Conformal isoparametric spacelike hypersurfaces in conformal space $Q_1^{n+1}$

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Abstract. In this paper, we study the conformal geometry of conformal isoparametric spacelike hypersurfaces in conformal space $Q_1^{n+1}$. We obtain the classification of the conformal isoparametric spacelike hypersurfaces in $Q_1^{n+1}$ with three distinct conformal principal curvatures, one of which is simple, and the classification of the conformal isoparametric spacelike hypersurfaces in $Q_6^1$.


Key words: conformal space; conformal isoparametric spacelike hypersurfaces; conformal second fundamental form; conformal form.

1 Introduction

Let $\langle \cdot, \cdot \rangle_s$ be the Lorentzian inner product with $s$ negative index of the $(n + s)$-dimensional Euclidean space $\mathbb{R}^{n+s}$; we denote

$$\langle X, Y \rangle_s = \sum_{i=1}^{n} x_i y_i - \sum_{i=n+1}^{n+s} x_i y_i, \quad \forall X = (x_i), Y = (y_i) \in \mathbb{R}^{n+s}.$$ 

Let $\mathbb{R}^{n+2}$ be the $(n + 2)$-dimensional real projective space. The quadric surface $Q_1^{n+1} = \{[\xi] \in \mathbb{R}^{n+2}|\langle \xi, \xi \rangle_2 = 0\}$ is called conformal space. We denote the Lorentzian space forms (the Lorentz space, the de Sitter sphere and the anti-de Sitter sphere), respectively, as follows:

$$\mathbb{R}_1^{n+1} = (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_1),$$
$$S_1^{n+1} = \{u \in \mathbb{R}^{n+2}|\langle u, u \rangle_1 = 1\},$$
$$H_1^{n+1} = \{u \in \mathbb{R}^{n+2}|\langle u, u \rangle_2 = -1\}.$$  

We denote as well $\pi = \{[x] \in Q_1^{n+1}|x_1 = x_{n+3}\}$, $\pi_+ = \{[x] \in Q_1^{n+1}|x_{n+3} = 0\}$ and $\pi_- = \{[x] \in Q_1^{n+1}|x_1 = 0\}$. We shall further consider the conformal diffeomorphisms

$$\sigma_0: \mathbb{R}_1^{n} \to Q_1^{n+1}\backslash \pi, \quad u \mapsto [(u, u)^{-1}, (u, u)^{1}],$$
$$\sigma_1: S_1^{n+1} \to Q_1^{n+1}\backslash \pi_+, \quad u \mapsto [(u, 1)];$$
$$\sigma_{-1}: H_1^{n+1} \to Q_1^{n+1}\backslash \pi_-, \quad u \mapsto [(1, u)].$$

From [11], we may regard $Q_1^{n+1}$ as the common compactified space of $\mathbb{R}_1^{n+1}, S_1^{n+1}$ and $\mathbb{H}_1^{n+1}$, while $\mathbb{R}_1^{n+1}, S_1^{n+1}$ and $\mathbb{H}_1^{n+1}$ are regarded as the subsets of $Q_1^{n+1}$.

Let $x : M \to Q_1^{n+1}$ be an $n$-dimensional immersed conformal regular spacelike hypersurface in the conformal space $Q_1^{n+1}$. From [9], we know that the conformal metric of the immersion $x$ can be defined by

$$g = \frac{n}{n-1} \left( \sum_{i,j} h_{ij}^2 - nH^2 \right) \langle dx, dx \rangle := e^{2\tau} \langle dx, dx \rangle,$$

which is a conformal invariant. Let

$$\Phi = \sum_{i=1}^n e^{\tau} C_i \theta_i, \quad A = \sum_{i,j=1}^n e^{2\tau} A_{ij} \theta_i \otimes \theta_j, \quad B = \sum_{i,j=1}^n e^{2\tau} B_{ij} \theta_i \otimes \theta_j, \quad D = A + \lambda B,$$

where $\lambda$ is a constant. We call $\Phi, A, B$ and $D$ the conformal form, the conformal Blaschke tensor, the conformal second fundamental form and the conformal para-Blaschke tensor of the immersion $x$, respectively. It is known that $\Phi, A, B$ and $D$ are conformal invariants.

The conformal geometry of regular hypersurfaces in the conformal space is determined by the conformal metric. The negative index of the conformal space $Q_1^{n+1}$ is 1. If the negative index is degenerate, then we obtain the Möbius geometry in the unit sphere, which has been studied by many authors (see [1]-[5], [7]-[15]). An eigenvalue of the conformal second fundamental form $B$, the conformal Blaschke tensor $A$ and the conformal para-Blaschke tensor $D$, are respectively called conformal principal curvature, Blaschke eigenvalue or para-Blaschke eigenvalue of the immersion $x$. A regular spacelike hypersurface $x : M \to Q_1^{n+1}$ is called conformal isoparametric spacelike hypersurface, if $\Phi \equiv 0$ and the conformal principal curvatures of the immersion $x$ are constant.

C.X. Nie et al. studied the conformal geometry of conformal isoparametric spacelike hypersurfaces in the conformal space $Q_1^{n+1}$ and obtained the following (see [11]):

**Theorem 1.1.** If $x : M \to Q_1^{n+1}$ is a conformal isoparametric spacelike hypersurface with two distinct principal curvatures, then $x$ is conformally equivalent to an open part of the following standard embeddings:

(i) the Riemannian product $S^m(c) \times H^{n-m}(\sqrt{c^2-r^2})$ in $S_1^{n+1}(r), c > r$; or
(ii) the Riemannian product $\mathbb{R}^n \times H^{n-m}(r)$ in $R_1^{n+1}$; or
(iii) the Riemannian product $\mathbb{H}^m(c) \times H^{n-m}(\sqrt{r^2-c^2})$ in $H_1^{n+1}(r), 0 < c < r$;

where $r^2 = \frac{n-1}{m(n-m)}$.

Recently, the first author and Su [14] obtained the classification of conformal isoparametric spacelike hypersurfaces in $Q_1^{3}$ and $Q_1^{1}$. In this paper, we continue to study the topic of conformal isoparametric spacelike hypersurfaces in $Q_1^{n+1}$. We obtain the classification of the conformal isoparametric spacelike hypersurfaces in $Q_1^{n+1}$ with three distinct conformal principal curvatures, one of which is simple, and the classification of the conformal isoparametric spacelike hypersurfaces in $Q_1^{6}$.

**Theorem 1.2.** Let $x : M \to Q_1^{n+1}$ $(n \geq 3)$ be a conformal isoparametric spacelike hypersurface in $Q_1^{n+1}$ with three distinct conformal principal curvatures, one of which
is simple. Then $x$ is conformally equivalent to an open part of the spacelike hypersurface $\mathbb{W} \mathbb{P}(p,q,c)$ given by Example 3.1, where $p,q,c$ are some constants, $p \geq 1, q \geq 1, p + q < n$ and $qc^2 + pd^2 = (qc^2 + pd^2)^2$, $d = \sqrt{c^2 - 1}$.

**Theorem 1.3.** Let $x : M \to \mathbb{Q}_1^n$ be a conformal isoparametric spacelike hypersurface in $\mathbb{Q}_1^n$. Then

a) $x$ is conformally equivalent to an open part of the standard embeddings:

(i) the Riemannian product $\mathbb{S}^m(c) \times \mathbb{H}^{n-m}((\sqrt{c^2 - r^2})$ in $\mathbb{S}_1^n(r)$, $c > r$, $m = 1, 2, 3, 4$, or

(ii) the Riemannian product $\mathbb{R}^m \times \mathbb{H}^{n-m}(r)$ in $\mathbb{R}_1^n$, $m = 1, 2, 3, 4$, or

(iii) the Riemannian product $\mathbb{H}^m(c) \times \mathbb{H}^{n-m}((\sqrt{c^2 - r^2})$ in $\mathbb{H}_1^n(r)$, $0 < c < r$, $m = 1, 2, 3, 4$, where $r^2 = m(\frac{1}{m^2 - m})$; or

(iv) the spacelike hypersurface $\mathbb{W} \mathbb{P}(p,q,c)$ given by Example 3.1, where $p,q,c$ are some constants, $p \geq 1, q \geq 1, p + q < 5$ and $qc^2 + pd^2 = (qc^2 + pd^2)^2$, $d = \sqrt{c^2 - 1}$;

b) $x$ is locally a Riemannian product $M^m_1 \times M^{5-m}_2$, $m = 3, 4$, where $M^{5-m}_2$ is a constant curvature Riemannian manifold.

