A structure by conformal transformations of Finsler functions on the projectivised tangent bundle of Finsler spaces with the Chern connection

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Abstract. It is shown that the projectivised tangent bundle of Finsler spaces with the Chern connection has a contact metric structure under a conformal transformation with certain condition of the Finsler function and moreover it is locally isometric to $E^m \times S^{m-1}(4)$ for $m > 2$ and flat for $m = 2$ if and only if the Cartan tensor vanishes, i.e., the Finsler space is a Riemannian manifold.


Key words: Finsler manifold, the projectivised tangent bundle, contact metric structure, conformal transformation.

1 Preliminaries

Let $M$ be an $m$-dimensional $C^\infty$ manifold and $x^i$ ($1 \leq i \leq m$) local coordinates on $M$. It is said to be a Finsler manifold if the length $s$ of any curve $t \mapsto (x^1(t), \ldots, x^m(t))$ $(a \leq t \leq b)$ is given by an integral

$$s = \int_a^b F \left( x^1(t), \ldots, x^m(t), \frac{dx^1}{dt}, \ldots, \frac{dx^m}{dt} \right) dt,$$

where $F$ has the first-degree homogeneity with respect to $\frac{dx^i}{dt}$.

Our convention for indices is as follows: Latin indices run from 1 to $m$ (except $m$). Greek indices run from 1 to $m$. Greek indices with bar run from 1 to $m-1$.

A Finsler manifold $M$ has a tangent bundle $\pi : TM \to M$. From $TM$ we obtain the projectivised tangent bundle of $M$, $PTM$, by identifying the non-zero vectors differing from each other by a real factor. Geometrically $PTM$ is the space of line elements on $M$. Then a non-zero tangent vector can be expressed as

$$X = y^i \partial_{x^i}, \quad (y^i \text{ not all zero}),$$
where we set $\partial_x := \frac{\partial}{\partial x^i}$ and $\partial_{y^i} := \frac{\partial}{\partial y^i}$. The $x^i, y^i$ are local coordinates on $TM$. They are also local coordinates on $PTM$ with $y^i$ being homogeneous coordinates (determined up to a real factor). We can consider $PTM$ as the base manifold of the vector bundle $p^*TM$, pulled back with the canonical projection map $p : PTM \to M$ defined by $p(x^i, y^i) = (x^i)$. The fibers of $p^*TM$ are the vector spaces of dimension $m$ and the base manifold $PTM$ is of dimension $2m - 1$.

From now on $f_{y^i}$, $f_{y^i y^j}$, etc. denote the partial derivative(s) of a smooth function $f$ with respect to the coordinates $y^i$. Adopt a similar notation for the partial derivatives with respect to the coordinates $x^i$. From the first-degree homogeneity of $F$, we have

$$y^i F_{y^i} = F \quad \text{and} \quad y^i F_{y^i y^j} = 0.$$  

A differential form on $PTM$ can be represented as one on $TM$ provided the latter is invariant under rescaling in the $y^i$ and yields zero when contracted with $y^i \partial_{y^i}$.

Our differential forms on $PTM$ will be so represented, and exterior differentiation on $PTM$ will be obtained by formal differentiation on $TM$. Then the Hilbert form

$$\omega = F_{y^i} dx^i$$

is intrinsically define on $PTM$.

Let

$$e_\alpha = u_{\alpha j} \partial_{x^j}$$

be an orthonormal frame field on the bundle $p^*TM$, and

$$\omega^\alpha = v_\alpha^k dx^k$$

its dual coframe field, so that

$$\langle e_\alpha, e_\beta \rangle = u_{\alpha \ell} g_{\ell k} u_{\beta k} = \delta_{\alpha \beta} \quad \text{and} \quad \langle e_\alpha, \omega^\beta \rangle = \delta^\beta_{\alpha}.$$  

(1.1) is the orthonormality condition with respect to the Finsler metric (positive definite)

$$G = g_{ij} dx^i \otimes dx^j$$

$$= \left( \frac{1}{2} F^2 \right)_{y^i y^j} dx^i \otimes dx^j$$

$$= (FF_{y^i y^j} + F_y F_{y^i}) dx^i \otimes dx^j$$

defined intrinsically on $PTM$, and (1.2) is the duality condition, which is equivalent

$$u_{\alpha k} v^\beta_k = \delta^\beta_{\alpha}.$$
We now distinguish the global sections

\[ e_m = \frac{y^i}{F} \partial_{x^i} =: \xi^i \partial_{x^i} \quad \text{and} \quad \omega^m = F y^i dx^i = \omega. \]

