A characterization of minimal surfaces in $S^5$
with parallel normal vector field

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Abstract. In this paper we proof that the Holomorphic angle for compact minimal surfaces in the sphere $S^5$ with constant Contact angle and with a parallel normal vector field must be constant.


Key words: contact angle, holomorphic angle, Clifford torus, parallel field.

1 Introduction

The notion of Kähler angle was introduced by Chern and Wolfson in [3] and [12]; it is a fundamental invariant for minimal surfaces in complex manifolds. Using the technique of moving frames, Wolfson obtained equations for the Laplacian and Gaussian curvature for an immersed minimal surface in $\mathbb{CP}^n$. Later, Kenmotsu in [7], Ohnita in [10] and Ogata in [11] classified minimal surfaces with constant Gaussian curvature and constant Kähler angle.

A few years ago, Li in [14] gave a counterexample to the conjecture of Bolton, Jensen and Rigoli (see [2]), according to which a minimal immersion (non-holomorphic, non anti-holomorphic, non totally real) of a two-sphere in $\mathbb{CP}^n$ with constant Kähler angle would have constant Gaussian curvature.

In [8] we introduced the notion of Contact angle, that can be considered as a new geometric invariant useful to investigate the geometry of immersed surfaces in $S^3$. Geometrically, the Contact angle ($\beta$) is the complementary angle between the contact distribution and the tangent space of the surface. Also in [8], we deduced formulas for the Gaussian curvature and the Laplacian of an immersed minimal surface in $S^3$, and we gave a characterization of the Clifford Torus as the only minimal surface in $S^3$ with constant Contact angle.

We define $\alpha$ to be the angle given by $\cos \alpha = \langle ie_1, v \rangle$, where $e_1$ and $v$ are defined in section 2. The Holomorphic angle $\alpha$ is the analogue of the Kähler angle introduced by Chern and Wolfson in [3].

Recently, in [9], we construct a family of minimal tori in $S^5$ with constant Contact and Holomorphic angle. These tori are parametrized by the following circle equation
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where \( a \) and \( b \) are given in Section 3 (equation (3.7)). In particular, when \( a = 0 \) in (1.1), we recover the examples found by Kenmotsu, in [6]. These examples are defined for \( 0 < \beta < \pi/2 \). Also, when \( b = 0 \) in (1.1), we find a new family of minimal tori in \( S^5 \), and these tori are defined for \( \pi/4 < \beta < \pi/2 \). Also, in [9], when \( \beta = \pi/2 \), we give an alternative proof of this classification of a Theorem from Blair in [1], and Yamaguchi, Kon and Miyahara in [13] for Legendrian minimal surfaces in \( S^5 \) with constant Gaussian curvature.

In this paper, we will classify minimal surfaces in \( S^5 \) with constant Contact angle and with a parallel normal vector field. We suppose that \( e_3 \) (in equation (3.1)) is a parallel normal vector field, and we get the following

**Theorem 1.** The Holomorphic angle \((0 < \alpha < \pi/2)\) is constant for compact minimal surfaces in \( S^5 \) with constant Contact angle \( \beta \) and null principal curvatures \( a, b \).

**Remark 1.** The Theorem 1 implies a more general classification in [9] that gives a family of minimal flat tori in \( S^5 \) with constant Contact angle and constant Holomorphic angle.

## 2 Contact Angle for Immersed Surfaces in \( S^{2n+1} \)

Consider in \( \mathbb{C}^{n+1} \) the following objects:

- the Hermitian product: \((z, w) = \sum_{j=0}^{n} z^j \overline{w}^j; \)
- the inner product: \((z, w) = \text{Re}(z, w); \)
- the unit sphere: \( S^{2n+1} = \{ z \in \mathbb{C}^{n+1}|(z, z) = 1 \}; \)
- the Reeb vector field in \( S^{2n+1} \), given by: \( \xi(z) = iz; \)
- the contact distribution in \( S^{2n+1} \), which is orthogonal to \( \xi \):

\[ \Delta_z = \{ v \in T_z S^{2n+1} | \langle \xi, v \rangle = 0 \}. \]

We observe that \( \Delta \) is invariant by the complex structure of \( \mathbb{C}^{n+1} \).

Let now \( S \) be an immersed orientable surface in \( S^{2n+1} \).

**Definition 1.** The Contact angle \( \beta \) is the complementary angle between the contact distribution \( \Delta \) and the tangent space \( TS \) of the surface.

Let \((e_1, e_2)\) be a local frame of \( TS \), where \( e_1 \in TS \cap \Delta \). Then \( \cos \beta = \langle \xi, e_2 \rangle \).

