Characterization of warped product submanifolds in Kenmotsu manifolds
Abdulqader Mustafa, Avik De and Siraj Uddin

Abstract. In the present paper, we show the existence of warped product semi-slant submanifolds in a Kenmotsu manifold by an example. We locally characterize the warped product semi-slant submanifolds in a Kenmotsu manifold. Such submanifold does not exist in Kähler, Sasakian and cosymplectic manifolds. Further, we search some geometric properties to construct an inequality for second fundamental form of the immersion of warped product submanifolds in Kenmotsu space forms. The equality case is also discussed.

Key words: Mean curvature; isometric immersion; warped products; semi-slant submanifolds; Kenmotsu space form.

1 Introduction

The idea of slant submanifolds of an almost Hermitian manifold was given by Chen [9] as a generalization of holomorphic and totally real submanifolds. Later on, N. Papaghiuc [21] introduced another class of submanifolds, called semi-slant submanifolds which generalize CR as well as slant submanifolds. On the other hand, warped products appeared in differential geometry, generalizing the class of Riemannian product manifolds [4]. The study of warped products are applied in general relativity to model the standard space time, specially in the neighborhood of massive stars and black holes. Let $N_1$ and $N_2$ be two Riemannian manifolds with Riemannian metrics $g_1$ and $g_2$, respectively, and $f > 0$ be a differential function on $N_1$. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$. Then their warped product $N_1 \times_f N_2$ is the Riemannian manifold $(N_1 \times N_2, g)$ equipped with the Riemannian structure such that

$$\|X\|^2 = \|\pi_{1*}(X)\|^2 + (f \circ \pi_1)^2 \|\pi_{2*}(X)\|^2,$$

for any tangent vector $X$ on $M$, where $*$ is the symbol for the tangent maps. It was proved in [4] that for a warped product manifold $M = N_1 \times_f N_2$, we have

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z,$$

for any $X \in TN_1$ and $Z \in TN_2$, where $\nabla$ denotes the Levi-Civita connection on $M$. A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function $f$ is constant. For a survey on warped products as Riemannian submanifolds we refer to ([2], [10], [12], [17]). Recently, Sahin [22] introduced the notion of semi-slant warped product in complex geometry.

On the other hand, Kenmotsu [15] studied a class of almost contact metric manifolds, so called Kenmotsu manifolds. He showed that Kenmotsu manifold is locally a warped product $I \times_f N$ of an interval $I$ and a Kähler manifold $N$ with warping function $f(t) = se^t$, where $s$ is a nonzero constant. Kenmotsu manifolds were studied by many authors such as De [13], Binh, Tamassy, De and Tarafdar [3], Ozgur ([19], [20]). Since Kenmotsu manifolds are themselves locally warped product spaces, it is interesting to study geometry of warped product submanifolds in the context of Kenmotsu manifolds. Several authors has studied warped product submanifolds of Kenmotsu manifolds (see for example, [1], [2], [18], [24] and the references therein). Non-existence of warped product semi-slant submanifolds in Kähler, cosymplectic and Sasakian manifolds was shown in [22], [16] and [23], respectively. On the contrary, there do exist warped product semi-slant submanifolds in Kenmotsu manifolds as given in Example 3.1.

Chen used Codazzi equation to construct a relation between the second fundamental form and the warping function for a CR-warped product in complex space forms [10]. Later on, it was extended for Sasakian and Kenmotsu space forms in [17] and [1], respectively. We use Gauss equation to establish an inequality in terms of the second fundamental form and the scalar curvatures.

This paper is organized as: In section 2, we enlist the basic definitions and equations which we need for the next sections. In Section 3, warped product semi-slant submanifolds in Kenmotsu manifolds are characterized. We also prove the existence of warped product semi-slant submanifolds in Kenmotsu manifolds by an example. In the last section, we discuss some geometric properties, specially $N_T$-minimality and using this result, we derive a general inequality. Finally, we establish an inequality for a more general type of warped product submanifolds $N_T \times_f N$ in a Kenmotsu space form.