Then, taking the exterior derivative of the Hilbert form \( \omega^m \) on \( PTM \), we have \((1.3)\)

\[ d\omega^m = \omega^\alpha \wedge \omega_\alpha^m, \]

where \( \omega_\alpha^m \) is

\[ \omega_\alpha^m = -u_\alpha^i F_{y^i} y^j dy^j + \frac{u_\alpha^i}{F} \left( F_{x^i} - y^j F_{y^j x^i} \right) \omega^m + u_\alpha^i u_\beta^j F_{x^i y^j} \omega^\beta + \lambda_{\alpha\beta} \omega^\beta \quad \text{(see [4] for } \lambda_{\alpha\beta}). \]

Define \( N^i_j \) and \( \delta y^j \) as follows:

\[ N^i_j = \frac{1}{F} G^i_j \quad \text{and} \quad \delta y^j = \frac{dy^j}{F} + N^j_k dx^k, \]

where \( G^i \) denotes

\[ G^i = g^{i\ell} \left\{ y^\ell \left( \frac{1}{2} F^2 \right)_{y^s x^s} - \left( \frac{1}{2} F^2 \right)_{x^s} \right\}. \]

Then the orthonormal vectors in \( T(TM \setminus 0) \) and the dual orthonormal vectors in \( T^*(TM \setminus 0) \) are given by

\[ \hat{e}_\alpha = u_\alpha^j \delta x^j \iff \omega^\alpha = v^\alpha_j dx^j \]

and

\[ \hat{e}_{m+\alpha} = u_\alpha^j \delta y^j \iff \omega_{m+\alpha} = v_{m+\alpha}^j dy^j, \]

where

\[ \delta x^i := \partial x^i - F N^j \partial_{y^j}, \]

and

\[ \delta y^i := F \partial_{y^i}. \]

The set \( \{ \delta x^i, \delta y^j \} \) is naturally dual to the set \( \{ dx^i, dy^j \} \), and these form local bases for \( T(TM \setminus 0) \) and \( T^*(TM \setminus 0) \), respectively.

Generally a \((2n+1)\)-dimensional manifold \( \tilde{M} \) is said to have a contact structure and is called a contact manifold if it carries a global \( 1 \)-form \( \eta \) such that

\[ \eta \wedge (d\eta)^n \neq 0 \quad \text{(1.4)}. \]
everywhere on $\tilde{M}$, where the exponent denotes the n-th exterior power. We call $\eta$ a contact form of $\tilde{M}$. A structure tensor $(\phi, \xi, \eta, g)$ on $(2n + 1)$-dimensional manifold $\tilde{M}$ said to be an almost contact metric structure if a tensor field of type (1,1) $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ satisfy

$$
\eta(\xi) = 1, \quad \phi^2 = -I + \xi \otimes \eta, \quad \phi \xi = 0, \quad \eta(\phi X) = 0,
$$

$$
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \quad \text{rank} \phi = 2n
$$

for any $X, Y \in \chi(\tilde{M})$, where $\chi(\tilde{M})$ is the Lie algebra of vector fields on $\tilde{M}$.

Let $\tilde{M}$ be a $(2n + 1)$-dimensional manifold with a contact form $\eta$. If $\tilde{M}$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ such that

$$
g(\phi X, Y) = d\eta(X, Y),
$$

then $\tilde{M}$ is said to have a contact metric structure and is called a contact metric manifold, that is

$$
\eta(\xi) = 1, \quad \phi^2 = -I + \xi \otimes \eta, \quad \phi \xi = 0, \quad \eta(\phi X) = 0,
$$

$$
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \quad \text{rank} \phi = 2n, \quad g(\phi X, Y) = d\eta(X, Y)
$$

for any $X, Y \in \chi(\tilde{M})$.

Let $\tilde{M}$ be a $(2n - 1)$-dimensional contact metric manifold with a contact metric structure $(\phi, \xi, \eta, g)$ and $R$ the curvature tensor field on $\tilde{M}$. It is well known that the condition $R(X, Y)\xi = 0$ for all $X, Y$ has a strong and interesting implication for a contact metric manifold, namely that $\tilde{M}$ is locally the product of Euclidean space $E^m$ and a sphere of constant curvature $+4$. D. E. Blair proved the following theorem.

**Theorem 1.1.** [2, 3] A contact metric manifold $\tilde{M}^{2m-1}$ satisfying $R(X, Y)\xi = 0$ is locally isometric to $E^m \times S^{m-1}(4)$ for $m > 2$ and flat for $m = 2$.

The following proposition is well known (cf. [2], [3], [6]).

**Proposition 1.2.** Let $\tilde{M}$ be a contact metric manifold with a contact metric structure $(\phi, \xi, \eta, g)$. Then $\tilde{M}$ is a $K$-contact manifold if and only if

$$
\nabla_X \xi = \phi X
$$

for any $X \in \chi(\tilde{M})$.

The following lemma is well known (cf. [4]).

**Lemma 1.3.** The Hilbert form on $PTM$ given by

$$
\omega^m = F_{\nu} dx^\nu = \omega
$$

satisfies the condition $\omega \wedge (d\omega)^{m-1} \neq 0$, that is $PTM$ has a contact structure with respect to Hilbert form $\omega$. 
Then S. S. Chern proved the following theorem.

