1}{Nk} \text{ on } \partial \Omega^* \cap \partial C.
\end{cases}
\end{equation}

4 Auxiliary results

In this section, we will state and prove some propositions which we will use in the Section 5. Let $\Omega$ (resp. $\Omega^*$) be the minimum of $J$ (resp. $j$). The two first propositions are given for $p \neq 2$. For $p = 2$, the proof is done if we replace The Hopf’s comparison principle by the maximum principle.

**Proposition 4.1.** Suppose that $C$ is of class $C^2$ and $|\nabla u_C| \geq \gamma$ in $\text{int}(C)$ (for some $\gamma > 0$) and $C$ satisfies the interior sphere condition. Then

1. either $\partial \Omega \cap \partial C \neq \emptyset$ and $|\nabla u_C(x)| \leq k$ on $\partial \Omega \cap \partial C$

2. or $C$ is strictly contained in $\Omega$.

**Proof.** Let $\partial \Omega \cap \partial C \neq \emptyset$ and suppose by contradiction there exists $x \in \partial \Omega \cap \partial C$ such that $|\nabla u_\Omega(x)| > k$. This together with (3) implies that $\partial \Omega \neq \partial C$.

Now, since

$$\Delta_p u_\Omega = -f = \Delta_p u_C \text{ in } \text{int}(C) \quad \text{and} \quad u_\Omega \geq 0 = u_C \text{ on } \partial C,$$

part 1. of Lemma 2.5 implies that

$$u_\Omega \geq u_C \text{ in } \text{int}(C).$$

But $u_\Omega \neq u_C$ in $\text{int}(C)$, then

$$u_\Omega > u_C \text{ in } \text{int}(C).$$

Now, since $C$ satisfies the interior sphere condition, $|\nabla u_C| \geq \gamma$ on $\text{int}(C)$ and

$$u_\Omega = u_C \text{ on } \partial \Omega \cap \partial C,$$
part 2. of Lemma 2.5, gives
\[ \frac{\partial u_\Omega}{\partial \nu} (x) < \frac{\partial u_C}{\partial \nu} (x) \]
or again, since \( |\nabla u_\Omega(x)| = -\frac{\partial u_\Omega(x)}{\partial \nu(x)} \),
\[ |\nabla u_C(x)| < |\nabla u_\Omega(x)|. \]
So \( |\nabla u_\Omega(x)| > k \) which contradicts (3.3).

If we replace, in the preceding proposition, \( \text{int}(C) \) by \( \Omega^* \), we can obtain

**Proposition 4.2.** Suppose that \( |\nabla u_\Omega^*| \geq \gamma \) in \( \Omega^* \) (for some \( \gamma > 0 \)) and \( \Omega^* \) satisfies the interior sphere condition. Then
1. either \( \partial \Omega^* \cap \partial \Omega \neq \emptyset \) and \( |\nabla u_\Omega^*(x)| \leq k \) on \( \partial \Omega \cap \partial \Omega^* \)
2. or \( \Omega^* \) is strictly contained in \( \Omega \).

**Proposition 4.3.** Suppose that \( C \) is of class \( C^2 \), then
1. either \( \partial \Omega^* \cap \partial C \neq \emptyset \) and \( H_{\partial \Omega^*} \geq \frac{f(x)}{Nk} \) on \( \partial \Omega^* \cap \partial C \)
2. or \( C \) is strictly contained in \( \Omega^* \).

**Proof.** Suppose there exists \( x \in \partial \Omega^* \cap \partial C \) such that \( H_{\partial \Omega^*} (x) < \frac{f(x)}{Nk} \). Since \( \text{int}(C) \subset \Omega^* \), \( x \in \partial \Omega^* \cap \partial C \) and \( C \) and \( \Omega^* \) are of class \( C^2 \), then
\[ \frac{f(x)}{Nk} \leq H_{\partial \Omega^*} \leq H_{\partial C} < \frac{f(x)}{Nk} \]
which is absurd. \( \square \)

## 5 Existence of free boundaries

**Theorem 5.1.** Suppose \( p \neq 2 \) and let \( \Omega \) and \( \Omega^* \) be as in Theorems 3.1 and 3.2. If \( |\nabla u_{\Omega^*}| > k \) on \( \Omega^* \) then \( \Omega \) is a solution of (FB) which strictly contains \( \Omega^* \).

**Remark 5.1.** For \( p = 2 \), we can obtain the same result if we replace the condition stated above by the following:
\[ |\nabla u_{\Omega^*}| > k \text{ on } \partial \Omega^* \]
**Proof.** This result is an immediate consequence of Proposition 4.1. \( \square \)

**Theorem 5.2.** Let \( \Omega \) and \( \Omega^* \) be as in Theorems 3.1 and 3.2.
1. If \( C \) is of class \( C^2 \) and \( H_{\partial C} < \frac{f}{Nk} \) on \( \partial C \) then
   (a) \( C \) is strictly contained in \( \Omega^* \)
(b) $\Omega^*$ is a minimal surface
(c) $\Omega$ is a solution of (FB) which contains $\Omega^*$

2. Furthermore, if $|\nabla u_\Omega| \leq k$ on $\partial \Omega^*$ or if $k|\partial \Omega^*| \geq \int_C f$ then $\Omega$ is a minimum of $j$ and so it is a minimal surface.