2 Preliminaries

A $(2m + 1)$-dimensional $C^\infty$ manifold $\tilde{M}$ is said to have an almost contact metric structure if there exist on $\tilde{M}$ a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ satisfying

$$
\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1,
$$

$$
\eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
$$

where $X$ and $Y$ are vector fields on $\tilde{M}$ [5].

A Riemannian manifold $\tilde{M}$ with an almost contact metric structure $(\phi, \xi, \eta, g)$ is called a Kenmotsu manifold if [15]

$$
(\tilde{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,
$$

for all $X, Y \in T\tilde{M}$. From (2.1) we also have $\tilde{\nabla}_X \xi = X - \eta(X)\xi$, for all $X \in T\tilde{M}$. 


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Let $M$ be a Riemannian manifold isometrically immersed in a Kenmotsu manifold $\tilde{M}$. Then, Gauss and Weingarten formulae are respectively given by \cite{8}

$$\nabla_X Y = \nabla_X Y + h(X, Y),$$

and

$$\nabla_X N = -A_N X + \nabla^\perp_X N,$$

for all vector fields $X, Y$ tangent to $M$, where $\nabla$ is the induced Riemannian connection on $M$, $N$ is a vector field normal to $M$, $h$ is the second fundamental form of $M$, $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and $A_N$ is the shape operator corresponding to $N$. Clearly, $g(A_N X, Y) = g(h(X, Y), N)$, where $g$ denotes the Riemannian metric on $\tilde{M}$ as well as the metric induced on $M$.

The equation of Gauss for the submanifold $M$ is given by \cite{8}

$$R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z))$$

$$- g(h(X, Z), h(Y, W)),$$

for all $X, Y, Z, W \in TM$, where $\tilde{R}$ and $R$ are the curvature tensors of $\tilde{M}$ and $M$ respectively.

For any $X \in TM$ and $N \in T^\perp M$, we write $\phi X = PX + FX$, and $\phi N = tN + fN$, where $PX$, $tN$ are the tangential components and $FX$, $fN$ are the normal components of $\phi X$ and $\phi N$, respectively. If $PX = 0$ (resp. $FX = 0$), then $M$ is called an invariant (resp. anti-invariant) submanifold.

The covariant derivatives of the tensor fields $P$ and $F$ are defined as

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y,$$

$$(\nabla_X F)Y = \nabla^\perp_X FY - F\nabla_X Y.$$

Also, we have

$$(\nabla_X P)Y = A_F Y X + th(X, Y) - g(X, PY)\xi - \eta(Y) PX,$$

$$(\nabla_X F)Y = fh(X, Y) - h(X, PY) - \eta(Y) FX.$$

We recall that the Riemannian curvature tensor of a Kenmotsu space form $\tilde{M}(c)$ of constant $\phi$-sectional curvature $c$ is given by \cite{15}

$$\tilde{R}(X, Y, Z, W) = \frac{c - 3}{4} \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}$$

$$+ \frac{c + 1}{4} \{\eta(Z)[\eta(Y)g(X, W) - \eta(X)g(Y, W)]$$

$$+ g(Y, Z)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - g(\phi X, W)g(\phi Y, Z)$$

$$+ g(\phi X, Z)g(\phi Y, W) + 2g(\phi X, Y)g(\phi Z, W)\},$$

for any vector fields $X, Y, Z, W$ tangent to $\tilde{M}$.

For an orthonormal frame $\{e_1, \cdots, e_n\}$ of $TM$, the mean curvature vector $H$ is given by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$
where $n = \dim M$. The submanifold $M$ is said to be \textit{totally geodesic} in $M$ if $h = 0$, and \textit{minimal} if $H = 0$. If $h(X, Y) = g(X, Y)H$ for all $X, Y \in TM$, then $M$ is called \textit{totally umbilical}. It is easy to check that a totally umbilical submanifold in Kenmotsu manifold is always totally geodesic. For differentiable function $\psi$ on $M$, the \textit{gradient} $\nabla \psi$ and the \textit{Laplacian} $\Delta \psi$ of $\psi$ are defined respectively by

$$g(\nabla \psi, X) = X \psi,$$

and

$$\Delta \psi = \sum_{i=1}^{n} ((\nabla_{e_i} \psi) - e_i e_i \psi),$$

for any vector field $X$ tangent to $M$. The scalar curvature of $M$ is defined by

$$\tau(TM) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by $e_i$ and $e_j$.