## 2 Fundamental formulas on conformal geometry

We firstly review the fundamental formulas on conformal geometry of spacelike hypersurfaces in $\mathbb{Q}_1^{n+1}$, and use the following range of indices throughout this paper: $1 \leq i, j, k, l, m \leq n$ (for more details, see [11] or [14]).

Let $x : M \to \mathbb{Q}_1^{n+1}$ be an $n$-dimensional conformal regular spacelike hypersurface with $\Phi \equiv 0$ in $\mathbb{Q}_1^{n+1}$. From the structure equations on $M$ (see [11]), we have

\begin{equation}
\omega_{ij} + \omega_{ji} = 0, \quad dw_i = \sum_j \omega_{ij} \wedge \omega_j, \tag{2.1}
\end{equation}

\begin{equation}
e^{2\tau}C_i = H\tau_i - \tau_i - \sum_j h_{ij}\tau_j, \quad e^\epsilon B_{ij} = h_{ij} - H I_{ij}, \tag{2.2}
\end{equation}

\begin{equation}
e^{2\tau}A_{ij} = \tau_i \tau_j - \tau_{i,j} - H h_{ij} - \frac{1}{2} \left( \sum_k r^k \tau_k - H^2 - \epsilon \right) I_{ij}, \tag{2.3}
\end{equation}

\begin{equation}
dw_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{jikl}, \tag{2.4}
\end{equation}

\begin{equation}
\sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = \frac{n-1}{n}, \quad tr A = \frac{1}{2n} (n^2 \kappa - 1), \tag{2.5}
\end{equation}

\begin{equation}
A_{ij,k} - A_{ik,j} = B_{ij} C_k - B_{ik} C_j, \quad B_{ij,k} - B_{ik,j} = \delta_{ij} C_k - \delta_{ik} C_j, \tag{2.6}
\end{equation}

\begin{equation}
C_{ij} - C_{ji} = \sum_k (B_{ik} A_{kj} - B_{kj} A_{ik}), \tag{2.7}
\end{equation}

\begin{equation}
R_{ijkl} = -(B_{ik} B_{jl} - B_{il} B_{jk}) + \delta_{ik} A_{jl} + \delta_{jl} A_{ik} - \delta_{il} A_{jk} - \delta_{jk} A_{il}, \tag{2.8}
\end{equation}
where $R_{ijkl}$ denotes the curvature tensor with respect to the conformal metric $g$ on $M$. Since the conformal form $\Phi \equiv 0$, we have for all indices $i,j,k$

\begin{equation}
A_{ij,k} = A_{ik,j}, \quad B_{ij,k} = B_{ik,j}, \quad \sum_k B_{ik}A_{kj} = \sum_k B_{kj}A_{ki}.
\end{equation}

The conformal $(0,2)$ para-Blaschke tensor is denoted by $\mathbf{D} = \sum_{i,j} D_{ij} \omega_i \otimes \omega_j$,

\begin{equation}
D_{ij} = L_{ij} + \lambda B_{ij}, \quad 1 \leq i,j \leq n,
\end{equation}

where $\lambda$ is a constant. From (2.9) and (2.10), we have for all indices $i,j,k$ that $D_{ij,k} = D_{ik,j}$.

3 Some results and examples

From Nie and Wu [10], Shu and Su [14], Nomizu [13], Li and Xie [6], we have the following:

**Theorem 3.1.** (see [10]) If $x : M \to \mathbb{Q}^{n+1}_1$ is a conformal regular spacelike hypersurface in $\mathbb{Q}^{n+1}_1$ with parallel conformal second fundamental form, then $x$ is conformally equivalent to an open part of these standard embeddings:

(i) the Riemannian product $\mathbb{S}^m(a) \times \mathbb{H}^{n-m}(\sqrt{a^2 - r^2})$ in $\mathbb{S}^{n+1}_1(r)$, $a > r$; or

(ii) the Riemannian product $\mathbb{H}^m(a) \times \mathbb{H}^{n-m}(r)$ in $\mathbb{H}^{n+1}_1(r)$; or

(iii) the Riemannian product $\mathbb{H}^m(\sqrt{a^2 - r^2})$ in $\mathbb{H}^{n+1}_1(r)$, $0 < a < r$, where $\sqrt{a^2 - r^2} = \frac{n-1}{m-n-m}$; or

(iv) the spacelike hypersurface $x = \sigma_0 \circ u : \mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d) \to \mathbb{Q}^{n+1}_1$ with $d = \sqrt{c^2 - 1}$, $p \geq 1$, $q \geq 1$, $p + q < n$, where

\[ u : \mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d) \to \mathbb{R}^{n+2}_1 \subset \mathbb{Q}^{n+1}_1, \quad u(u', t, u'', u'''') = (tu', u'', tu'''), \]

for all $u' \in \mathbb{S}^p(c)$, $t \in \mathbb{R}^+$, $u'' \in \mathbb{R}^{n-p-q-1}$, $u''' \in \mathbb{H}^q(d)$.

**Proposition 3.2.** (see [14]) Let $x : M \to \mathbb{Q}^{n+1}_1$ be an $n$-dimensional conformal isoparametric spacelike hypersurface in $\mathbb{Q}^{n+1}_1$ with constant normalized conformal scalar curvature $\kappa$ and $\kappa \neq 1$. Then $x$ is an $n$-dimensional Euclidean isoparametric spacelike hypersurface.

**Proposition 3.3.** (see [13], [6]). Let $x$ be a Euclidean isoparametric spacelike hypersurface in Lorentzian space form. Then $x$ can have at most two distinct Euclidean principal curvatures.

**Example 3.1.** (see [10]). For any natural number $p, q, p + q < n$ and real number $c \in (1, +\infty)$ and $d = \sqrt{c^2 - 1}$, consider the immersed hypersurface $u : \mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d) \to \mathbb{R}^{n+2}_1 \subset \mathbb{Q}^{n+1}_1$: $u(u', t, u'', u''') = (tu', u'', tu''')$, $u' \in \mathbb{S}^p(c)$, $t \in \mathbb{R}^+$, $u'' \in \mathbb{R}^{n-p-q-1}$, $u''' \in \mathbb{H}^q(d)$, then $x = \sigma_0 \circ u : \mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d) \to \mathbb{Q}^{n+1}_1$ is a conformal regular spacelike hypersurface in $\mathbb{Q}^{n+1}_1$, which is denoted by $WP(p, q, c) = x(\mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d))$. From [10], by a direct calculation, we know that $WP(p, q, c)$ has three distinct constant conformal principal curvatures and the conformal second fundamental form is parallel. We may also
calculate that \( WP(p, q, c) \) is of parallel conformal Blaschke tensor. Thus, the conformal Blaschke eigenvalues are constants, from (2.5), we know that the normalized conformal scalar curvature \( \kappa \) is constant. If \( \kappa \neq 1 \), from Proposition 3.2 and (2.2), we see that \( WP(p, q, c) \) is of three distinct constant Euclidean principal curvatures, this contradicts Proposition 3.3. If \( \kappa = 1 \), we know that the normalized Euclidean scalar curvature \( R = \kappa = 1 \). From Gaussian equation \( n(n-1)(R-1) = \sum_{i,j} b_{ij}^2 - n^2 H^2 \), we see that \( \sum_{i,j} h_{ij}^2 = n^2 H^2 \), this is equivalent to \( qe^4 + pd^4 = (qe^2 + pd^2)^2 \) (see Example 2.1 of [10]).