Finally, let \( v \) be the unit vector in the direction of the orthogonal projection of \( e_2 \) on \( \Delta \), defined by the following relation

\[ (2.1) \quad e_2 = \sin \beta v + \cos \beta \xi. \]
3 Equations for Gaussian curvature and Laplacian of a minimal surface in $S^5$

In this section, we deduce the equations for the Gaussian curvature and for the Laplacian of a minimal surface in $S^5$ in terms of the Contact angle and the Holomorphic angle. Consider the normal vector fields

$$
\begin{align*}
  e_3 &= i \csc \alpha e_1 - \cot \alpha v \\
  e_4 &= \cot \alpha e_1 + i \csc \alpha v \\
  e_5 &= \csc \beta \xi - \cot \beta e_2
\end{align*}
$$

where $\beta \neq 0, \pi$ and $\alpha \neq 0, \pi$. We will call $(e_j)_{1 \leq j \leq 5}$ an adapted frame.

Using (2.1) and (3.1), we get

$$
\begin{align*}
  v &= \sin \beta e_2 - \cos \beta e_5, \\
  iv &= \sin \alpha e_4 - \cos \alpha e_1 \\
  \xi &= \cos \beta e_2 + \sin \beta e_5
\end{align*}
$$

It follows from (3.1) and (3.2) that

$$
\begin{align*}
  i e_1 &= \cos \alpha \sin \beta e_2 + \sin \alpha e_3 - \cos \alpha \cos \beta e_5 \\
  i e_2 &= -\cos \beta z - \cos \alpha \sin \beta e_1 + \sin \alpha \sin \beta e_4
\end{align*}
$$

Consider now the dual basis $(\theta^j)$ of $(e_j)$. The connection forms $(\theta^j_k)$ are given by

$$
De_j = \theta^j_k e_k,
$$

and the second fundamental form with respect to this frame are given by

$$
II^j = \theta^j_1 \theta^1 + \theta^j_2 \theta^2; \quad j = 3, ..., 5.
$$

Using (3.3) and differentiating $v$ and $\xi$ on the surface $S$, we get

$$
\begin{align*}
  D\xi &= -\cos \alpha \sin \beta \theta^2 e_1 + \cos \alpha \sin \beta \theta^1 e_2 + \sin \alpha \theta^1 e_3 + \sin \alpha \sin \beta \theta^2 e_4 \\
  Dv &= \sin \beta \theta^2_2 e_1 + \cos \beta \theta^2_2 e_2 + \sin \beta \theta^2_2 e_3 \\
  &+ (\sin \beta \theta^2_4 - \cos \beta \theta^2_4) e_4 + \sin \beta (d\beta + \theta^2_2) e_5.
\end{align*}
$$

Differentiating $e_3$, $e_4$ and $e_5$, we have
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\[ \theta_1^3 = -\theta_1^3 \]
\[ \theta_2^3 = \sin \beta (d\alpha + \theta_1^4) - \cos \beta \sin \alpha \theta_1^1 \]
\[ \theta_3^3 = \csc \beta \theta_1^2 - \cot \alpha (\theta_1^3 + \cot \beta \theta_1^2) \]
\[ \theta_4^3 = \cot \beta \theta_1^2 - \csc \beta \sin \alpha \theta_1^4 \]
\[ \theta_5^3 = -\theta_1^4 \]
\[ \theta_2^4 = \csc \beta \theta_1^2 + \cot \alpha (\theta_1^3 + \cot \beta \theta_1^2) \]
\[ \theta_3^4 = \cot \beta \theta_1^2 - \sin \alpha \theta_2^2 \]
\[ \theta_4^4 = -d\alpha - \csc \beta \theta_1^2 + \sin \alpha \cot \beta \theta_1^4 \]
\[ \theta_5^4 = -d\alpha - \csc \beta \sin \alpha \theta_1^4 \]
\[ \theta_5^5 = \cot \beta \left( d\alpha \circ J - \cos \alpha \theta_1^2 \right) \]
\[ \theta_3^5 = \cot \beta (d\alpha \circ J - \cos \alpha \cot \beta \csc \beta \theta_4^1 + (-b \csc \beta \cot \beta + \sin \alpha (\cot^2 \beta - 1)) \theta_2^2 \]

The conditions of minimality and of symmetry are equivalent to the following equations:

\[ \theta_1^3 \wedge \theta_1^4 + \theta_2^3 \wedge \theta_2^2 = 0 = \theta_1^3 \wedge \theta_2^2 - \theta_2^3 \wedge \theta_1^4. \]

On the surface \( S \), we consider

\[ \theta_3^1 = a \theta_1^1 + b \theta_1^2 \]