Let $M$ be a submanifold of an almost contact metric manifold $M$. For each non zero vector $X$ tangent to $M$ at $x$, such that $X$ is not proportional to $\xi$, if the angle $\theta(X) \ (0 \leq \theta(X) \leq \pi/2)$ between $\phi X$ and $T_x M$ is constant for all $X \in T_x M - (\xi)$ and $x \in M$, then $M$ is said to be a slant submanifold [6]. Obviously, if $\theta = 0$, $M$ is invariant and if $\theta = \pi/2$, $M$ is an anti-invariant submanifold. A slant submanifold is said to be \textit{proper slant} if it is neither invariant nor anti-invariant. For $\xi \in T^\perp M$, following the same procedure as in [6] we can easily verify the following:

**Theorem 2.1.** Let $M$ be a submanifold of an almost contact metric manifold $M$ such that $\xi \in T^\perp M$. Then $M$ is slant if and only if there exists a constant $\delta \in [0, 1]$ such that $P^2 X = -\delta X$. Furthermore, if $\theta$ is slant angle, then $\delta = \cos^2 \theta$. Also, for all $X, Y \in TM$,

$$(2.8) \quad g(\gamma X, P Y) = \cos^2 \theta g(X, Y),$$

$$(2.9) \quad g(\phi X, FY) = \sin^2 \theta g(X, Y),$$

**Definition 2.1.** [7] A submanifold $M$ of an almost contact manifold $M$ is said to be a \textit{proper semi-slant submanifold} if there exist two distributions $D$ and $D_\theta$ such that

(i) $TM = D \oplus D_\theta \oplus \langle \xi \rangle$

(ii) $D$ is invariant i.e., $\phi D \subseteq TM$.

(iii) $D_\theta$ is slant with slant angle $0 \neq \theta \neq \frac{\pi}{2}$.

A semi-slant submanifold $M$ is said to be \textit{mixed geodesic} if $h(X, Z) = 0$, for any $X \in D$ and $Z \in D_\theta$.

If $\nu_x$ is the maximal invariant subspace of the normal space $T^\perp_x M$, $x \in M$, then in the case of semi-slant submanifold, $\nu : x \rightarrow \nu_x$, $x \in M$, forms a subbundle of the normal bundle $T^\perp M$. Then, $T^\perp M$ can be decomposed as $T^\perp M = FD_\theta \oplus \nu$. 
3 Characterization for warped products

In this section, we prove a local characterization for warped product semi-slant submanifolds $M = N_T \times_f N_\theta$, with $\xi$ tangent to $N_T$, in a Kenmotsu manifold. We first prove the following:

**Lemma 3.1.** Let $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold in a Kenmotsu manifold $M$ such that $\xi$ is tangent to $N_T$, where $N_T$ and $N_\theta$ are invariant and slant submanifolds of $M$, respectively. Then

(i) $X\ln f - \eta(X) = 0$,
(ii) $A_{FZ}X = th(X, Z) = 0$

for all $X \in TN_T$ and $Z \in TN_\theta$.

**Proof.** From (2.3) and (2.5) we have

$$A_{FZ}X + th(X, Z) = (\nabla_X P)Z = (X\ln f)PZ - (X\ln f)PZ = 0,$$

which implies

$$A_{FZ}X + th(X, Z) = 0,$$

and thus for all $Z, W \in TN_\theta$,

$$g(h(X, W), FZ) = g(h(X, Z), FW).$$

Again, from (2.3) and (2.5) we get

$$th(X, Z) - \eta(X)PZ = (\nabla_Z P)X = (PX\ln f)Z - (X\ln f)PZ,$$

which implies

$$\{X\ln f - \eta(X)\}g(PZ, W) = (PX\ln f)g(Z, W) + g(h(X, Z), FW).$$

Interchanging $Z$ and $W$ in the above equation we obtain

$$\{X\ln f - \eta(X)\}g(PW, Z) = (PX\ln f)g(Z, W) + g(h(X, W), FZ).$$

From (3.3) and (3.4) we conclude that $X\ln f - \eta(X) = 0$, and it directly implies $PX\ln f = 0$, for all $X \in TN_T$. Therefore, from (3.1) and (3.3) we obtain the lemma.