**Example 3.2.** (see [14]). Spacelike hypersurface \( x : S^m(c) \times \mathbb{H}^{n-m} (\sqrt{c^2 - r^2}) \to S_1^{n+1}(r), r < c \). Let \( x = (x_1, x_2) \in S^m(c) \times \mathbb{H}^{n-m} (\sqrt{c^2 - r^2}) \subset \mathbb{R}^{m+1} \times \mathbb{R}^{n-m+1} \), \( \langle x_1, x_1 \rangle = c^2 \), \( \langle x_2, x_2 \rangle = -(c^2 - r^2) \). By a direct calculation, we see that \( x \) has two distinct principal curvatures \( \frac{d}{r} \) and \( -\frac{1}{r^2} \) with multiplicities \( m \) and \( n - m \) and the conformal second fundamental form of \( x \) is parallel, where \( d = \sqrt{c^2 - r^2} \).

**Example 3.3.** (see [14]). Spacelike hypersurface \( x : \mathbb{R}^m \times \mathbb{H}^{n-m} (r) \to \mathbb{R}^{n+1} \). Let \( x = (x_1, x_2) \), \( x_1 \in \mathbb{R}^m, x_2 \in \mathbb{H}^{n-m} (r) \subset \mathbb{R}^{m+1} \times \mathbb{R}^{n-m+1} \), \( \langle x_1, x_1 \rangle = -c^2 \), \( \langle x_2, x_2 \rangle = -r^2 \). By a direct calculation, we see that \( x \) has two distinct principal curvatures \( \frac{d}{r} \) and \( -\frac{1}{r^2} \) with multiplicities \( m \) and \( n - m \) and the conformal second fundamental form of \( x \) is parallel.

**Example 3.4.** (see [14]). Spacelike hypersurface \( x : \mathbb{H}^m(c) \times \mathbb{H}^{n-m} (\sqrt{r^2 - c^2}) \to \mathbb{H}_1^{n+1}(r), 0 < c < r \). Let \( x = (x_1, x_2) \in \mathbb{H}^m(c) \times \mathbb{H}^{n-m} (\sqrt{r^2 - c^2}) \subset \mathbb{R}^{m+1} \times \mathbb{R}^{n-m+1} \), \( \langle x_1, x_1 \rangle = -c^2 \), \( \langle x_2, x_2 \rangle = -r^2 \). By a direct calculation, we see that \( x \) has two distinct principal curvatures \( \frac{d}{r} \) and \( -\frac{1}{r^2} \) with multiplicities \( m \) and \( n - m \) and the conformal second fundamental form of \( x \) is parallel, where \( d = \sqrt{r^2 - c^2} \).

### 4 Proof of theorem 1.2

Throughout this section, we shall make the following convention on the ranges of indices: \( 1 \leq a, b \leq m_1, m_1 + 1 \leq p, q \leq m_1 + m_2, m_1 + m_2 + 1 \leq \alpha, \beta \leq m_1 + m_2 + m_3 = n, 1 \leq i, j, k \leq n \). Let \( A, B \) and \( D \) denote the \( n \times n \)-symmetric matrices \( (A_{ij}), (B_{ij}) \) and \( (D_{ij}) \), respectively. From (2.9) and (2.10), we know that \( BA = AB, DA = AD \) and \( BD = DB \). Thus, we may always choose a local orthonormal basis \( \{ E_1, E_2, \ldots, E_n \} \) such that

\[
A_{ij} = A_0 \delta_{ij}, \quad B_{ij} = B_i \delta_{ij}, \quad D_{ij} = D_i \delta_{ij},
\]

where \( A_i, B_i \) and \( D_i \) are the conformal Blaschke eigenvalues, the conformal principal curvatures and the conformal para-Blaschke eigenvalues of the immersion \( x \).

**Proof of Theorem 1.2.** If the conformal second fundamental form of \( x \) is parallel, since \( x \) has three distinct conformal principal curvatures, from Theorem 3.1, Example 3.1–Example 3.4, we know that \( x \) is conformally equivalent to an open part of the spacelike hypersurface \( WP(p, q, c) \) for some constants \( p, q, c \) given by Example 3.1.

If the conformal second fundamental form of \( x \) is not parallel, denote by \( B_1, B_2 \) and \( B_3 \) the three distinct constant conformal principal curvatures of \( x \) with multiplicities
From the definition of the covariant derivative of $B_{ij}$ (see (2.7) of [14]), we have

\begin{equation}
\sum_k B_{ij,k} \omega_k = (B_i - B_j) \omega_{ij}, \quad B_{ij,k} = \Gamma^i_{ik}(B_i - B_j), \tag{4.2}
\end{equation}

where $\Gamma^i_{ik}$ is the Levi-Civita connection for the conformal metric $g$ given by $\omega_{ij} = \sum_k \Gamma^j_{ik} \omega_k$. By (4.2), it follows that for any $a, b, p, q, \alpha, \beta, k$

\begin{equation}
B_{ab,k} = B_{pq,k} = B_{\alpha\beta,k} = 0. \tag{4.3}
\end{equation}

Thus, from (4.2) and (4.3), we have

\begin{equation}
R_{ijkl} = E_i (\Gamma^j_{ik}) - E_k (\Gamma^j_{il}) + \sum_m (\Gamma^j_{im} \Gamma^m_{kl} - \Gamma^j_{im} \Gamma^m_{kl}) + \Gamma^j_{im} \Gamma^m_{kl} - \Gamma^j_{im} \Gamma^m_{kl} + \Gamma^j_{im} \Gamma^m_{kl}. \tag{4.4}
\end{equation}

From (2.4) and (2.1), the curvature tensor of $x$ may be given by (see [8])

\begin{equation}
\Gamma^p_{ab} = \Gamma^\alpha_{ab} = 0, \quad \Gamma^\alpha_{pq} = \Gamma^\alpha_{pq} = 0, \quad \Gamma^\alpha_{\alpha\beta} = \Gamma^\alpha_{\alpha\beta} = 0, \quad \Gamma^\alpha_{p;\alpha} = \Gamma^\alpha_{p;\alpha} = \frac{B_{ap,a}}{B_1 - B_2}, \tag{4.5}
\end{equation}

\begin{equation}
\Gamma^p_{ab} = \frac{B_{ap,b}}{B_1 - B_2}, \quad \Gamma^p_{ab} = \frac{B_{bp,n}}{B_1 - B_2}, \quad \Gamma^p_{ab} = \frac{B_{bp,n}}{B_1 - B_2}, \quad \Gamma^p_{ab} = \frac{B_{bp,n}}{B_1 - B_2}. \tag{4.6}
\end{equation}

From (4.5) and (4.6), we have

\begin{equation}
\Gamma^p_{an} = \frac{B_{ap,n}}{B_1 - B_2}, \quad \Gamma^p_{ab} = \frac{B_{ap,n}}{B_1 - B_2}, \quad \Gamma^p_{ab} = \frac{B_{bp,n}}{B_1 - B_2}, \quad \Gamma^p_{ab} = \frac{B_{bp,n}}{B_1 - B_2}. \tag{4.7}
\end{equation}

Thus, from (4.4), we have

\begin{equation}
R_{apbq} = \Gamma^p_{an} \Gamma^p_{bq} - \Gamma^p_{an} \Gamma^p_{bq} - \Gamma^p_{an} \Gamma^p_{bq} = \frac{B_{ap,n} B_{bp,n} + B_{aq,n} B_{bp,n}}{(B_1 - B_2)(B_2 - B_3)}. \tag{4.9}
\end{equation}

On the other hand, from (2.8), we have

\begin{equation}
R_{apbq} = (-B_a B_p + A_a + A_p) \delta_{ab} \delta_{pq}. \tag{4.10}
\end{equation}