**Theorem 1.4.** [4] There exists a torsion-free and an almost metric-compatible linear connection \( p^*TM \to PTM \), that is the Chern connection

\[
D : \Gamma(p^*TM) \to \Gamma(p^*TM \otimes PTM)
\]

given by

\[
De_\alpha = \omega^\beta_\alpha e_\beta, \quad \omega_m^m = 0,
\]

that is \( d\omega^\alpha = \omega^\beta \wedge \omega^\alpha_\beta \) and

\[
(1.8) \quad \omega_\alpha^\beta + \omega_\beta^\alpha = -2A_{\alpha\beta\gamma}\omega_m^\gamma.
\]

In particular

\[
(1.9) \quad \omega_\alpha^\beta + \omega_\beta^\alpha = 0,
\]

where \( \omega_\alpha^\beta = \omega_\alpha^\gamma\delta_\gamma^\beta \) and the Cartan tensor \( A = A_{\alpha\beta\gamma}\omega_\alpha \otimes \omega_\beta \otimes \omega_\gamma \) is given by

\[
A_{\alpha\beta\gamma} = \frac{F}{2} \left( \frac{1}{2} F^2 \right) \eta_{\gamma\beta} u_\alpha^i u_\beta^j u_\gamma^k.
\]

Next we define the Chern connection in natural coordinates as follows:

\[
D : \Gamma(p^*TM) \to \Gamma(p^*TM \otimes T^*(TM\setminus 0))
\]

given by

\[
D\partial x^i = \omega^j_i \partial x^j,
\]

where \( \omega^j_i \) are the components of the connection matrix in natural coordinates. Since the Chern connection is torsion-free, we can see that (see [1] and [4])

\[
(1.10) \quad dx^i \wedge \omega^j_i = 0,
\]

which is equivalent to the torsion-free condition of the Chern connection in natural coordinates. Wedge product of \( \omega^j_i \) and \( dx^i \) is zero in (1.10), so they are linearly dependent. We can write \( \omega^j_i \) in terms of \( dx^i \) as

\[
\omega^j_i = \Gamma^j_i dx^i,
\]

where the quantities

\[
\Gamma^i_{jk} = \frac{g_{is}}{2} (\delta_x^i g_{sj} - \delta_x^s g_{jk} + \delta_x^j g_{ks})
\]

are called the Christoffel symbols of the first. Then we obtain

\[
(1.11) \quad \Gamma^i_{jk} \ell^j = N^i_k.
\]
By using the Cartan formula, we obtain the following Lie bracket (cf. [1]):

\[ [\delta x^k, \delta y^l] = \left\{ \dot{A}^i_{kl} + \frac{\ell^i}{F} (FF_y^k)_x^l - \ell^k N_{kl} \right\} \delta y^i, \]

where the quantities \( \dot{A}^i_{kl} \) are

\[ \dot{A}^i_{kl} := (\delta_x \dot{A}^i_{kl} + A^h_{kl} \Gamma^i_{hs} - A^i_{hl} \Gamma^h_{ks} - A^i_{kh} \Gamma^h_{ls}) \ell^s. \]

On the other hand, by straightforward calculations we obtain

\[ [\delta x^k, \delta y^l] = \frac{1}{2} G^{i}_{y^k y^l} \delta y^i = \left\{ \dot{A}^i_{kl} + \Gamma^i_{kl} \right\} \delta y^i. \]

On \( PTM \), there are the quantities which are homogeneous of degree zero in the \( y^i \).

Let \( f \) be a smooth function on \( PTM \). Using the Euler’s theorem, we have

\[ \ell^i \delta y^i f = y^i f_y = 0. \]

From (1.11), (1.12), (1.13) and (1.14), it follows that

\[ N^{i}_{j} \delta y^j f = \ell^i \Gamma^i_{kj} \delta y^j f = 0. \]

Then, by (1.15), we can see that the orthonormal vectors in \( T(PTM) \) and the dual orthonormal vectors in \( T^\ast(PTM) \) are given by

\[ \tilde{e}_\alpha = u_{\alpha}^j \partial x^j \iff \omega_\alpha = v_{\alpha}^j dx^j \]

and

\[ \tilde{e}_{m+\alpha} = u_{\alpha}^j \delta y^j \iff \omega^\alpha = v_{\alpha}^j \delta y^j. \]

2 Theorem

Now, let us consider the conformal transformation:

\[ F = e^{\sigma(x)} F, \]

of the fundamental function \( F \), where \( \sigma(x) \) is a local differentiable function on the base manifold \( M \) (cf. [5]).

