Biminimal submanifolds in contact 3-manifolds

Jun-ichi Inoguchi

Abstract. We study biminimal submanifolds in contact 3-manifolds. In particular, biminimal curves in homogeneous contact Riemannian 3-manifolds and biminimal Hopf cylinders in Sasakian 3-space forms are investigated.

Key words: biminimal immersion, biharmonic map, harmonic map, contact manifolds.

1 Introduction

A smooth map \( \phi : (M, g) \rightarrow (N, h) \) between Riemannian manifolds is said to be **biharmonic** if it is a critical point of the bienergy functional:

\[
E_2(\phi) = \int_M |\tau(\phi)|^2dv_g,
\]

where \( \tau(\phi) = \text{tr} \nabla d\phi \) is the tension field of \( \phi \). Clearly, if \( \phi \) is harmonic, i.e., \( \tau(\phi) = 0 \), then \( \phi \) is biharmonic. A biharmonic map is said to be **proper** if it is not harmonic.

B. Y. Chen and S. Ishikawa [7] studied biharmonic curves and surfaces in semi-Euclidean space (see also [11]–[12]). In particular, Chen and Ishikawa proved the non-existence of proper biharmonic surfaces in Euclidean 3-space \( \mathbb{R}^3 \). R. Caddeo, S. Montaldo and C. Oniciuc generalized this non-existence theorem to surfaces in 3-dimensional space forms of non-positive curvature [5].

Biharmonic submanifolds in the 3-sphere \( S^3 \) are classified by Caddeo, Montaldo and Oniciuc [4]. Since, \( S^3 \) is a typical example of contact Riemannian 3-manifold, it is interesting to study biharmonic submanifolds in contact Riemannian manifolds. In our previous paper [13], we have studied biharmonic Legendre curves and Hopf cylinders in Sasakian 3-space forms. J. T. Cho, J. E. Lee and the present author [8]–[9] studied biharmonic curves in unimodular homogeneous contact Riemannian 3-manifolds.

K. Arslan, R. Ezentas, C. Murathan and T. Sasahara studied biharmonic submanifolds in 3 or 5-dimensional contact Riemannian manifolds [1], [2], [19].
On the other hand, in [14], E. Loubeau and S. Montaldo introduced the notion of biminimal immersion.

An isometric immersion $\phi : (M, g) \to (N, h)$ is said to be biminimal if it is a critical point of the bienergy functional under all normal variations. Thus the biminimality is weaker than biharmonicity for isometric immersions, in general.

In this paper, we study biminimal submanifolds in contact 3-manifolds. In particular we study biminimality of Legendre curves and Hopf cylinders (anti-invariant surfaces) in Sasakian 3-space forms.

### 2 Preliminaries

#### 2.1

Let $(M^m, g)$ and $(N^n, h)$ be Riemannian manifolds and $\phi : M \to N$ a smooth map. Denote by $\nabla^\phi$ the connection of the vector bundle $\phi^*TN$ induced from the Levi-Civita connection $\nabla^h$ of $(N, h)$. The second fundamental form $\nabla^d\phi$ is defined by

$$(\nabla^d\phi)(X, Y) = \nabla^\phi_X d\phi(Y) - d\phi(\nabla_X Y), \ X, Y \in \Gamma(TM).$$

Here $\nabla$ is the Levi-Civita connection of $(M, g)$. The tension field $\tau(\phi)$ is a section of $\phi^*TN$ defined by

$$\tau(\phi) = \text{tr} \ \nabla d\phi.$$

A smooth map $\phi$ is said to be harmonic if its tension field vanishes. It is well known that $\phi$ is harmonic if and only if $\phi$ is a critical point of the energy:

$$E(\phi) = \frac{1}{2} \int |d\phi|^2 dv_g$$

over every compact region of $M$. Now let $\phi : M \to N$ be a harmonic map. Then the Hessian $H_\phi$ of $E$ is given by

$$H_\phi(V, W) = \int h(J_\phi(V), W)dv_g, \ V, W \in \Gamma(\phi^*TN).$$

Here the Jacobi operator $J_\phi$ is defined by

$$J_\phi(V) := \tilde{\Delta}_\phi V - R_\phi(V), \ V \in \Gamma(\phi^*TN),$$

where $R_N$ and $\{e_i\}$ are the Riemannian curvature of $N$, and a local orthonormal frame field of $M$, respectively. For general theory of harmonic maps, we refer to Urakawa’s monograph [21].