**Theorem 3.2.** Let $M$ be a proper semi-slant submanifold of Kenmotsu manifold $\tilde{M}$. Then $M$ is locally a warped product of invariant and slant submanifolds if and only if

$$A_{FZ}X = 0,$$

for any $X \in D \oplus \langle \xi \rangle$ and any $Z \in D_\theta$, where $D \oplus \langle \xi \rangle$ and $D_\theta$ are invariant and slant distributions of $M$, respectively.
Proof. Let \( M \) be a semi-slant submanifold of a Kenmotsu manifold \( \tilde{M} \) such that (3.5) holds. Let \( Y \) be a vector field in \( D \oplus \langle \xi \rangle \), \( Z \in D_\theta \) and \( V \in TM \). Then, from (2.9) we have

\[
\sin^2 \theta g(\nabla_Y Z, Z) = g(F \nabla_Y Z, FZ) = -g((\nabla_Y F)Z, FZ) = g(fh(V, Y) - h(V, PY) - \eta(Y)FV, FZ) = -g(A_{fFZ}Y + A_{FZ}PY, V) + \sin^2 \theta \eta(Y)g(V, Z). \tag{3.6}
\]

Now, \(-Z = \phi^2 Z = P^2 Z + tFZ + fFZ + FPZ\), implies that \( fFZ = -FPZ \). Hence, from (3.5) and (3.6) we conclude \( g(\nabla_Y Z, Z) = \eta(Y)g(V, Z) \), since \( \sin^2 \theta \neq 0 \).

So, if \( V \in D \oplus \langle \xi \rangle \), we obtain \( \nabla_Y Z \perp D_\theta \), which implies \( D \oplus \langle \xi \rangle \) is integrable and each of its leaves \( N_T \) is totally geodesic in \( M \).

Next, if we consider \( V \in D_\theta \), then \( g(\nabla_Y Z, Y) = -g(V, Z)g(\xi, Y) \), for all \( Y \in D \oplus \langle \xi \rangle \). Hence, \( D_\theta \) is integrable. Let us consider \( N_\theta \) to be a leaf of \( D_\theta \) and \( h^0 \) be the second fundamental form of the immersion of \( N_\theta \) in \( M \). Then we have \( g(h^0(V, Z), Y) = g(\nabla_Y Z, Y) = -g(V, Z)g(\xi, Y) \), for all \( Y \in D \oplus \langle \xi \rangle \). Hence \( N_\theta \) is totally umbilical in \( M \) with mean curvature vector \( \xi \). Moreover, if \( \tilde{\nabla} \) is the normal connection of the immersion \( N_\theta \) in \( M \), then \( g(\tilde{\nabla}_Z \xi, Y) = g(\nabla_Y \xi, Y) = g(Z - \eta(\xi)\xi, Y) = 0 \), implying the mean curvature vector of \( N_\theta \) is parallel. Thus the leaves of \( D \oplus \langle \xi \rangle \) are totally geodesic and the leaves of \( D_\theta \) are totally umbilical with parallel mean curvature vector. Hence by a result of Hiepko [14], \( M \) is a warped product of the type \( M = N_T \times_f N_\theta \), \( \xi \) tangent to \( N_T \), for some function \( f \) defined on \( N_T \).

The converse part is obvious from Lemma 3.1. \( \square \)

Now we provide an example of warped product semi-slant submanifold of a Kenmotsu manifold.