With respect to (2.1) we have the conformal transformation:

\[ \bar{g}_{ij} := \left( \frac{1}{2} F^2 \right) y^i y^j = e^{2\sigma(x)} \left( \frac{1}{2} F^2 \right) y^i y^j =: e^{2\sigma(x)} g_{ij}, \]

of the fundamental tensor field.

On the manifold \( TM \setminus \{0\} \) we locally define the tensor field :

\[ g_{ij} dx^i \otimes dx^j + \bar{g}_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}. \]
For \( \{ \tilde{e}_\alpha (\text{resp. } \omega^\alpha), \tilde{e}_{m+\alpha} (\text{resp. } \omega_m^\alpha) \} \) in \( T(PTM) \) (resp. \( T^*(PTM) \)), we can rewrite it as

\[
\delta_{\alpha\beta} \omega^\alpha \otimes \omega^\beta + e^{2\sigma(x)} \delta_{m+\alpha} \omega_m^\alpha \otimes \omega_m^\beta.
\] (2.4)

We now distinguish the global sections \( \tilde{e}_m := e^{-\sigma(x)} \tilde{e}_m \) and \( \omega^m := e^{\sigma(x)} \omega_m = e^{\sigma(x)} \omega \) (=: \( \tilde{\omega} \)).

Putting \( \tilde{\omega}^\alpha := \omega^\alpha \), we locally define the tensor field:

\[
\delta_{\alpha\beta} \omega^\alpha \otimes \omega^\beta + e^{2\sigma(x)} \delta_{m+\alpha} \omega_m^\alpha \otimes \omega_m^\beta.
\] (2.5)

We consider the following tensor field \( \phi \) of \((1,1)\) type:

\[
\phi \tilde{e}_\alpha = -e^{-\sigma(x)} \tilde{e}_{m+\alpha}, \quad \phi \tilde{e}_m = 0 \quad \text{and} \quad \phi \tilde{e}_{m+\alpha} = e^{\sigma(x)} \tilde{e}_\alpha.
\] (2.6)

For the conformal transformation, we get the following theorem.

**Theorem 2.1.** A structure tensor \( (\phi, \tilde{e}_m, \omega, g_s) \) is an almost contact metric structure on \( PTM \). Moreover \( \tilde{\omega} \) is a contact form on \( PTM \) and \( (\phi, \tilde{e}_m, \tilde{\omega}, \tilde{g}^s) \) is a contact metric structure if and only if \( \sigma(x) \) is a function satisfying \( d\sigma = \omega^m \).

**Proof.** It is evident that \( \tilde{\omega}(\tilde{e}_m) = 1 \). From (1.16), (1.17) and (2.5) we have

\[
\tilde{g}^s(\tilde{e}_m, \tilde{e}_m) = \delta_{\alpha\beta} \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta(\tilde{e}_m, \tilde{e}_m) = \delta_{mm} = 1,
\]

from which

\[
\tilde{g}^s(\tilde{e}_m, \tilde{e}_m) = \omega(\tilde{e}_m) = 1.
\] (2.7)

Using the argument similar to (2.7), we get

\[
\tilde{g}^s(\tilde{e}_\alpha, \tilde{e}_m) = \tilde{\omega}(\tilde{e}_\alpha) = 0
\] (2.8)

and

\[
\tilde{g}^s(\tilde{e}_{m+\alpha}, \tilde{e}_m) = \tilde{\omega}(\tilde{e}_{m+\alpha}) = 0.
\] (2.9)

By (2.7), (2.8) and (2.9), we get

\[
\tilde{g}^s(X, \tilde{e}_m) = \tilde{\omega}(X)
\] (2.10)

for any \( X \in \chi(PTM) \).

From (2.6) we see that

\[
\phi^2 \tilde{e}_\alpha = -\phi e^{-\sigma(x)} \tilde{e}_{m+\alpha} = -\tilde{e}_\alpha, \quad \phi^2 \tilde{e}_m = 0
\]

and

\[
\phi^2 \tilde{e}_{m+\alpha} = \phi e^{\sigma(x)} \tilde{e}_\alpha = -\tilde{e}_{m+\alpha}.
\]
Then it follows that
\[ \phi^2 X = -X + \varpi(X)\tilde{e}_m \]
for any \( X \in \chi(PTM) \). Moreover, we get
\[
\phi \mapsto \begin{pmatrix}
0 & \cdots & 0 & 0 & e^\sigma(x) & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & e^\sigma(x) \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
e^{-\sigma(x)} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -e^{-\sigma(x)} & 0 & 0 & \cdots & 0
\end{pmatrix},
\]
from which we have
\[ \text{rank } \phi = 2(m - 1). \]

It is clear that \( \varpi(\phi \tilde{e}_m) = 0 \). Moreover we have
\[
\varpi(\phi \tilde{e}_\alpha) = -e^{-\sigma(x)} \varpi^m(\tilde{e}_{m+\alpha}) = 0
\]
and
\[
\varpi(\phi \tilde{e}_{m+\alpha}) = e^{\sigma(x)} \varpi^m(\tilde{e}_\alpha) = e^{2\sigma(x)} \delta^m_\alpha = 0.
\]
It follows that
\[ \varpi(\phi X) = 0 \]
for any \( X \in \chi(PTM) \).