J. Eells and J. H. Sampson [10] suggested to study polyharmonic maps. In this paper, we only consider polyharmonic maps of order 2. Such maps are frequently called biharmonic maps.
Definition 2.1. A smooth map \( \phi : (M, g) \to (N, h) \) is said to be biharmonic if it is a critical point of the bienergy functional:

\[
E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, dv_g,
\]

with respect to all compactly supported variation.

The Euler-Lagrange equation of \( E_2 \) is

\[
\tau_2(\phi) := -\mathcal{J}_\phi(\tau(\phi)) = 0.
\]

The section \( \tau_2(\phi) \) is called the bitension field of \( \phi \). If \( \phi \) is an isometric immersion, then \( \tau(\phi) = mH \), where \( H \) is the mean curvature vector field. Hence \( \phi \) is harmonic if and only if \( \phi \) is a minimal immersion. As is well known, an isometric immersion \( \phi : M \to N \) is minimal if and only if it is a critical point of the volume functional \( \mathcal{V} \). The Euler-Lagrange equation of \( \mathcal{V} \) is \( H = 0 \).

Motivated by this coincidence, the following notion was introduced by Loubeau and Montaldo:

Definition 2.2. ([14]) An isometric immersion \( \phi : (M^m, g) \to (N^n, h) \) is called a biminimal immersion if it is a critical point of the bienergy functional \( E_2 \) with respect to all normal variation with compact support. Here, a normal variation means a variation \( \{ \phi_t \} \) through \( \phi = \phi_0 \) such that the variational vector field \( V = d\phi_t/dt|_{t=0} \) is normal to \( M \).

The Euler-Lagrange equation of this variational problem is \( \tau_2(\phi)^\perp = 0 \). Here \( \tau_2(\phi)^\perp \) is the normal component of \( \tau_2(\phi) \). Since \( \tau(\phi) = mH \), the Euler-Lagrange equation is given explicitly by

\[
\\{ \bar{\triangle}_g H - \mathcal{R}_\phi(H) \}^\perp = 0
\]

(2.2.1)

Obviously, every biharmonic immersion is biminimal, but the converse is not always true.

Submanifolds with harmonic mean curvature \( \Delta H = 0 \) or normal harmonic mean curvature \( \Delta^\perp H = 0 \) have been studied extensively. Here \( \Delta^\perp \) is the Laplace-Beltrami operator of the normal bundle (and called the normal Laplacian). More generally, submanifolds with property \( \Delta H = \lambda H \) or \( \Delta^\perp H = \lambda H \) have been studied extensively by many authors (See references in [13]). Analogously, we may generalize the notion of biminimal immersion to the following one:

Definition 2.3. An isometric immersion \( \phi : M \to N \) is called a \( \lambda \)-biminimal immersion if it is a critical point of the functional:

\[
E_{2,\lambda}(\phi) = E_2(\phi) + \lambda E(\phi), \quad \lambda \in \mathbb{R}
\]

The Euler-Lagrange equation for \( \lambda \)-biminimal immersions is

\[
\tau_2(\phi)^\perp = \lambda \tau(\phi).
\]

More explicitly,

\[
\{ \bar{\triangle}_g H - \mathcal{R}_\phi(H) \}^\perp = -\lambda H
\]

or equivalently

\[
\mathcal{J}_\phi(H)^\perp = -\lambda H.
\]
To close this section, we here collect fundamental ingredients of contact Riemannian geometry from [3] for our use.