Example 3.1. Let \( \tilde{M}_1, \tilde{M}_2 \) be two Kähler manifolds. Then \( \tilde{M}_1 \times \tilde{M}_2 \) is a Kähler manifold. Let \( N_\theta \subset \tilde{M}_2 \) be a slant submanifold. Note that, \( \tilde{M} = \mathbb{R} \times_{f} (\tilde{M}_1 \times \tilde{M}_2) \), \( f = e^0 \) is a Kenmotsu manifold.

Then \( N_T = \mathbb{R} \times_{f} \tilde{M}_1 \subset \tilde{M} \) is an invariant submanifold of \( \tilde{M} \) and \( N_\theta \subset \tilde{M} \) is a slant submanifold of \( \tilde{M} \).

Therefore, \( N_T \times_{f} N_\theta = \mathbb{R} \times_{f} (\tilde{M}_1 \times N_\theta) \subset \tilde{M} \) is a warped product semi-slant submanifold of the Kenmotsu manifold \( \tilde{M} \).

4 Inequality for warped products \( N_T \times_{f} N \)

In this section, we consider an \( n \)-dimensional warped product submanifold \( M = N_T \times_{f} N \) of a \((2m+1)\)-dimensional Kenmotsu manifold \( \tilde{M} \) such that \( \xi \) is tangent to \( N_T \), where \( N_T \) is an invariant submanifold of \( \tilde{M} \) and \( N \) is a Riemannian submanifold of \( \tilde{M} \). Let the dimension of \( N_T \) be \( n_1 \) and the dimension of \( N \) be \( n_2 \), then \( n = n_1 + n_2 \).

We consider the orthonormal frame \( \{e_1, \ldots, e_{2m+1}\} \) of \( T\tilde{M} \) where \( e_1, \ldots, e_s, e_{s+1} = \phi e_1, \ldots, e_{n_1-1} = \phi e_s, e_{n_1} = \xi \) are tangent to \( N_T \), \( e_{n_1+1} \), \ldots, \( e_n = e_{n_1+n_2} \) are tangent to \( N \), \( e_{n_1} = F e_{n_1+1}, \ldots, e_{n+n_2} = F e_n \).

In the beginning of this section, we prove the following lemma for later use.
Lemma 4.1. Let $M = N_T \times_f N$, with $\xi$ tangent to $N_T$ be a warped product submanifold in a Kenmotsu manifold $M$, such that $N_T$ is invariant in $M$, and $N$ is a Riemannian submanifold of $M$. Then, for any $X,Y \in TN_T$, $Z \in TN$ and the normal vector $\zeta \in \nu$, the following holds:

(i) $g(h(X,Y), FZ) = 0$,

(ii) $g(h(X,X), \zeta) = -g(h(\phi X, \phi X), \zeta)$.

Proof. We have,

$$g(h(X,Y), FZ) = g(\nabla_X Y, FZ)$$

$$= g(\nabla_X Y, \phi Z) - g(\nabla_X Y, PZ). \tag{4.1}$$

Since, $g(Y, PZ) = g(Y, \phi Z) = -g(\phi Y, Z) = 0$, for $Y \in TN_T$, $Z \in TN$, from the above equation (4.1) and the Kenmotsu structure equation (2.1) we obtain

$$g(h(X,Y), FZ) = g(\phi Y, \nabla_X Z) + g(Y, \nabla_X PZ) \tag{4.2}$$

Hence, using (1.1) in the above equation, we get the required result (i). To prove the second part, we make use of (1.1) and (2) to obtain

$$\nabla_X \phi X + h(\phi X, X) - \phi \nabla_X X - \phi h(X, X) = -g(X, X)\xi + \eta(X)X.$$ 

Taking the inner product with $\phi \zeta$, we deduce

$$g(h(\phi X, X), \phi \zeta) = g(h(X, X), \zeta). \tag{4.3}$$

Interchanging $X$ with $\phi X$ in (4.3) and using (1.1), we obtain

$$g(h(\phi X, \phi X), \zeta) = -g(h(X, \phi X), \phi \zeta) + \eta(X)g(h(\xi, \phi X), \phi \zeta).$$

Since for a Kenmotsu manifold $h(\xi, \phi X) = 0$, hence we get

$$g(h(\phi X, \phi X), \zeta) = -g(h(X, \phi X), \phi \zeta). \tag{4.4}$$

Thus the result follows from (4.3) and (4.4). \qed

Definition 4.1. [11] An immersion $\varphi : N_1 \times_f N_2 \to \bar{M}$ is called $N_1$-totally geodesic if the partial second fundamental form $h_i$ vanishes identically. It is called $N_i$-minimal if the partial mean curvature vector $H_i$ vanishes, for $i = 1$ or 2.