From (2.5), (2.6) and (2.8) we see that
\[
\bar{g}^a(\phi \tilde{e}_\gamma, \phi \tilde{e}_\mu) = e^{-2\sigma(x)} \bar{g}^a(\tilde{e}_{m+\gamma}, \tilde{e}_{m+\mu})
= \delta_{m+\alpha} - m+\beta \varpi^m(\tilde{e}_{m+\gamma}, \tilde{e}_{m+\mu})
= \delta_{m+\alpha} - m+\beta \delta^\alpha_\gamma \delta^\beta_\mu = \delta_{m+\gamma} m+\mu.
\]
Since we have
\[
\bar{g}^a(\tilde{e}_\gamma, \tilde{e}_\mu) = \delta_{\alpha\beta} \varpi^\alpha \otimes \varpi^\beta(\tilde{e}_\gamma, \tilde{e}_\mu) = \delta_{\gamma\mu},
\]
we get
\[ \bar{g}^a(\phi \tilde{e}_\gamma, \phi \tilde{e}_\mu) = \bar{g}^a(\tilde{e}_\gamma, \tilde{e}_\mu) - \varpi(\tilde{e}_\gamma) \varpi(\tilde{e}_\mu). \]
Similarly we obtain
\[ \bar{g}^a(\phi \tilde{e}_{m+\mu}) = \bar{g}^a(\tilde{e}_{m+\mu}) - \varpi(\tilde{e}_{m+\mu}) \varpi(\tilde{e}_{m+\mu}) \]
and
Using (2.14)~(2.16) it follows that
\begin{equation}
\gamma^s(\phi e_{m+\gamma}, \phi e_{m+\mu}) = \gamma^s(e_{m+\gamma}, e_{m+\mu}) - \varpi(e_{m+\gamma})\varpi(e_{m+\mu}).
\end{equation}

By means of (2.14)~(2.16) we obtain
\begin{equation}
\gamma^s(\phi X, \phi Y) = \gamma^s(X, Y) - \varpi(X)\varpi(Y)
\end{equation}
for any $X, Y \in \chi(PTM)$, so that we find that $\phi$ is skew-symmetric. Hence we see that a structure tensor $(\phi, \varpi, \varpi, \gamma^s)$ is an almost contact metric structure on $PTM$. From the exterior derivative of the form $\varpi$ on $PTM$, we see that
\begin{equation}
d\varpi = d\left(e^{\sigma(x)}\omega^m\right) = de^{\sigma(x)}\wedge\omega^m + e^{\sigma(x)}\omega^{\alpha}\wedge\omega^m_{\alpha},
\end{equation}
from which
\begin{equation}
\varpi \wedge (d\varpi)^{m-1} = e^{m\sigma(x)}\omega \wedge (d\omega)^{m-1} \neq 0.
\end{equation}
Hence $\varpi$ is a contact form of $PTM$.

Using (1.9), (2.6) and (2.18) we have
\begin{equation}
\gamma^s(\phi e_{\gamma}, e_{\mu}) = -e^{-\sigma(x)}\gamma^s(e_{m+\gamma}, e_{m+\mu}) = 0
\end{equation}
and
\begin{equation}
d\varpi(e_{\gamma}, e_{\mu}) = (de^{\sigma(x)}\wedge\omega^m - e^{\sigma(x)}\omega^{\alpha}\wedge\omega^m_{\alpha})(e_{\gamma}, e_{\mu}) = 0.
\end{equation}
Thus we find that
\begin{equation}
\gamma^s(\phi e_{\gamma}, e_{\mu}) = d\varpi(e_{\gamma}, e_{\mu}).
\end{equation}
Using the similar techniques, we obtain
\begin{equation}
\gamma^s(\phi e_{m+\gamma}, e_{m+\mu}) = d\varpi(e_{m+\gamma}, e_{m+\mu}),
\end{equation}
\begin{equation}
\gamma^s(\phi e_{\gamma}, e_{m+\mu}) = d\varpi(e_{\gamma}, e_{m+\mu})
\end{equation}
and
\begin{equation}
\gamma^s(\phi e_{m+\gamma}, e_{m+\mu}) = d\varpi(e_{m+\gamma}, e_{m+\mu}).
\end{equation}
Using (2.5) we get
\begin{equation}
\gamma^s(\phi X, e_{m}) = 0.
\end{equation}
On the other hand, by (2.18), we obtain
\begin{equation}
d\varpi(X, e_{m}) = X e^{\sigma(x)} = \omega^m(X) e_{m} e^{\sigma(x)}.
\end{equation}
By (2.19)~(2.22), we get
\begin{equation}
\gamma^s(\phi X, Y) = d\varpi(X, Y)
\end{equation}
for any $X, Y \in \chi(PTM)$ if and only if
\begin{equation}
d\varpi(X, e_{m}) = 0,
\end{equation}
or equivalently,
\begin{equation}
X e^{\sigma(x)} = \omega^m(X) e_{m} e^{\sigma(x)} \iff de^{\sigma(x)} = \omega^m.
\end{equation}
This proves the theorem.
Moreover, by the definition of Lie bracket, we get
\[ \text{in particular} \]
\[ (2.27) \]
\[ \text{and} \]
\[ (2.26) \]
\[ \text{for any } X, Y, Z \in \chi(PTM). \]