Let $M$ be a 3-dimensional manifold. A one-form $\eta$ is called a contact form on $M$ if it satisfies $d\eta \wedge \eta \neq 0$ on $M$. A 3-manifold $M$ together with a contact form $\eta$ is called a contact 3-manifold (in the restricted sense). The contact distribution $\mathcal{D}$ of $(M, \eta)$ is defined by

$$\mathcal{D} = \{ X \in TM \mid \eta(X) = 0 \}.$$ 

On a contact 3-manifold $(M, \eta)$, there exist a unique vector field $\xi$ such that

$$\eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0.$$ 

This vector field $\xi$ is called the Reeb vector field of $(M, \eta)$. Moreover, there exists an endomorphism field $\varphi$ and a Riemannian metric $g$ such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad g(\xi, \cdot) = \eta,$$

for all vector fields $X, Y$ on $M$. A contact 3-manifold $(M, \eta)$ together with structure tensors $(\xi, \varphi, g)$ is called a contact Riemannian 3-manifold.

**Definition 2.4.** A contact Riemannian 3-manifold $(M, \eta; \xi, \varphi, g)$ is said to be a 3-dimensional Sasaki manifold (or Sasaki 3-manifold) if

$$(\nabla X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

for all vector fields $X, Y$ on $M$. Here $\nabla$ denotes the Levi-Civita connection of $(M, g)$.

Let $(M, \eta; \xi, \varphi, g)$ be a contact Riemannian 3-manifold. A tangent plane at a point $p$ is said to be holomorphic if it is invariant under $\varphi$. The sectional curvature of a holomorphic tangent plane is called a holomorphic sectional curvature. If the sectional curvature function is constant on all holomorphic planes in $TM$, then $M$ is said to be of constant holomorphic sectional curvature. In particular, complete Sasaki 3-manifolds of constant holomorphic sectional curvature are called Sasaki 3-space forms.

A contact Riemannian 3-manifold $M$ is said to be regular if $\xi$ generates a one-parameter group $K$ of isometries on $M$ such that the action of $K$ on $M$ is simply transitive. If $M$ is regular, then $\varphi$ and $\eta$ are invariant under $K$-action. Moreover the contact Riemannian structure on $M$ induces an almost Kähler structure $(\tilde{g}, J)$ on the orbit space $M := M/K$. The natural projection $\pi : M \to \overline{M}$ is a Riemannian submersion.

Now let $M$ be a regular Sasaki 3-manifold. Take a regular curve $\tilde{\gamma}$ parametrized by the arclength with signed curvature function $\kappa$. Then the inverse image $S_{\gamma} := \pi^{-1}\{\gamma\}$ is a flat surface in $M$ with mean curvature $H = (\kappa \circ \pi)/2$. This flat surface is called the Hopf cylinder over $\tilde{\gamma}$. 
3 Biminimal curves

First of all we recall the following well known result (cf. [14]).

Lemma 3.1.

i) A curve $\gamma$ in a Riemannian 2-manifold of Gaussian curvature $K$ is biminimal if and only if its signed curvature $\kappa$ satisfies:

$$\kappa'' - \kappa^3 + \kappa K = 0.$$  

(3.3.1)

ii) A curve $\gamma$ in a Riemannian 3-manifold of constant sectional curvature $c$ is biminimal if and only if its curvature $\kappa$ and torsion $\tau$ fulfill the system:

$$\begin{cases} 
\kappa'' - \kappa^3 - \kappa^2 \tau^2 + \kappa c = 0 \\
\kappa^2 \tau = \text{constant}
\end{cases}$$ 

(3.3.2)

Note that $\gamma$ is biharmonic if and only if $\gamma$ is biminimal and additionally satisfies $\kappa \kappa' = 0$. Thus a non-geodesic biharmonic curve has constant curvature $\kappa$.