Theorem 4.2. In a Kenmotsu manifold $\bar{M}$, every isometric immersion $\varphi : M = N_T \times_f N \to \bar{M}$, with $\xi$ tangent to $N_T$, is $N_T$-minimal, where $N_T$ is an invariant submanifold of $M$, and $N$ is a Riemannian submanifold of $M$.

Proof. By definition, the squared norm of the mean curvature vector $H$ restricted to $N_T$ is given by

$$||H||^2 = \frac{1}{n_T^2} \sum_{r = n+1}^{2n+1} (h'_{11} + \cdots + h'_{n_1, n_1})^2.$$
Using the frames of $TN_T$ and $TN$ and the fact that for a Kenmotsu manifold $h(\xi, \xi) = 0$, the above definition can be expanded as

$$
\|H_1\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} (h_{r1}^r + \cdots + h_{rs}^r h_{s+1}^r + \cdots + h_{2s}^r)^2,
$$

where $h_{ij}^r = g(h(e_i, e_j), e_r), 1 \leq i, j \leq n, n + 1 \leq r \leq 2m + 1$.

Using Lemma 4.1, we obtain $\|H_1\|^2 = 0$.

From the above proof, we obtain the following result.

**Corollary 4.3.** Let $\varphi$ be an isometric immersion $\varphi : M = N_T \times_f N \rightarrow \tilde{M}$, with $\xi$ tangent to $N_T$, such that $N_T$ is an invariant submanifold of $\tilde{M}$, and $N$ is a Riemannian submanifold of $\tilde{M}$. Then, the squared mean curvature of $M$ is

$$
\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} (h_{n1}^{r1} + \cdots + h_{nn}^r)^2.
$$

Now, we construct a general inequality for the warped product submanifold $M = N_T \times_f N$ in a Kenmotsu manifold $\tilde{M}$ by applying Gauss equation and the preceding theory.

**Theorem 4.4.** Let $\varphi : M = N_T \times_f N \rightarrow \tilde{M}$, be an isometric immersion of an $n$-dimensional warped product submanifold $M$ into a $(2m+1)$-dimensional Kenmotsu manifold $\tilde{M}$ such that $N_T$ is an $n_1$-dimensional invariant submanifold tangent to $\xi$, and $N$ is an $n_2$-dimensional Riemannian submanifold of $\tilde{M}$. Then, we have

(i) $\frac{1}{2} \|h\|^2 \geq \tau(TM) - \tau(TN_T) - \tau(TN) - \frac{n^2\Delta_f}{\tilde{f}}$.

(ii) If the equality in (i) holds, then $N_T$ and $N$ are totally geodesic and totally umbilical submanifolds in $\tilde{M}$, respectively.

**Proof.** Putting $X = W = e_i, Y = Z = e_j$ in the Gauss equation (2.2) and taking summation over $1 \leq i, j \leq n (i \neq j)$, we obtain

$$
2\tau(TM) = 2\tau(TM) + n^2\|H\|^2 - \|h\|^2.
$$

In view of (2.8), we get

$$
\|h\|^2 = -2 \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} K(e_i \wedge e_j) - 2\tau(TN_T) - 2\tau(TN) + 2\tau(TM) + n^2\|H\|^2.
$$