Let \( f \) be a smooth function on \( PTM \). By the definition of Lie bracket and
\[ \omega_{\alpha}^\beta = v_i^\beta (du_\alpha^i + u_\alpha^j \omega_j^i) = v_j^\beta (du_\alpha^i + u_\alpha^j \Gamma^i_{jk} dx^k), \]
we get
\[ [\tilde{e}_\alpha, \tilde{e}_\beta] (f) = \left[ u_\alpha^i \partial_{x^i}, u_\beta^j \partial_{x^j} \right] (f) \]
\[ = u_\alpha^i u_\beta^j \partial_{x^i}(\partial_{x^j} f) + u_\alpha^i (\partial_{x^i} u_\beta^j) \partial_{x^j} f - u_\beta^j (\partial_{x^j} u_\alpha^i) \partial_{x^i} f \]
\[ = \left( u_\alpha^i \partial_{x^i} u_\beta^j - u_\beta^j \partial_{x^j} u_\alpha^i \right) \partial_{x^i} f \]
\[ = (u_\alpha^i \omega_\beta^j (\tilde{e}_\alpha) - u_\alpha^i \omega_\alpha^j (\tilde{e}_\beta)) \]
\[ = (\omega_{\alpha}^\gamma (\tilde{e}_\alpha) - \omega_{\alpha}^\gamma (\tilde{e}_\beta)) \tilde{e}_\gamma (f). \]
from which
\[ (2.25) \]
\[ [\tilde{e}_\alpha, \tilde{e}_\beta] = \left( \omega_{\beta}^\gamma (\tilde{e}_\alpha) - \omega_{\alpha}^\gamma (\tilde{e}_\beta) \right) \tilde{e}_\gamma. \]

Similarly, by straightforward calculations, using (1.16) and (1.17), we have the followings:
\[ (2.26) \]
\[ [\tilde{e}_\alpha, \tilde{e}_{m+\beta}] = \omega_\beta^\gamma (\tilde{e}_\alpha) \tilde{e}_{m+\gamma} - \omega_\alpha^\gamma (\tilde{e}_{m+\beta}) \tilde{e}_\gamma \]
and
\[ (2.27) \]
\[ [\tilde{e}_{m+\alpha}, \tilde{e}_{m+\beta}] = \left( \omega_\beta^\gamma (\tilde{e}_{m+\alpha}) - \omega_\alpha^\gamma (\tilde{e}_{m+\beta}) \right) \tilde{e}_{m+\gamma}, \]
in particular
\[ [\tilde{e}_\alpha, \tilde{e}_m] = -\omega_\alpha^\gamma (\tilde{e}_m) \tilde{e}_\gamma, \]
\[ [\tilde{e}_m, \tilde{e}_{m+\alpha}] = \omega_\alpha^\gamma (\tilde{e}_m) \tilde{e}_{m+\gamma} - \tilde{\tilde{e}}_\alpha. \]
Moreover, by the definition of Lie bracket, we get
\[ [\tilde{e}_\alpha, \tilde{e}_m] (f) = [\tilde{e}_\alpha, e^{-\sigma(x)} \tilde{e}_m] (f) \]
\[ = e^{-\sigma(x)} [\tilde{e}_\alpha, \tilde{e}_m] (f) + (\tilde{e}_\alpha e^{-\sigma(x)}) \tilde{e}_m (f) \]
\[ = -e^{-\sigma(x)} \omega_\alpha^\gamma (\tilde{e}_m) \tilde{e}_\gamma (f) - e^{-2\sigma(x)} (de^\sigma(x) (\tilde{e}_\alpha)) \tilde{e}_m (f), \]
from which
\begin{equation}
(2.28) \quad \left[\tilde{e}_\alpha, \tilde{e}_m\right] = -e^{-\sigma(x)}\omega_\alpha \tilde{\gamma}(\tilde{e}_m)\tilde{e}_\gamma.
\end{equation}

Similarly, by straightforward calculations we have
\begin{equation}
(2.29) \quad \left[\tilde{e}_m, \tilde{e}_{m+\alpha}\right] = e^{-\sigma(x)}\omega_\alpha \tilde{\gamma}(\tilde{e}_m)\tilde{e}_{m+\gamma} - e^{-\sigma(x)}\tilde{e}_\alpha.
\end{equation}