Corollary 3.1. (1) A non-geodesic curve in a Riemannian 2-manifold is biharmonic if and only if $\gamma$ is a Riemannian circle of signed curvature $\kappa$. The signed curvature $\kappa$ satisfies $K = \kappa^2 > 0$. Thus proper biharmonic curves can be exist only in positive curvature 2-manifolds.

(2) There are no proper biharmonic curves in Riemannian 3-manifolds of constant nonpositive curvature.

Proper biharmonic curves in $S^3$ are classified in [4].

Corollary 3.2. A non-geodesic curve $\gamma$ in a Riemannian 2-manifold is $\lambda$-biminimal if and only if

$$\kappa'' - \kappa^3 + \kappa (K - \lambda) = 0.$$

4 Biminimal curves in homogeneous contact 3-manifolds

4.1

A contact Riemannian 3-manifold is said to be homogeneous if there exists a connected Lie group $G$ acting transitively as a group of isometries on it which preserve the contact form.

D. Perrone [18] has proven that simply connected homogeneous contact Riemannian 3-manifolds are Lie group together with a left invariant contact Riemannian structure.

Now let $M$ be a 3-dimensional unimodular Lie group with left invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$. Then $M$ admits its compatible left-invariant contact Riemannian structure if and only if there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{m}$ such that (cf. [18]):
\[ [e_1, e_2] = 2e_3, \ [e_2, e_3] = c_2e_1, \ [e_3, e_1] = c_3e_2. \]

The Reeb vector field \( \xi \) is obtained by left translation of \( e_3 \). The contact distribution \( \mathcal{D} \) is spanned by \( e_1 \) and \( e_2 \).

By the Koszul formula, one can calculate the Levi-Civita connection \( \nabla \) in terms of the basis \( \{e_1, e_2, e_3\} \) as follows:

\[
\begin{align*}

\nabla_{e_1}e_2 &= \frac{1}{2}(c_3 - c_2 + 2)e_3, & \nabla_{e_1}e_3 &= -\frac{1}{2}(c_3 - c_2 + 2)e_2, \\

\nabla_{e_2}e_1 &= \frac{1}{2}(c_3 - c_2 - 2)e_3, & \nabla_{e_2}e_3 &= -\frac{1}{2}(c_3 - c_2 - 2)e_1, \\

\nabla_{e_3}e_1 &= \frac{1}{2}(c_3 + c_2 - 2)e_2, & \nabla_{e_3}e_2 &= -\frac{1}{2}(c_3 + c_2 - 2)e_1,
\end{align*}
\]

(4.4.1)

all others are zero.

In particular, \( M \) is a Sasaki manifold if and only if \( c_2 = c_3 \), and it is of constant holomorphic sectional curvature \( c = -3 + 2c_2 \) (cf. [18]). The Riemannian curvature \( R \) is given by

\[
\begin{align*}

R(e_1, e_2)e_2 &= \{\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\}e_1, \\

R(e_1, e_3)e_3 &= \{-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}(c_3^2 - c_2^2) + 1 - c_2 + c_3\}e_1, \\

R(e_2, e_1)e_1 &= \{\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\}e_2, \\

R(e_2, e_3)e_3 &= \{\frac{1}{4}(c_3 + c_2)^2 - c_2^2 + 1 + c_2 - c_3\}e_2, \\

R(e_3, e_1)e_1 &= \{-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}(c_3^2 - c_2^2) + 1 - c_2 + c_3\}e_3, \\

R(e_3, e_2)e_2 &= \{\frac{1}{4}(c_3 + c_2)^2 - c_2^2 + 1 + c_2 - c_3\}e_3.
\end{align*}
\]

4.2

Now we study biharmonic curves in homogeneous contact Riemannian 3-manifold \( M \).