It follows from (3.6) that

\[ \theta_1^3 = a \theta_1^1 + b \theta_1^2 \]
\[ \theta_3^1 = a \theta_1^1 - a \theta_2^2 \]
\[ \theta_2^4 = d\alpha + (b \csc \beta - \sin \alpha \cot \beta) \theta_1^3 - a \csc \beta \theta_1^2 \]
\[ \theta_3^4 = d\alpha - J - a \csc \beta \theta_1^4 - (b \csc \beta - \sin \alpha \cot \beta) \theta_2^2 \]
\[ \theta_4^5 = d\beta - \cot \alpha \theta_2^2 \]
\[ \theta_5^5 = -d\beta - \cos \alpha \theta_1^1 \]

where \( J \) is the complex structure of \( S \) is given by \( Je_1 = e_2 \) and \( Je_2 = -e_1 \). Moreover, the normal connection forms are given by:

\[ \theta_3^4 = \cot \beta (d\alpha \circ J - \cos \alpha \cot \beta \csc \beta \theta_4^1 + \left( -b \csc \beta \cot \beta + \sin \alpha \cot^2 \beta \theta_2^2 \right) \]

while the Gauss equation is equivalent to the equation:

\[ d\theta_1^3 + \theta_2^1 \wedge \theta_2^2 = \theta_1^4 \wedge \theta_2^2. \]

Therefore, using equations (3.7) and (3.9), we have
This implies that

\[ K = 1 - |\nabla \beta|^2 - 2 \cos \alpha \beta_1 - \cos^2 \alpha - (1 + \csc^2 \beta)(a^2 + b^2) + 2b \sin \alpha \csc \beta \cot \beta + 2 \sin \alpha \cot \alpha_1 - |\nabla \alpha|^2 \]
\[ + 2a \csc \beta \alpha_2 - 2b \csc \beta \alpha_1 - \sin^2 \alpha \cot^2 \beta \]

Using (3.5) and the complex structure of \( S \), we get

(3.10)

\[ \theta_1^1 = \tan \beta(d \beta \circ J - 2 \cos \alpha \theta^2) \]

Differentiating (3.11), we conclude that

\[ d\theta_2^1 = (- (1 + \tan^2 \beta)|\nabla \beta|^2 - \tan \beta \Delta \beta - 2 \cos \alpha(1 + 2 \tan^2 \beta)\beta_1 + 2 \tan \beta \sin \alpha \alpha_1 - 4 \tan^2 \beta \cos^2 \alpha) \theta^1 \wedge \theta^2 \]

where \( \Delta = tr \nabla^2 \) is the Laplacian of \( S \). The Gaussian curvature is therefore given by:

(3.12)

\[ K = -(1 + \tan^2 \beta)|\nabla \beta|^2 - \tan \beta \Delta \beta - 2 \cos \alpha(1 + 2 \tan^2 \beta)\beta_1 + 2 \tan \beta \sin \alpha \alpha_1 - 4 \tan^2 \beta \cos^2 \alpha. \]

From (3.10) and (3.12), we obtain the following formula for the Laplacian of \( S \):

(3.13)

\[ \tan \beta \Delta \beta = (1 + \tan^2 \beta)(a^2 + b^2) + 2b \csc \beta \alpha_1 - \sin \alpha \cot \beta - 2a \csc \beta \alpha_2 - \tan^2 \beta(|\nabla \beta + 2 \cos \alpha \alpha_1|^2 - |\cot \beta \nabla \alpha + \sin \alpha(1 - \cot^2 \beta \alpha_1|^2)) + \sin^2 \alpha(1 - \tan^2 \beta) \]

4 Gauss-Codazzi-Ricci equations for a minimal surface in \( S^5 \) with constant Contact angle \( \beta \)

In this section, we will compute Gauss-Codazzi-Ricci equations for a minimal surface in \( S^5 \) with constant Contact angle \( \beta \).

Using the connection form (3.7) and (3.8) in the Codazzi-Ricci equations, we have

\[ d\theta_3^1 + \theta_3^2 \wedge \theta_1^1 + \theta_3^1 \wedge \theta_4^1 + \theta_3^1 \wedge \theta_1^1 = 0 \]

This implies that

(4.1) \( (b_1 - a_2) + (a^2 + b^2) \cot \alpha \csc \beta \cot^2 \beta - a \cot \alpha(\csc^2 \beta + \cot^2 \beta)\alpha_2 + b(\cot \alpha(\csc^2 \beta + \cot^2 \beta)\alpha_1 - \cos \alpha \cot \beta(\csc^2 \beta + \cot^2 \beta - 3 \sec^2 \beta(1 + \sin^2 \beta))) - \cos \alpha \csc \beta(2(\cot \beta - \tan \beta)\alpha_1 - \sin \alpha(\cot^2 \beta - 3)) + \cot \alpha \csc \beta|\nabla \alpha|^2 = 0 \]