Again, using Gauss equation (2.2), we calculate

$$
\tau(TN_T) = \tau(TM) + \sum_{r=n+1}^{2m+1} \sum_{1 \leq i < k \leq n_1} (h_{i1}^r h_{k1}^r - (h_{i1}^r)^2),
$$

and

$$
\tau(TN) = \tau(TM) + \sum_{r=n+1}^{2m+1} \sum_{1 \leq j < t \leq n} (h_{j1}^r h_{t1}^r - (h_{j1}^r)^2).
$$
In view of (4.8) and (4.9), the equation (4.7) transforms into

\[
||h||^2 = -2 \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} K(e_i \land e_j) - 2\tilde{\sigma}(TN_T) - 2\tilde{\sigma}(TN) + 2\tilde{\sigma}(TM)
\]

\[
-2 \sum_{r=n+1}^{2m+1} \sum_{1 \leq i < k \leq n_1} (h^r_{ik}h^r_{kk} - (h^r_{ik})^2) + n^2||H||^2
\]

\[
- 2 \sum_{r=n+1}^{2m+1} \sum_{1 \leq j < t \leq n} (h^r_{jj}h^r_{tt} - (h^r_{jt})^2),
\]

which is equivalent to the following form

\[
||h||^2 = -2 \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} K(e_i \land e_j) - 2\tilde{\sigma}(TN_T) - 2\tilde{\sigma}(TN) + 2\tilde{\sigma}(TM)
\]

\[
- \sum_{r=n+1}^{2m+1} \sum_{1 \leq i \neq k \leq n_1} (h^r_{ik}h^r_{kk} - (h^r_{ik})^2) + n^2||H||^2
\]

\[
- \sum_{r=n+1}^{2m+1} \sum_{1 \leq j \neq t \leq n} (h^r_{jj}h^r_{tt} - (h^r_{jt})^2).
\]

(4.10)

Since \( \varphi \) is an \( N_T \)-minimal immersion, we have

\[
\sum_{r=n+1}^{2m+1} \sum_{1 \leq i \neq k \leq n_1} (h^r_{ik}h^r_{kk} - (h^r_{ik})^2) = -\sum_{r=n+1}^{2m+1} \sum_{1 \leq j \neq t \leq n} (h^r_{jt})^2.
\]

Hence, (4.10) takes the following form:

\[
||h||^2 = -2 \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} K(e_i \land e_j) - 2\tilde{\sigma}(TN_T) - 2\tilde{\sigma}(TN) + 2\tilde{\sigma}(TM)
\]

\[
+ \sum_{r=n+1}^{2m+1} \sum_{1 \leq i \neq k \leq n_1} (h^r_{ik})^2 + n^2||H||^2
\]

\[
- n^2||H||^2 + \sum_{r=n+1}^{2m+1} \sum_{1 \leq j \neq t \leq n} (h^r_{jt})^2.
\]

(4.11)

Next, we use the following formula for general warped products [4]

\[
\sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} K(e_i \land e_j) = \frac{n_2 \Delta f}{f}.
\]

(4.12)

Then from (4.11) and (4.12), the inequality (i) follows immediately.

Now, if the equality holds in (i), then we must have \( h(X, Y) = 0 \), for both \( X, Y \in TN_T \), and for both \( X, Y \in TN \). Hence, the immersion \( N_T \rightarrow M \) is totally geodesic and the immersion \( N \rightarrow M \) is totally umbilical.
For a warped product semi-slant submanifold $M = N_T \times_f N_\theta \rightarrow M$, we have $\nabla \ln f = \xi$ by Lemma 3.1. Further, since $\frac{\Delta f}{f} = \Delta \ln f - g(\nabla \ln f, \nabla \ln f)$, we calculate

\[
\sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} K(e_i \wedge e_j) = n_2 \Delta \ln f - n_2 g(\nabla \ln f, \nabla \ln f)
\]

\[
= -n_2 \sum_{i=1}^{n_1} g(\nabla e_i, \xi) - n_2 g(\xi, \xi)
\]

\[= -n_1 n_2.
\]

Hence, we obtain the following:

**Corollary 4.5.** Let $\varphi : M = N_T \times_f N_\theta \rightarrow M$ be an isometric immersion of an $n$-dimensional warped product semi-slant submanifold $M$ into a $(2m+1)$-dimensional Kenmotsu manifold $\bar{M}$ such that $N_T$ is an $n_1$-dimensional invariant submanifold tangent to $\xi$ and $N_\theta$ is an $n_2$-dimensional proper slant submanifold of $M$. Then, we have

(i) $\frac{1}{2} ||h||^2 \geq \tau(TN_T) - \tau(TM) + n_1 n_2$.