Using (2.24)~(2.29) and (1.8), we obtain
\begin{align*}
2\mathcal{G}^\gamma (\nabla_{\tilde{e}_{m+\alpha}} \tilde{e}_{m+\beta}, \tilde{e}_\gamma) &= \tilde{e}_{m+\alpha} (\mathcal{G}^\gamma (\tilde{e}_{m+\beta}, \tilde{e}_\gamma)) + \tilde{e}_{m+\beta} (\mathcal{G}^\gamma (\tilde{e}_{m+\alpha}, \tilde{e}_\gamma)) \\
&\quad - \tilde{e}_\gamma (\mathcal{G}^\gamma (\tilde{e}_{m+\alpha}, \tilde{e}_{m+\beta})) \\
&\quad + \mathcal{G} (\tilde{e}_{m+\alpha}, \tilde{e}_{m+\beta}, \tilde{e}_\gamma) - \mathcal{G} (\tilde{e}_{m+\beta}, \tilde{e}_\gamma, \tilde{e}_{m+\alpha}) \\
&\quad + \mathcal{G} (\tilde{e}_\gamma, \tilde{e}_{m+\alpha}, \tilde{e}_{m+\beta}) \\
&= e^{2\sigma(x)} (\omega_\beta \tilde{\gamma}(\tilde{e}_\gamma) + \omega_\gamma \tilde{\gamma}(\tilde{e}_\gamma)) = -2e^{2\sigma(x)} A_{\alpha\beta} \omega_\gamma \tilde{G}(\tilde{e}_\gamma) \\
&= 0.
\end{align*}
Moreover we get
\begin{align*}
\mathcal{G}^\gamma (\nabla_{\tilde{e}_{m+\alpha}} \tilde{e}_{m+\beta}, \tilde{e}_m) &= -e^{\sigma(x)} \delta_\alpha \beta \\
\mathcal{G}^\gamma (\nabla_{\tilde{e}_{m+\alpha}} \tilde{e}_{m+\beta}, \tilde{e}_{m+\gamma}) &= e^{2\sigma(x)} \{ \omega_\beta \tilde{\gamma}(\tilde{e}_\gamma) + A_{\alpha\beta} \tilde{\gamma}(\tilde{e}_\gamma) \}.
\end{align*}
Thus we find that
\begin{equation}
(2.30) \quad \nabla_{\tilde{e}_{m+\alpha}} \tilde{e}_{m+\beta} = -e^{\sigma(x)} \delta_\alpha \beta \tilde{e}_m + \omega_\beta \tilde{\gamma}(\tilde{e}_{m+\alpha})\tilde{e}_{m+\gamma} + A_{\alpha\beta} \tilde{G}(\tilde{e}_{m+\gamma}).
\end{equation}
Using the similar techniques, we have
\begin{equation}
(2.31) \quad \nabla_{\tilde{e}_{m+\alpha}} \tilde{e}_\beta = \omega_\beta \tilde{\gamma}(\tilde{e}_{m+\alpha})\tilde{e}_\gamma + A_{\alpha\beta} \tilde{G}(\tilde{e}_{m+\gamma}),
\end{equation}
\begin{equation}
(2.32) \quad \nabla_{\tilde{e}_\alpha} \tilde{e}_{m+\beta} = A_{\alpha\beta} \tilde{G}(\tilde{e}_\alpha)\tilde{e}_{m+\gamma} + \omega_\beta \tilde{\gamma} \tilde{e}_{m+\gamma}
\end{equation}
and
\begin{equation}
(2.33) \quad \nabla_{\tilde{e}_\alpha} \tilde{e}_\beta = \omega_\beta \tilde{\gamma}(\tilde{e}_\alpha)\tilde{e}_\gamma - A_{\alpha\beta} \tilde{G}(\tilde{e}_{m+\gamma}).
\end{equation}
From (2.24)~(2.29) and (1.8), it follows that
\begin{equation}
\mathcal{G}^\gamma (\nabla_{\tilde{e}_\alpha} \tilde{e}_m, \tilde{e}_\gamma) = \mathcal{G}^\gamma (\nabla_{\tilde{e}_\alpha} \tilde{e}_m, \tilde{e}_{m+\gamma}) = 0.
\end{equation}
Thus we find that
\begin{equation}
(2.34) \quad \nabla_{\tilde{e}_\alpha} \tilde{e}_m = 0.
\end{equation}
Using the similar techniques, we have
Similarly, replacing

\[ \omega \]

for any

\[ (2.37) \]

Also, setting

\[ \sigma \]

with respect to

\[ \mathbf{PTM} \]

for any

\[ (2.38) \]

From Proposition 1.2 and (2.36), we obtain the following theorem.