It follows from (4.9) and (4.10) that

\begin{equation}
\frac{B_{ap,n} B_{bp,n} + B_{aq,n} B_{bp,n}}{(B_1 - B_3)(B_2 - B_3)} = (-B_a B_p + A_a + A_p) \delta_{ab} \delta_{pq}, \tag{4.11}
\end{equation}

\begin{equation}
\frac{2B_{ap,n} B_{aq,n}}{(B_1 - B_3)(B_2 - B_3)} = (-B_1 B_2 + A_a + A_p) \delta_{pq}, \quad \text{if} \quad a = b, \tag{4.12}
\end{equation}

\begin{equation}
\frac{2B_{ap,n} B_{bp,n}}{(B_1 - B_3)(B_2 - B_3)} = (-B_1 B_2 + A_a + A_p) \delta_{ab}, \quad \text{if} \quad p = q, \tag{4.13}
\end{equation}

\begin{equation}
\frac{2B_{ap,n} B_{aq,n}}{(B_1 - B_3)(B_2 - B_3)} = (-B_1 B_2 + A_a + A_p) \delta_{pq}, \quad \text{if} \quad m_1 = 1. \tag{4.14}
\end{equation}
Since the conformal second fundamental form is not parallel, we may prove that there exists exactly one \( p \), such that \( B_{1p,n} \neq 0 \). In fact, if there exist at least two \( p_1, p_2 \), \( (p_1 \neq p_2) \) such that \( B_{1p_1,n} \neq 0, B_{1p_2,n} \neq 0 \), from (4.14), we have \( B_{1p_1,n}B_{1p_2,n} = 0 \), this follows that \( B_{1p_1,n} = 0 \), or \( B_{1p_2,n} = 0 \), a contradiction. Thus, we know that there exists exactly one \( p \), such that \( B_{1p,n} \neq 0 \).

If \( m_2 = 1 \), it follows that 

\[
2B_{am_1+1,n}B_{am_1+1,n} = (-B_1B_2 + A_a + A_{m_1+1})\delta_{ab}.
\]

The same reason implies that there exists exactly one \( a \), such that \( B_{am_1+1,n} \neq 0 \).

If \( m_1 \geq 2 \) and \( m_2 \geq 2 \), we may prove that there exists exactly one \( a \) and exactly one \( p \) such that \( B_{ap,n} \neq 0 \). In fact, if there exist at least two \( a_1, a_2 \), \( (a_1 \neq a_2) \) such that \( B_{ap,n} \neq 0 \), \( B_{2ap,n} \neq 0 \), from (4.13), we see that \( B_{ap,n}B_{2ap,n} = 0 \), this follows that \( B_{ap,n} = 0 \), or \( B_{2ap,n} = 0 \), a contradiction. Thus, we know that there exists exactly one \( a \), such that \( B_{ap,n} \neq 0 \). By the same reason, we may prove that there exists exactly one \( p \), such that \( B_{ap,n} \neq 0 \).

Combining the above three cases, we see that if \( m_1 \geq 1 \) and \( m_2 \geq 1 \), there exists exactly one \( a \) and exactly one \( p \), say \( a_1 \) and \( p_1 \), such that

\[
B_{a_1p_1,n} \neq 0, \quad B_{ap,n} = 0, \quad \text{for} \quad a \neq a_1, \forall p, \quad \text{or} \quad \forall a, p \neq p_1.
\]

By (4.10), (4.12) and (4.15), we get

\[
R_{a_1p_1a_1p_1} = -B_1B_2 + A_{a_1} + A_{p_1} = \frac{2B_{a_1p_1,n}}{(B_1 - B_3)(B_2 - B_3)},
\]

\[
R_{apap} = -B_1B_2 + A_a + A_p = 0, \quad a \neq a_1, \quad p \neq p_1,
\]

\[
R_{ap_1p_1} = -B_1B_2 + A_{a_1} + A_{p_1} = 0, \quad a \neq a_1,
\]

\[
R_{a_1p_1p_1} = -B_1B_2 + A_{a_1} + A_p = 0, \quad p \neq p_1.
\]

From (4.2), (4.3), (4.4), (2.8), (4.10) and for the reason above, we get

\[
R_{a_1n_1a_1n_1} = -B_1B_2 + A_{a_1} + A_n = \frac{2B_{a_1p_1,n}}{(B_1 - B_2)(B_3 - B_2)},
\]

\[
R_{aan} = -B_1B_2 + A_a + A_n = 0, \quad a \neq a_1,
\]

\[
R_{p_1np_1} = -B_1B_2 + A_{p_1} + A_n = \frac{2B_{a_1p_1,n}}{(B_2 - B_1)(B_3 - B_1)},
\]

\[
R_{pppn} = -B_1B_2 + A_p + A_n = 0, \quad p \neq p_1.
\]

Thus, from (4.16)–(4.23), we see that the normalized conformal scalar curvature

\[
\kappa = \frac{1}{(n+1)} \sum_{i \neq j} R_{ijij} = 0 \neq 1.
\]

Since (2.2) implies that the matrix \((B_{ij})\) and \((h_{ij})\) are commutative, we can choose a local orthonormal basis such that \(B_{ij} = B_i\delta_{ij}\) and \(h_{ij} = \lambda_i\delta_{ij}\), where \(\lambda_i\) are the Euclidean principal curvatures of \(x\). From (2.2) and Proposition 3.2, we know that \(x\) is an \(n\)-dimensional Euclidean isoparametric spacelike hypersurface with three distinct Euclidean principal curvatures, this contradicts Proposition 3.3. Thus, the case that the conformal second fundamental form of \(x\) is not parallel does not occur. This completes the proof of Theorem 1.2.

5 Proof of theorem 1.3

**Proposition 5.1.** (see [12]). Two regular spacelike hypersurface \(x : M \to \mathbb{Q}_1^{n+1}\) and \(\tilde{x} : M \to \mathbb{Q}_1^{n+1}\) in \(\mathbb{Q}_1^{n+1}(n \geq 3)\) are conformally equivalent if and only if there
exists a diffeomorphism $f : M \to \tilde{M}$ which preserves the conformal metric $g$ and the conformal second fundamental form $B$.

From [12], we also know the definition that a spacelike hypersurface with vanishing conformal form is called a conformal para-isotropic spacelike hypersurface if there is a function $\mu$ such that $A + \lambda B + \mu g \equiv 0$. We have the following:

**Proposition 5.2.** (see [12]). A conformal para-isotropic spacelike hypersurface in $\mathbb{Q}^{n+1}_1$ is conformally equivalent to one of the spacelike hypersurfaces with constant mean curvature and constant scalar curvature in Lorentzian space form.

**Proof of Theorem 1.3.** From (2.5), we see that the number $\gamma$ of distinct conformal principal curvatures can only take the values $\gamma = 2, 3, 4, 5$.

(1) If $\gamma = 2$, from Theorem 1.1, we know that Theorem 1.3 is true.

(2) If $\gamma = 3$, we see that at least one of the conformal principal curvatures is simple. From Theorem 1.2, we know that Theorem 1.3 is true.