Replacing the following (3.8) in the Codazzi-Ricci equations

\[ d\theta_3^1 + \theta_3^2 \wedge \theta_1^1 + \theta_3^1 \wedge \theta_4^1 + \theta_3^1 \wedge \theta_1^1 = 0 \]
\[ d\theta_4^1 + \theta_4^2 \wedge \theta_1^1 + \theta_4^1 \wedge \theta_5^1 + \theta_4^1 \wedge \theta_1^1 = 0 \]
\[ d\theta_5^1 + \theta_5^2 \wedge \theta_1^1 + \theta_5^1 \wedge \theta_6^1 + \theta_5^1 \wedge \theta_1^1 = 0 \]
\[ d\theta_2^1 + \theta_2^2 \wedge \theta_1^1 + \theta_2^1 \wedge \theta_3^1 + \theta_2^1 \wedge \theta_1^1 = 0 \]
\[ d\theta_1^1 + \theta_1^2 \wedge \theta_1^1 + \theta_1^1 \wedge \theta_4^1 + \theta_1^1 \wedge \theta_1^1 = 0 \]
We get
\[(a_1 + b_2) + b \cot \alpha \alpha_2 + a(\cot \alpha \alpha_1 + 6 \tan \beta \cos \alpha)\]
\[(4.2) \quad -2 \sec \beta \cos \alpha \alpha_2 = 0\]

Using the connection form (3.8) in the Codazzi-Ricci equations
\[d\theta^2_1 + \theta^1_2 \wedge \theta^1_1 + \theta^1_3 \wedge \theta^1_2 + \theta^1_4 \wedge \theta^1_3 = 0\]
\[d\theta^3_4 + \theta^1_1 \wedge \theta^1_3 + \theta^1_2 \wedge \theta^1_4 + \theta^1_5 \wedge \theta^1_4 = 0\]
\[d\theta^4_5 + \theta^1_1 \wedge \theta^1_4 + \theta^1_2 \wedge \theta^1_5 + \theta^1_3 \wedge \theta^1_5 = 0\]

We have
\[(a_2 - b_1) - (a^2 + b^2) \cot \alpha \sin \beta \cot \beta + a \cot \alpha \alpha_2\]
\[+ b(-\cot \alpha \alpha_1 + 2 \cos \alpha(\cot \beta - 3 \tan \beta)) + 2 \cos \alpha \sin \beta(\cot \beta - \tan \beta)\alpha_1\]
\[+ \sin \alpha \cos \alpha \sin \beta(5 - \cot^2 \beta) + \sin \beta \Delta \alpha = 0\]
(4.3)

Codazzi-Ricci equations
\[d\theta^2_2 + \theta^2_2 \wedge \theta^1_1 + \theta^2_3 \wedge \theta^1_2 + \theta^2_4 \wedge \theta^1_3 = 0\]
\[d\theta^3_4 + \theta^2_4 \wedge \theta^1_2 + \theta^3_5 \wedge \theta^1_3 + \theta^3_1 \wedge \theta^4_1 = 0\]

give the following equation
\[(a^2 + b^2)(1 + \csc^2 \beta) + 2b \csc \beta(\alpha_1 - \cot \beta \sin \alpha) - 2a \csc \beta \alpha_2\]
\[+ |\nabla \alpha|^2 + 2 \sin \alpha(\tan \beta - \cot \beta)\alpha_1 - 4 \tan^2 \beta \cos^2 \alpha\]
\[(4.4) \quad - \sin^2 \alpha(1 - \cot^2 \beta) = 0\]

The following Codazzi equation is automatically verified
\[d\theta^5_2 + \theta^5_2 \wedge \theta^1_2 + \theta^5_3 \wedge \theta^1_3 + \theta^5_4 \wedge \theta^2_2 = 0\]

\section{5 Proof of the Theorem 1}

In this section, we will give a proof of the theorem, using Gauss-Codazzi-Ricci equations for a minimal surface in $S^5$ with constant Contact angle and null principal curvatures $a, b$.

Suppose that $a, b$ are nulls and the Contact angle $\beta$ is constant, then using the Codazzi equation (4.1), we have
\[\cos \alpha(2(\cot \beta - \tan \beta)\alpha_1 - \sin \alpha(\cot^2 \beta - 3)) - \cot \alpha |\nabla \alpha|^2 = 0\]
\[(5.1)\]

On the other hand, Codazzi equation (4.3) with $a, b$ nulls and constant Contact angle implies
\[2 \cos \alpha(\cot \beta - \tan \beta)\alpha_1 + \sin \alpha \cos \alpha(5 - \cot^2 \beta) + \Delta \alpha = 0\]
\[(5.2)\]
Using equations (5.1) and (5.2), we obtain a new Laplacian equation of $\alpha$

$$\Delta \alpha = -\sin(2\alpha) - \cot \alpha |\nabla \alpha|^2$$  

(5.3) 

Now suppose that $(0 < \alpha < \frac{\pi}{2})$. Using the Hopf’s Lemma, we get the Theorem 1. □

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References


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