**Theorem 2.2.** \( \mathbf{PTM} \) has a non-\( K \)-contact, contact metric structure \((e, \tilde{m}, \varpi, \bar{g})\) with respect to \( \bar{g}^s \) satisfying \( d\sigma = \omega^m \).

**Remark 2.3.** \( \mathbf{PTM} \) gives us an example of non-\( K \)-contact, contact metric manifold with respect to \( \bar{g}^s \) satisfying \( d\sigma = \omega^m \).

The curvature tensor field \( \overline{R} \) on \( \mathbf{PTM} \) is given by

\[ \overline{R}(X, Y) e_m := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \]

for any \( X, Y, Z \in \chi(\mathbf{PTM}) \). From (2.36) and (2.37) it follows that

\[ \overline{R}(X, Y) e_m = -3e^{-\sigma(x)} \sum_{\alpha} (\omega^m(X) \bar{g}^s(Y, \tilde{e}_{m+\alpha}) - \omega^m(Y) \bar{g}^s(X, \tilde{e}_{m+\alpha})) e_{\alpha} \]

\[ + e^{-3\sigma(x)} \sum_{\alpha} (\bar{g}^s(Y, \nabla_X \tilde{e}_{m+\alpha}) - \bar{g}^s(X, \nabla_Y \tilde{e}_{m+\alpha})) e_{\alpha} \]

\[ + e^{-3\sigma(x)} \sum_{\alpha} (\bar{g}^s(Y, \tilde{e}_{m+\alpha}) \nabla_X \tilde{e}_{\alpha} - \bar{g}^s(X, \tilde{e}_{m+\alpha}) \nabla_Y \tilde{e}_{\alpha}) \]

for any \( X, Y \in \chi(\mathbf{PTM}) \).

Setting \( X = \tilde{e}_\alpha \) and \( Y = \tilde{e}_\beta \) in (2.38), by (2.32), we get

\[ \overline{R}(\tilde{e}_\alpha, \tilde{e}_\beta) e_m = -3e^{-\sigma(x)} \sum_{\alpha} (\bar{g}^s(\tilde{e}_\beta, \nabla_{\tilde{e}_\alpha} \tilde{e}_{m+\alpha}) - \bar{g}^s(\tilde{e}_\alpha, \nabla_{\tilde{e}_\beta} \tilde{e}_{m+\alpha})) \tilde{e}_{\alpha} \]

\[ + e^{-3\sigma(x)} \sum_{\alpha} (\bar{g}^s(\tilde{e}_\beta, \tilde{e}_{m+\alpha}) \nabla_{\tilde{e}_\alpha} \tilde{e}_{\alpha} - \bar{g}^s(\tilde{e}_\alpha, \tilde{e}_{m+\alpha}) \nabla_{\tilde{e}_\beta} \tilde{e}_{\alpha}) \]

\[ = e^{-3\sigma(x)} \sum_{\alpha} (A_{\alpha\beta}^{\gamma} \tilde{e}_{\gamma} - A_{\beta\alpha}^{\gamma} \tilde{e}_{\gamma}) \tilde{e}_{\alpha} = 0. \]

Similarly, replacing \( X \) by \( \tilde{e}_\alpha \) and \( Y \) by \( \tilde{e}_{m+\beta} \) and using (2.30) and (2.32), we obtain

\[ \overline{R}(\tilde{e}_\alpha, \tilde{e}_{m+\beta}) e_m = -e^{-\sigma(x)} A_{\alpha\beta}^{\gamma} \tilde{e}_{m+\gamma}. \]

Also, setting \( X = \tilde{e}_{m+\alpha} \) and \( Y = \tilde{e}_{m+\beta} \) in (2.38), by (2.30) and (2.31), we have

\[ \overline{R}(\tilde{e}_{m+\alpha}, \tilde{e}_{m+\beta}) e_m = 0. \]
Hence we obtain
\begin{equation}
R(X,Y)\tilde{e}_m = -e^{-3\sigma(x)} \sum_{\alpha} \sum_{\beta} \left( g^s(X,\tilde{e}_\alpha)g^s(Y,\tilde{e}_{m+\beta}) \right. \\
- \\
\left. g^s(Y,\tilde{e}_\alpha)g^s(X,\tilde{e}_{m+\beta}) \right) A^\gamma_{\alpha\beta} \tilde{e}_\gamma
\end{equation}
(2.39)
for all $X, Y \in \chi(PTM)$.

From Theorem 1.1 and (2.39), we obtain

**Theorem 2.4.** A $(2m-1)$-dimensional contact metric manifold $PTM$ with respect to $\tilde{g}^s$ satisfying $d\sigma = \omega^m$ is locally isometric to $E^m \times S^{m-1}(4)$ for $m > 2$ and flat for $m = 2$ if and only if the Cartan tensor $A = 0$, i.e., $M$ is a Riemannian manifold.

**References**


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