(3) If $\gamma = 4$, from Theorem 3.1, Example 3.1–Example 3.4, we know that the conformal second fundamental form of $x$ is not parallel. Let $B_1, B_2, B_3, B_4, B_5$ be the constant conformal principal curvatures of $x$. Without loss of generality, we may assume that $B_1 \neq B_2 \neq B_3 \neq B_4 \neq B_5$. From (4.2), we have

\[(5.1) \quad B_{i,k} = 0, \quad B_{45,k} = 0, \quad \text{for all } i, k, \quad \omega_{ij} = \sum_k B_{ij,k} \omega_k, \quad \text{for } B_i \neq B_j.\]

By the similar method in [4], we have the following Lemmas (see Lemma 3.1 and Lemma 3.2 in [4]):

**Lemma 5.3.** Under the assumptions above, we have

\[(5.2) \quad \frac{B_{12,4}B_{12,5}}{(B_1 - B_2)(B_4 - B_2)} = \frac{B_{13,4}B_{13,5}}{(B_1 - B_3)(B_4 - B_3)},\]

\[(5.3) \quad \frac{B_{12,4}B_{12,5}}{(B_2 - B_1)(B_4 - B_1)} = \frac{B_{23,4}B_{23,5}}{(B_2 - B_3)(B_4 - B_3)}.\]

**Lemma 5.4.** Let $i, j, k$ be the three distinct elements of $\{1, 2, 3\}$ with arbitrarily given order. Then

\[(5.4) \quad R_{ijj} = \frac{2B_{ij,4}^2}{(B_j - B_i)(B_k - B_j)} + \frac{2B_{ij,5}^2}{(B_k - B_i)(B_4 - B_j)},\]

\[(5.5) \quad R_{4i4} = \frac{2B_{ij,4}^2}{(B_j - B_i)(B_4 - B_k)} + \frac{2B_{ik,4}^2}{(B_k - B_i)(B_4 - B_i)},\]

\[(5.6) \quad R_{5i5} = \frac{2B_{ij,5}^2}{(B_j - B_i)(B_5 - B_k)} + \frac{2B_{ik,5}^2}{(B_k - B_i)(B_5 - B_i)}.\]

**Lemma 5.5.** Under the assumptions above, we have

(i) for any distinct $i, j \in \{1, 2, 3\}$ and any distinct $\alpha, \beta \in \{4, 5\}$, if $B_{i,\alpha}B_{j,\beta} \neq 0$, then $B_{12,\beta} = B_{13,\beta} = B_{23,\beta} = 0$.

(ii) $B_{12,4}B_{12,5} = B_{13,4}B_{13,5} = B_{23,4}B_{23,5} = 0$. 


Proof. (i) Without loss of generality, we may only prove that for any distinct \(i, j \in \{1, 2, 3\}\), if \(B_{12,3}B_{1,4} \neq 0\), then \(B_{12,5} = B_{13,5} = B_{23,5} = 0\). In fact, if \(B_{12,5} \neq 0\), from the definition of the covariant derivative of \(L_{ij}\) and \(B_{ij}\) (see (2.6) and (2.7) of [14]), we have

\[
A_{ij,k} = E_k(A_i)\delta_{ij} + \Gamma_{ik}^j(A_i - A_j), \quad B_{ij,k} = E_k(B_i)\delta_{ij} + \Gamma_{ik}^j(B_i - B_j).
\]

Thus, from (5.7), we see that for any distinct \(i, j \in \{1, 2, 3\}\), \(\frac{A_{12,3}}{B_{12,3}} = \frac{A_{12,5}}{B_{12,5}} = \frac{A_{13,5}}{B_{13,5}} = \frac{A_{23,5}}{B_{23,5}}\). If \(i = 2, j = 3\), we see that there is a function \(\lambda\) such that

\[
\frac{A_1 - A_2}{B_1 - B_2} = \frac{A_1 - A_3}{B_1 - B_3} = \frac{A_2 - A_4}{B_2 - B_4} = \frac{A_1 - A_5}{B_1 - B_5} = -\lambda.
\]

Thus, from (5.8), we also see that there is another function \(\mu\) such that

\[
A_1 + \lambda B_1 = A_2 + \lambda B_2 = A_3 + \lambda B_3 = A_4 + \lambda B_4 = A_5 + \lambda B_5 = -\mu.
\]

Thus, we see that \(x\) is a conformal para-isotropic spacelike hypersurface, and from [12], we know that \(\lambda\) and \(\mu\) are constant. From Proposition 5.2, we know that \(x\) is conformally equivalent to one of the spacelike hypersurfaces with constant mean curvature and constant scalar curvature in Lorentzian space form, which, from [12], is also a conformal para-isotropic spacelike hypersurface denoted by \(\tilde{x}\). From Proposition 5.1, we know that \(\tilde{x}\) also has four distinct constant conformal principal curvatures \(\tilde{B}_i(i = 1, 2, 3, 4)\). Since \(\tilde{x}\) has constant mean curvature and constant scalar curvature, from Gaussian equation and \(e^\tilde{B}_i = \tilde{\lambda}_i - \tilde{H}\), we see that \(\tilde{x}\) is a Euclidean isoparametric spacelike hypersurface with four distinct Euclidean principal curvatures \(\lambda_i(i = 1, 2, 3, 4)\) in Lorentzian space form, this contradicts Proposition 3.3. Thus, we must have \(B_{12,5} = 0\). By the similar reason, we may prove that \(B_{13,5} = 0\) and \(B_{23,5} = 0\).

(ii) Suppose that \(B_{12,4}B_{12,5} \neq 0\), by Lemma 5.3, we have \(B_{13,4}B_{13,5} \neq 0\) and \(B_{23,4}B_{23,5} \neq 0\). By the similar method in the proof of (i), we shall conclude. \(\square\)

Now, we return to consider the case \(\gamma = 4\), since the conformal second fundamental form is not parallel, from (5.1), we should notice that the possible nonzero elements of \(B_{i,j,k}, 1 \leq i, j, k \leq 5\), may be \(\{B_{12,3}, B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}\).

We may consider two cases: \(B_{12,3} = 0\) and \(B_{12,3} \neq 0\).

Case (i). If \(B_{12,3} = 0\), since \(B\) is not parallel, we know that there is at least one nonzero element in \(\{B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}\), without loss of generality, we may assume that \(B_{12,4} \neq 0\). By Lemma 5.5, we have \(B_{12,5} = 0\) and there are at most two nonzero elements in \(\{B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}\).

Subcase (i). If \(B_{13,4} = B_{13,5} = B_{23,4} = B_{23,5} = 0\), since \(B_{12,4} \neq 0\), \(B_{12,3} = 0\) and \(B_{12,5} = 0\), from (5.4), (5.6) and (2.8), we have

\[
A_2 + A_3 - B_2B_5 = 0, \quad A_3 + A_5 - B_3B_5 = 0,
\]

\[
A_2 + A_4 - B_2B_4 = \frac{2B_{12,4}^2}{(B_1 - B_2)(B_1 - B_4)}, \quad A_3 + A_4 - B_3B_4 = 0.
\]
From (5.10) and (5.11), we have $A_2 - A_3 - (B_2 - B_3)B_4 = \frac{2B_{13,4}^2}{(B_1 - B_2)(B_1 - B_3)}, A_2 - A_3 - (B_2 - B_4)B_5 = 0$. Since $B_4 = B_5$, we see that $\frac{2B_{13,4}^2}{(B_1 - B_2)(B_1 - B_3)} = 0$, that is $B_{12,4} = 0$, a contradiction. Thus, subcase (i) does not occur.

Subcase (ii). If exactly one of $\{B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$ is nonzero, the symmetry of indices 1 and 2 implies that we need only to consider two cases: $B_{23,4} \neq 0$ with $B_{13,4} = B_{13,5} = B_{23,5} = 0$, or $B_{23,4} \neq 0$ with $B_{13,4} = B_{13,5} = B_{23,4} = 0$.

If $B_{23,4} \neq 0$ with $B_{13,4} = B_{13,5} = B_{23,5} = 0$, since $B_{12,4} \neq 0$, $B_{12,3} = B_{12,5} = 0$, from Lemma 5.4, we have

\begin{align*}
R_{1212} &= \frac{2B_{12,4}^2}{(B_4 - B_1)(B_4 - B_2)}, \\
R_{1313} &= 0, \\
R_{2424} &= \frac{2B_{23,4}^2}{(B_1 - B_2)(B_1 - B_4)} + \frac{2B_{23,4}^2}{(B_3 - B_4)(B_3 - B_4)}, \\
R_{3434} &= \frac{2B_{23,4}^2}{(B_2 - B_3)(B_2 - B_4)}, \\
R_{2525} &= 0, \\
R_{3535} &= 0.
\end{align*}

From (5.1), we have $\omega_{15} = \omega_{25} = \omega_{35} = 0$. From (2.8), we know that if three of \{i, j, k, l\} are either the same or distinct, then

\begin{equation}
R_{ijkl} = 0.
\end{equation}

By (2.4), (5.16), $\omega_{ij} = \sum_k \Gamma_{ik}^j \omega_k$, $\omega_{15} = \omega_{25} = \omega_{35} = 0$ and $R_{1515} = R_{2525} = R_{3535} = 0$, we obtain $0 = 2\omega_{15} + \sum_k \omega_{1k} \wedge \omega_{k5} = 0$. From (5.12)–(5.15) and $R_{4545} = 0$, we have $R_{4545} = 0$. From (2.2) and Proposition 3.2, we know that $x$ is a Euclidean isoparametric spacelike hypersurface with four distinct Euclidean principal curvatures, this contradicts Proposition 3.3.