Proof. (1)

- (a) is an immediate consequence of Proposition 4.3.
- (b) The optimality condition (4) gives $H_{\partial \Omega^*} = 0$ which implies that $\Omega^*$ is a minimal surface.
- (c) According to (a), $C$ is strictly contained in $\Omega^*$ but $\Omega^* \subset \Omega$. So $C$ is strictly contained in $\Omega$ and the optimality conditions (2) and (3) imply $|\nabla u_\Omega| = k$ on $\partial \Omega$

(2) If in addition $\Omega^*$ verifies one of the two conditions stated above, then $j(\Omega^*) \geq 0$. But $\Omega \in \mathcal{O}_k$ and by (c) $j(\Omega) = 0$ so $j(\Omega) \leq j(\Omega^*)$. Then we can conclude by (b). $\square$

Replace in the expressions of $J$ and $j$, $f$ by $1 + f$ and denote by $J_1$ and $j_1$ the corresponding functionals of domains. We obtain

**Theorem 5.3.** Let $\Omega_1$ (resp. $\Omega_1^*$) be the minimum of $J_1$ (resp. of $j_1$). If $C$ is of class $C^2$ and $H_{\partial C} < \frac{1+f}{Nk}$ on $\partial C$ then

1. $C$ is strictly contained in $\Omega_1^*$
2. $\Omega_1^*$ is a ball with radius $Nk$
3. $\Omega_1$ is a solution of (FB) which contains $\Omega_1^*$

Reasoning like in Theorem 5.3, the first and the third items are immediate. For the second item one can replace, in the optimality conditions (3.4), $f$ by $1 + f$. Then using the fact that $C$ is strictly contained in $\Omega_1^*$, one can obtain $H_{\partial \Omega_1^*} = \frac{1}{Nk}$ which says that $\Omega_1^*$ is a ball with radius $Nk$ thanking to the Alexandrov result [1].

**Remark 5.2.** A simple calculation shows that we cannot put conditions on $\Omega_1^*$ (as in (2) of Theorem 5.3) and so $\Omega_1$ cannot be a minimum of $j_1$.

**Remark 5.3.** In one hand $\Omega_1^*$ is a ball, so it satisfies the Geometric Normal Property w.r.t its center. In the other hand $\Omega_1^*$ has the $C$-GNP. Therefore, the center of $\Omega_1^*$ belongs to $C$.

6 Concluding remarks

**Remark 6.1.** The aim of Theorem 2.8 is to give the $C^2$ regularity of the minimum $\Omega$ (resp. $\Omega^*$) of $J$ (resp. $j$). This in order to use the shape derivative and so to resolve Problem (FB). The proof of this theorem uses the following Lemma (see [2]):
Lemma 6.1. Let $L$ be a compact subset of $\mathbb{R}^N$. Let $f_n$ be a sequence of functions defined as Theorem 2.8. Suppose that $\Omega$ is an open subset of $L$ such that

$$\Omega = \{ x \in L : h(x) > 0 \} \quad \text{and} \quad \partial \Omega = \{ x \in L : h(x) = 0 \},$$

where $h$ is a continuous function defined in $L$. If the $f_n$ converge uniformly to $h$ in $L$, then the $\Omega_n$ converge in the compact sense, to $\Omega$.

Remark 6.2. a) The hypothesis in Theorem 2.8 about the local regularity is not too restrictive because of, for instance, results due to E. DiBenditto [10], J.L. Lewis [15] and G.M. Lieberman [16].

b) When $p = 2$ Proposition 4.1 and Theorem 5.1 can be extended to the divergence operator $\text{div}(a(x)\nabla u)$. For this kind of operator the continuity result is a simple consequence of Mosco convergence (see for instance [7]).

c) According to the results obtained by Bucur and Trebeschi in [7], Proposition 4.3 and Theorem 5.3 can be extended to other divergence operators like $\text{div}(a(x,Du))$.

d) Let $f = a\chi_{B_R}$ where $a > 0$, $B_R \subset \mathbb{R}^2$ is some ball of radius $R$ and $\chi_{B_R}$ is its characteristic function. The condition stated in Theorem 5.3 becomes $aR > 2k$. Now if $\Omega$ is a regular solution of (FB), then Green formula implies $aR > 2k$, i.e this condition is necessary and sufficient for solving (FB) in this case.

e) Consider the case of (FB) where $\mu$ is the uniform density $\delta_{[-1,1] \times \{0\}}$. Let $C$ be the ball of radius 1 and of center 0. According to the preceding remark, $a > 2k$ is a necessary and sufficient condition of existence for a free boundary which contains strictly the segment line $[-1,1] \times \{0\}$. Notice that in [12], the authors gave $a > 24\pi k$ as sufficient condition of existence for this problem while in [5], the author proposes $a > 3.92k$.

f) Let $\Omega$ be a solution of (FB) in the case where $\mu \equiv 1$. Using the same arguments as in Theorem 3.2, we can prove the existence of a minimum $\Omega^*$ of $j$ on some class of admissible domains (for instance the domains which are contained in $\Omega$ and satisfy the $\varepsilon$-cone property). If both $\Omega$ and $\Omega^*$ are of class $C^2$ then by the optimality condition, $\Omega^*$ is a ball with radius $Nk$ and $j(\Omega^*) = 0 = j(\Omega)$. Therefore $\Omega$ is a minimum of $j$ and so $\Omega = B_{Nk}$, i.e it is the solution of Serrin’s problem [19]. Now according to Remark e), this result can be extended to other divergence operators like $\text{div}(a(x,Du))$ and according to Remark e), it cannot be obtained when $\mu$ is nonconstant.

References


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