(ii) If the equality in (i) holds, then $N_T$ and $N$ are totally geodesic and totally umbilical submanifolds in $M$, respectively.

Now, we can derive the following relation for a Kenmotsu space form:

**Theorem 4.6.** Let $\varphi : M = N_T \times_f N \rightarrow \bar{M}(c)$ be an isometric immersion from a warped product submanifold $M$ into a Kenmotsu space form $\bar{M}(c)$ with constant $\phi$-sectional curvature $c$ where $N_T$ is an $n_1$-dimensional invariant submanifold tangent to $\xi$ and $N$ is an $n_2$-dimensional Riemannian submanifold of $\bar{M}(c)$. Then, we have

(i) $||h||^2 \geq \frac{(c-3)}{2} n_1 n_2 + \frac{(c+1)}{2} n_2 - 2n_2 \Delta f$.

(ii) If the equality in (i) holds, then $N_T$ and $N$ are totally geodesic and totally umbilical submanifolds in $M$, respectively.

**Proof.** Putting $X = W = e_i, Y = Z = e_j$ in the curvature equation (2.7) for Kenmotsu space form, we obtain

\[
2\tau(TN_T) = \frac{(c-3)}{4} \sum_{1 \leq i \neq j \leq n_1} \{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)g(e_j, e_i)\}
\]

\[
- \frac{(c+1)}{4} \sum_{1 \leq i \leq n_1} \eta(e_{n_1})g(e_i, e_i) - \eta(e_i)g(e_{n_1}, e_i)
\]

\[
+ \eta(e_{n_1}) \sum_{1 \leq j \leq n_1} [g(e_j, e_j)\eta(e_{n_1}) - g(e_{n_1}, e_j)\eta(e_j)]
\]

\[
- \sum_{1 \leq i \neq j \leq n_1} g(\phi e_i, e_i)g(\phi e_j, e_j) + 3 \sum_{1 \leq i \neq j \leq n_1} g(\phi e_i, e_j)g(\phi e_j, e_i)
\]

\[= \frac{(c-3)}{4} n_1(n_1 - 1) + \frac{(c+1)}{4} (n_1 - 1).
\]

Similarly, we obtain

\[2\tau(TN) = \frac{(c-3)}{4} n_2(n_2 - 1) + 3\frac{(c+1)}{4} \sum_{n_1+1 \leq \alpha \neq \beta \leq n} g(\phi e_\alpha, e_\beta)^2.
\]
and

\[
2\tau(TM) = \frac{(c-3)}{4} n(n-1) + \frac{(c+1)}{4} \left[ (n-1) + 3 \sum_{n+1 \leq \alpha, \beta \leq n} g(\phi e_\alpha, e_\beta)^2 \right].
\]

Therefore, using Theorem 4.4, we obtain the required results. \(\square\)

We have the following consequence of the above theorem.

**Corollary 4.7.** Let \(\varphi : M = N_T \times_f N_0 \rightarrow \hat{M}(c)\) be an isometric immersion from a warped product submanifold \(M\) into a Kenmotsu space form \(\hat{M}(c)\) with constant section curvature \(c\) where \(N_T\) is an \(n_1\)-dimensional invariant submanifold tangent to \(\xi\) and \(N_0\) is an \(n_2\)-dimensional proper slant submanifold of \(\hat{M}(c)\). Then, we have

(i) \[\|h\|^2 \geq \frac{(c+1)}{2} n_2(n_1+1),\]

(ii) If the equality in (i) holds, then \(N_T\) and \(N\) are totally geodesic and totally umbilical submanifolds in \(\hat{M}\), respectively.

**References**


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