If $B_{23,5} \neq 0$ with $B_{13,4} = B_{13,5} = B_{23,4} = 0$, since $B_{12,4} \neq 0, B_{12,3} = B_{12,5} = 0$, from Lemma 5.4, we have $R_{3434} = 0$. By (5.1), we have $\omega_{15} = \omega_{13} = 0$. Thus, from (2.4), $\omega_{ij} = \sum_k \Gamma_{ik}^j \omega_k$, (5.16) and $R_{3434} = 0$, we obtain $0 = -\omega_{12} \wedge \omega_{24} - \omega_{35} \wedge \omega_{54}, - \Gamma_{35}^3 \Gamma_{21}^4$, this implies that $\Gamma_{35}^3 \Gamma_{21}^4 = 0$, a contradiction. Thus, subcase (iii) does not occur.

Subcase (iii). If exactly two of $\{B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$ are nonzero, the symmetry of indices 1 and 2 and (ii) of Lemma 5.5 imply that we need only to consider three cases: $B_{23,4} \neq 0, B_{13,4} \neq 0$ with $B_{13,5} = B_{23,5} = 0$, or $B_{23,4} \neq 0, B_{13,5} \neq 0$ with $B_{13,4} = B_{23,5} = 0$, or $B_{13,4} \neq 0, B_{13,5} \neq 0$ with $B_{13,4} = B_{23,4} = 0$.

If $B_{23,4} \neq 0, B_{13,4} \neq 0$ with $B_{13,5} = B_{23,5} = 0$, since $B_{12,4} \neq 0, B_{12,3} = B_{12,5} = 0$, from (5.1), we see that $\omega_{12} = \frac{B_{12,4} \omega_4}{B_{12,4} \omega_4}, \omega_{13} = \frac{B_{13,4} \omega_4}{B_{13,4} \omega_4}, \omega_{14} = \frac{B_{12,4} \omega_2 + B_{13,4} \omega_3}{B_{12,4} \omega_2 + B_{13,4} \omega_3}, \omega_{15} = 0, \omega_{23} = \frac{B_{23} \omega_4}{B_{23} \omega_4}, \omega_{24} = \frac{B_{12,4} \omega_1}{B_{12,4} \omega_1 + \omega_3 \omega_2 + \omega_4 \omega_3}, \omega_{25} = 0, \omega_{34} = \frac{B_{13,4} \omega_1}{B_{13,4} \omega_1 + \omega_3 \omega_2 + \omega_4 \omega_3}$. From (5.1), we have $\omega_{15} = \omega_{25} = \omega_{35} = 0$. From (2.8), we know that if three of \{i, j, k, l\} are either the same or distinct, then

\begin{equation}
R_{ijkl} = 0.
\end{equation}

By (2.4), (5.16), $\omega_{ij} = \sum_k \Gamma_{ik}^j \omega_k$, $\omega_{15} = \omega_{25} = \omega_{35} = 0$ and $R_{1515} = R_{2525} = R_{3535} = 0$, we obtain $0 = 2\omega_{15} + \sum_k \omega_{1k} \wedge \omega_{k5} = 0$. From (5.12)–(5.15) and $R_{4545} = 0$, we have $R_{4545} = 0$. From (2.2) and Proposition 3.2, we know that $x$ is a Euclidean isoparametric spacelike hypersurface with four distinct Euclidean principal curvatures, this contradicts Proposition 3.3.

If $B_{23,5} \neq 0$ with $B_{13,4} = B_{13,5} = B_{23,4} = 0$, since $B_{12,4} \neq 0, B_{12,3} = B_{12,5} = 0$, from Lemma 5.4, we have $R_{3434} = 0$. By (5.1), we have $\omega_{15} = \omega_{13} = 0$. Thus, from (2.4), $\omega_{ij} = \sum_k \Gamma_{ik}^j \omega_k$, (5.16) and $R_{3434} = 0$, we obtain $0 = -\omega_{12} \wedge \omega_{24} - \omega_{35} \wedge \omega_{54}, - \Gamma_{35}^3 \Gamma_{21}^4$, this implies that $\Gamma_{35}^3 \Gamma_{21}^4 = 0$, a contradiction. Thus, subcase (iii) does not occur.
Conformal isoparametric spacelike hypersurfaces

Consider the equation $B_{23,4}^4 \omega_2 = \omega_{35} = 0$. From (2.4) and (2.1), we have

\begin{equation}
R_{2323} \omega_2 \wedge \omega_3 = -\frac{1}{2} \sum_{k,l} R_{23kl} \omega_k \wedge \omega_l = d\omega_{23} - \sum_k \omega_{2k} \wedge \omega_{k3}
\end{equation}

\begin{equation}
= \frac{B_{23,4}}{B_2 - B_3} \omega_1 \wedge \left( \frac{B_{12,4}}{B_1 - B_4} \omega_2 + \frac{B_{13,4}}{B_1 - B_4} \omega_3 \right)
\end{equation}

\begin{equation}
+ \frac{B_{23,4}}{B_2 - B_3} \omega_2 \wedge \left( \frac{B_{12,4}}{B_2 - B_4} \omega_1 + \frac{B_{23,4}}{B_2 - B_4} \omega_3 \right)
\end{equation}

\begin{equation}
+ \frac{B_{23,4}}{B_2 - B_3} \omega_3 \wedge \left( \frac{B_{13,4}}{B_3 - B_4} \omega_1 + \frac{B_{23,4}}{B_3 - B_4} \omega_2 \right)
\end{equation}

\begin{equation}
+ \left\{ \frac{B_{12,4}}{B_2 - B_4} \omega_1 + \frac{B_{23,4}}{B_2 - B_4} \omega_3 \right\} \wedge \left\{ \frac{B_{13,4}}{B_3 - B_4} \omega_1 + \frac{B_{23,4}}{B_3 - B_4} \omega_2 \right\}.
\end{equation}

Comparing the coefficients of $\omega_1 \wedge \omega_2$ and $\omega_1 \wedge \omega_3$ on both sides of the above equation, we obtain

$\frac{1}{(b_2 - b_3)(b_1 - b_4)}\left(\frac{1}{b_2 - b_3}\frac{1}{b_1 - b_4}\right) - \frac{1}{(b_3 - b_4)(b_1 - b_4)}\left(\frac{1}{b_3 - b_4}\frac{1}{b_1 - b_4}\right) = 0$, this implies $\frac{1}{(b_2 - b_3)(b_1 - b_4)}\left(\frac{1}{b_2 - b_3}\frac{1}{b_1 - b_4}\right) = 0$, a contradiction.

If $B_{23,4} \neq 0$, $B_{13,5} \neq 0$ with $B_{13,4} = B_{23,5} = 0$, since $B_{12,4} \neq 0$, $B_{12,3} = B_{12,5} = 0$, from (5.7) and for the reason in the proof of Lemma 5.5, we see that (5.9) holds for $\lambda$ and $\mu$, that is, $x$ is a conformal para-isotropic spacelike hypersurface. By reasoning as in the proof of Lemma 5.5 again, we have a contradiction.

If $B_{23,5} \neq 0$, $B_{13,5} \neq 0$ with $B_{13,4} = B_{23,4} = 0$, since $B_{12,4} \neq 0$, $B_{12,3} = B_{12,5} = 0$, by reasoning as in the proof of Lemma 5.5, we see that $x$ is a conformal para-isotropic spacelike hypersurface and we also have a contradiction. Thus, subcase (iii) does not occur.

To sum up, we know that case (i) does not occur.

\textit{Case (ii).} If $B_{12,3} \neq 0$, by Lemma 5.3 and Lemma 5.5, we see that there are at most three nonzero elements in $(B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5})$.

\textit{Subcase (i).} If $B_{12,4} = B_{12,5} = B_{13,4} = B_{13,5} = B_{23,4} = B_{23,5} = 0$, since $B_{12,3} \neq 0$, from Lemma 5.4 and (2.8), we have

\begin{equation}
A_1 + A_2 - B_1 B_2 = \frac{2B_{12,3}^2}{(B_3 - B_1)(B_3 - B_2)}.
\end{equation}

\begin{equation}
A_1 + A_4 - B_1 B_3 = \frac{2B_{12,3}^2}{(B_2 - B_1)(B_2 - B_3)}.
\end{equation}

\begin{equation}
A_2 + A_3 - B_2 B_3 = \frac{2B_{12,3}^2}{(B_1 - B_2)(B_1 - B_3)}.
\end{equation}

\begin{equation}
A_1 + A_4 - B_1 B_4 = 0, \quad A_2 + A_4 - B_2 B_4 = 0, \quad A_3 + A_4 - B_3 B_4 = 0,
\end{equation}

\begin{equation}
A_1 + A_5 - B_1 B_5 = 0, \quad A_2 + A_5 - B_2 B_5 = 0, \quad A_3 + A_5 - B_3 B_5 = 0.
\end{equation}
Since $B_4 = B_5$, from (5.21) and (5.22), we get $A_4 = A_5$. From (5.18)–(5.21) we obtain

\begin{equation}
(B_1 - B_2)(B_3 - B_4) = \frac{2(B_1 + B_2 - 2B_3)B_{12,3}^2}{(B_2 - B_1)(B_2 - B_3)(B_1 - B_3)},
\end{equation}

\begin{equation}
2A_1 - B_1B_2 - B_1B_3 + B_2B_3 = \frac{4B_{12,3}^2}{(B_3 - B_1)(B_1 - B_2)}.
\end{equation}

From (5.23), we see that $B_{12,3}$ is constant. Thus, from (5.24), (5.18)–(5.22), we know that $A_1, A_2, A_3, A_4, A_5$ are constants. By (5.21), we see that $A_1 - B_1B_1 = A_2 - B_2B_2 = A_3 - B_3B_3 = A_4$. On the other hand, we have $A_4 - B_4B_4 = A_5 - B_4B_5 = \text{constant}: = \nu$. We may prove that $\nu \neq A_4$. In fact, if $\nu = -A_4$, denote by $D = A + (-B_4)B$ the conformal para-Blaschke tensor of the immersion $x$, we see that $x$ is a conformal para-isotropic spacelike hypersurface. By reasoning as in the proof of Lemma 5.5, we have a contradiction. Thus, we know that $x$ must be a conformal spacelike hypersurface with two distinct constant conformal para-Blaschke eigenvalues. Let $\zeta$ and $\eta$ be the two distinct constant conformal para-Blaschke eigenvalues of $x$ with multiplicities $m$ and $5 - m$ respectively. From the definition of the covariant derivative of $D_{ij}$, we have $\sum_k D_{ij,k}\omega_k = (D_i - D_j)\omega_{ij}$. Thus $D_{ij,k} = 0$ for $1 \leq i, j \leq m$, or $m + 1 \leq i, j \leq 5$. From the symmetry of $D_{ij,k}$, we see that $D_{ij,k} = 0$ for all $i, j, k$, that is, the conformal para-Blaschke tensor of $x$ is parallel. Thus, we have $\omega_{ij} = 0$, for $1 \leq i \leq m, m + 1 \leq j \leq 5$. Hence, we know that the distributions of the eigenspaces with respect to $\zeta$ and $\eta$ are integrable. Since the number of distinct conformal para-Blaschke eigenvalues of $x$ is two, we see that $x$ is locally a Riemannian product $M^m_1 \times M^{5-m}_2$, where $M^m_1$ and $M^{5-m}_2$ are the Riemannian integrable manifold corresponding to $\zeta$ and $\eta$ respectively. Since $\omega_{ij} = 0$, for $1 \leq i \leq m, m + 1 \leq j \leq 5$, we have $R_{ijij} = 0$ for $1 \leq i \leq m, m + 1 \leq j \leq 5$. Thus, from (2.8) and $D = A + (-B_4)B$, we have $(B_i - B_4)(B_j - B_4) - (\zeta + \eta) - B_4^2 = 0$ for $1 \leq i \leq m, m + 1 \leq j \leq 5$.

If $m = 1$, we have $(B_1 - B_4)(B_{j1} - B_{j2}) = 0$ for $2 \leq j_1, j_2 \leq 5, j_1 \neq j_2$. Since $B_1 \neq B_4$, we obtain that $B_2 = B_3 = B_4 = B_5$, a contradiction.

If $m = 2$, we have $(B_{i1} - B_{i2})(B_j - B_4) = 0$ for $1 \leq i_1, i_2 \leq 2, 1 \leq j \leq 5$. Since $B_1 \neq B_2$, we obtain that $B_3 = B_4 = B_5$, a contradiction. Thus, we must have $m \geq 3$. From (2.8), we may easily obtain that $R_{ijkl} = (2\eta + B_4^2)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$ for $m + 1 \leq i, j, k, l \leq 5$, that is, $M^{5-m}_2$ is a constant curvature Riemannian manifold.

Subcase (ii). If exactly one of $\{B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$ is nonzero, without loss of generality, we may assume that $B_{12,4} \neq 0$. Since $B_{12,3} \neq 0, B_{12,5} = B_{13,4} = B_{13,5} = B_{23,4} = B_{23,5} = 0$, from (5.5), (5.6) and (2.8), we have $A_2 + A_4 - B_2B_4 = \frac{2B_{12,4}^2}{(B_1 - B_2)(B_1 - B_3)}, A_2A_5 - B_2B_5 = 0, A_3 + A_4 - B_3B_4 = 0, A_3 + A_5 - B_3B_5 = 0$, this implies that $\frac{2B_{12,4}^2}{(B_1 - B_2)(B_1 - B_3)} = 0$ and $B_{12,4} = 0$, a contradiction. Thus, subcase (ii) does not occur.

Subcase (iii). If exactly two of $\{B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$ are nonzero, without loss of generality, we may assume that $B_{12,4} \neq 0$. By Lemma 5.5, we have $B_{12,5} = B_{13,4} = B_{23,5} = 0$. Thus exactly one of $\{B_{13,4}, B_{23,4}\}$ is nonzero, without loss of generality, we may assume that $B_{13,4} \neq 0, B_{23,4} = 0$. From (5.1),
(2.4), (2.1) and for the reason above, we see that
\begin{equation}
-R_{2323} \omega_2 \wedge \omega_3 = -\frac{1}{2} \sum_{k,l} R_{23kl} \omega_k \wedge \omega_l = d \omega_{23} - \sum_{k} \omega_{2k} \wedge \omega_{k3}
\end{equation}

\[ \begin{align*}
&= \frac{dB_{12,3}}{B_2 - B_3} \wedge \omega_1 + \frac{B_{12,3}}{B_2 - B_3} \left( \frac{B_{12,3}}{B_3 - B_2} \omega_3 + \frac{B_{12,4}}{B_1 - B_2} \omega_4 \right) \wedge \omega_2 \\
&+ \frac{B_{12,3}}{B_2 - B_3} \left( \frac{B_{12,3}}{B_1 - B_3} \omega_2 + \frac{B_{13,4}}{B_1 - B_2} \omega_4 \right) \wedge \omega_3 \\
&+ \frac{B_{12,3}}{B_2 - B_3} \left( \frac{B_{12,4}}{B_1 - B_4} \omega_2 + \frac{B_{13,4}}{B_1 - B_4} \omega_3 \right) \wedge \omega_4 \\
&+ \left( \frac{B_{12,3}}{B_1 - B_3} \omega_1 + \frac{B_{12,4}}{B_1 - B_3} \omega_2 \right) \wedge \left( \frac{B_{12,3}}{B_1 - B_3} \omega_2 + \frac{B_{13,4}}{B_1 - B_3} \omega_4 \right).
\end{align*} \]

Comparing the coefficients of $\omega_2 \wedge \omega_4$ and $\omega_3 \wedge \omega_4$ on both sides of the above equation, we obtain $(B_2 - B_1)(B_1 - B_3) = (B_2 - B_3)(B_1 - B_2) = (B_1 - B_2)(B_1 - B_3) = 0$, this implies $(B_1 - B_2)(B_1 - B_3) = 0$, a contradiction. Thus, subcase (iii) does not occur.

**Subcase (iv).** If exactly three of $\{B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$ are nonzero, we may consider the following cases:

If all of $B_{12,5}, B_{13,5}, B_{23,5}$ are zero, then it must have $B_{12,4} \neq 0$, $B_{13,4} \neq 0$, $B_{23,4} \neq 0$. From (5.1), Lemma 5.4 and (2.8), we have $\omega_{15} = \omega_{25} = \omega_{35} = 0$ and $R_{1515} = 0$, $R_{2525} = 0$, $R_{3535} = 0$. Therefore, from (2.4) and (4.3), we obtain
\[ 0 = d \omega_{15} - \sum_{k,l} \omega_{1k} \wedge \omega_{k5} = -\omega_{14} \wedge \omega_{45} = -(\Gamma^k_{14} \omega_2 + \Gamma^k_{14} \omega_l) \wedge \omega_{45}, 0 = -\omega_{24} \wedge \omega_{45} = -(\Gamma^k_{14} \omega_1 + \Gamma^k_{14} \omega_l) \wedge \omega_{45}, 0 = -\omega_{34} \wedge \omega_{45} = -(\Gamma^k_{14} \omega_3 + \Gamma^k_{14} \omega_l) \wedge \omega_{45}.
\]

This follows that $\omega_{45} = 0$. Combining $\omega_{15} = \omega_{25} = \omega_{35} = 0$, we obtain $R_{1545} = 0$. From (5.4), (5.5) and $R_{i55} = 0, i = 1, 2, 3, 4$, we get $\kappa = \frac{1}{25} \sum_{i \neq j} R_{i j i j} = 0 \neq 1$. From (2.2) and Proposition 3.2, we know that $x$ is a Euclidean isometric spacelike hypersurface with four distinct Euclidean principal curvatures, this contradicts Proposition 3.3.

If two of $\{B_{12,5}, B_{13,5}, B_{23,5}\}$ are zero, without loss of generality, we may assume that $B_{12,5} = B_{13,5} = 0$ and $B_{23,4} \neq 0$. From (ii) of Lemma 5.5, we must have $B_{23,4} = 0$. Thus, it must follow that $B_{12,4} \neq 0$, $B_{13,4} \neq 0$. From (5.7) and for the reason in the proof of Lemma 5.5, we see that $x$ is a conformal para-isotropic spacelike hypersurface and we have a contradiction.

If one of $\{B_{12,5}, B_{13,5}, B_{23,5}\}$ is zero, without loss of generality, we may assume that $B_{12,5} = 0$, $B_{13,5} \neq 0$ and $B_{23,5} \neq 0$. From (ii) of Lemma 5.5, we must have $B_{13,4} = B_{23,4} = 0$. Thus, it must follow that $B_{12,4} \neq 0$. By reasoning as in the proof of Lemma 5.5, we see that $x$ is a conformal para-isotropic hypersurface and we have a contradiction.

To sum up, we know that case (ii) does not occur.

(4) If $\gamma = 5$, from Theorem 3.1, Example 3.1–Example 3.4, we know that $B$ is not parallel. Without loss of generality, we may assume that $B_{12,3} \neq 0$. Since $B_1 \neq B_2 \neq B_3 \neq B_4 \neq B_5$, from (4.2), we have $B_{ik} = 0$ for all $i, k$.

By a similar method as in the proof of [4], we have the following (see Lemma 4.1 in [4]):

**Lemma 5.6.** Let $B_1, B_2, B_3, B_4, B_5$ be the constant conformal principal curvatures of $x : M \to \mathbb{R}^6_1$ with $B_1 \neq B_2 \neq B_3 \neq B_4 \neq B_5$ and $i,j,k,l,s$ be the five distinct
elements of \{1, 2, 3, 4, 5\} with arbitrarily given order. Then

\begin{equation}
R_{ijij} = \frac{2B^2_{ij,k}}{(B_k - B_j)(B_k - B_i)} + \frac{2B^2_{ij,l}}{(B_l - B_i)(B_l - B_j)} + \frac{2B^2_{ij,s}}{(B_s - B_i)(B_s - B_j)}.
\end{equation}

Now, we return to consider the case \(\gamma = 5\), since \(B\) is not parallel, we may consider the following two cases:

Case (i). If \(B_{12,3} \neq 0\) and all of \(\{B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}\) are zero, in this case, we may prove that at most one of \(\{B_{14,5}, B_{24,5}, B_{34,5}\}\) is zero. In fact, without loss of generality, if \(B_{14,5} = B_{24,5} = 0\), by Lemma 5.6, we obtain \(\kappa = \frac{1}{20} \sum_{i \neq j} R_{iiij} = 0 \neq 1\). From (2.2) and Proposition 3.2, we know that \(x\) is a Euclidean isoparametric spacelike hypersurface with five distinct Euclidean principal curvatures, this contradicts Proposition 3.3.

We now assume that \(B_{24,5} \neq 0, B_{34,5} \neq 0\). Since \(B_{12,3} \neq 0\), by the similar method as in the proof of Lemma 5.5, we see that there exist \(\lambda\) and \(\nu\) such that \(A_2 + \lambda B_2 = A_3 + \lambda B_3 = A_4 + \lambda B_4 = A_5 + \lambda B_5, A_1 + \nu B_1 = A_2 + \nu B_2 = A_3 + \nu B_3\), this implies that \(\lambda = \nu\) and

\begin{equation}
A_1 + \lambda B_1 = A_2 + \lambda B_2 = A_3 + \lambda B_3 = A_4 + \lambda B_4 = A_5 + \lambda B_5.
\end{equation}

From (5.27), we see that \(x\) is a conformal para-isotropic spacelike hypersurface. By reasoning as in the proof of Lemma 5.5, we have a contradiction.

Case (ii). If \(B_{12,3} \neq 0\) and at least one of \(\{B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}\) is nonzero, without loss of generality, we may assume that \(B_{12,4} \neq 0\). We consider the following two subcases:

Subcase (i). If all of \(\{B_{12,5}, B_{13,5}, B_{23,5}, B_{14,5}, B_{24,5}, B_{34,5}\}\) are zero, since \(B_{12,3} \neq 0\) and \(B_{12,4} \neq 0\), by Lemma 5.6, we obtain \(\kappa = \frac{1}{20} \sum_{i \neq j} R_{iiij} = 0 \neq 1\). From (2.2) and Proposition 3.2, we know that \(x\) is a Euclidean isoparametric spacelike hypersurface with five distinct Euclidean principal curvatures, this contradicts Proposition 3.3.

Subcase (ii). If at least one of \(\{B_{12,5}, B_{13,5}, B_{23,5}, B_{14,5}, B_{24,5}, B_{34,5}\}\) is nonzero, without loss of generality, we may assume that \(B_{12,5} \neq 0\). Since \(B_{12,3} \neq 0\) and \(B_{12,4} \neq 0\), by the similar method as in the proof of Lemma 5.5, we see that \(x\) is a conformal para-isotropic spacelike hypersurface and we have a contradiction. This completes the proof of Theorem 1.3. \(\Box\)

References


Conformal isoparametric spacelike hypersurfaces


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