Let \( \gamma : I \to M \) be a curve parametrized by arc-length with Frenet frame \( (t, n, b) \). Expand \( t, n, b \) as \( t = T_1e_1 + T_2e_2 + T_3e_3, \ n = N_1e_1 + N_2e_2 + N_3e_3, \ b = B_1e_1 + B_2e_2 + B_3e_3 \) with respect to the basis \( \{e_1, e_2, e_3\} \). Since \( (t, n, b) \) is positively oriented, \( B_1 = T_2N_3 - T_3N_2, B_2 = T_3N_1 - T_1N_3, B_3 = T_1N_2 - T_2N_1. \)

Direct computation shows
Proposition 4.1. Let \( \gamma \) be a Legendre curve in a unimodular homogeneous contact Riemannian 3-manifold. Then \( \gamma \) is biminimal if and only if

\[
(k'' - k^3 - k\tau^2) + \kappa \left\{ \frac{1}{4} (c_3 - c_2)^2 - 3 + c_2 + c_3 \right\} = 0
\]

and

\[
2\tau \kappa' + \kappa\tau' = 0.
\]

Corollary 4.1. Let \( \gamma \) be a non-geodesic Legendre curve in a unimodular homogeneous contact Riemannian 3-manifold. Then \( \gamma \) is biharmonic if and only if \( \gamma \) is a helix such that
\[ \kappa^2 + \tau^2 = \frac{1}{4}(c_3 - c_2)^2 - 3 + c_2 + c_3. \]

In particular, there are no proper biharmonic Legendre curves in homogeneous contact 3-manifold with \( \frac{1}{4}(c_3 - c_2)^2 - 3 + c_2 + c_3 \leq 0 \).

Note that Corollary 4.1 is a special case of [2]. In fact, every homogeneous contact Riemannian 3-manifold is a \((\kappa, \mu)\)-space.

**Example 4.1.** (Solvable Lie groups) Choose \( c_3 = 0 \) and \( c_2 > 0 \). Then \( M \) is the Euclidean motion group \( E(2) \). Hence, if \( c_2 > 2 \), then \( M = E(2) \) admits proper biharmonic Legendre helices. On the other hand, if \( c_3 = 0 \) and \( c_2 < 0 \), then \( M \) is the Minkowski motion group \( E(1, 1) \). In this case, \( M = E(1, 1) \) admits proper biharmonic Legendre helices if and only if \( c_2 < -6 \). Note that \( E(1, 1) \) with left invariant metric \( c_2 < -6 \) is not isomorphic to the model space \( \text{Sol} \left( c_2 = -2 \right) \) of the solvegeometry in the sense of W. Thurston. Hence \( \text{Sol} \) admits no proper biharmonic Legendre curves.

## 5 Biminimal submanifolds in Sasakian 3-space forms

### 5.1

Let us denote by \( \mathcal{M}^3(c) \) a Sasakian 3-space form of constant holomorphic sectional curvature \( c \). Then \( \mathcal{M}^3(c) \) is regular and its orbit space \( \overline{\mathcal{M}}^2 \) is a complex space form of constant curvature \( (c + 3) \). Take a curve \( \overline{\gamma}(s) \) in the orbit space \( \overline{\mathcal{M}}^2 \) parametrized by arclength \( s \). Denote by \( \{ t, n \} \) the Frenet frame of \( \overline{\gamma} \). The arclength parameter \( s \) is also an arclength parameter of the horizontal lift \( \overline{\gamma}^* \) in \( \mathcal{M}^3(c) \). Thus the Frenet frame of \( \overline{\gamma}^* \) is given by \( \{ p_1, p_2, p_3 \} \), where

\[ p_1 = t = \overline{t}', \quad p_2 = n = \overline{n}' = \varphi t, \quad p_3 = \pm \xi. \]

Without loss of generality, we may assume that \( p_1 = \xi \).

### 5.2

Let \( \gamma : I \to \mathcal{M}^3(c) \) be a curve in a Sasakian 3-space form which is not a geodesic. Then the bitension field of \( \gamma \) is computed as ([13], p. 175):

\[ \tau_2(\gamma) = -3\kappa'p_1 + (\kappa'' - \kappa^3 - \kappa \tau^2)p_2 + (2\kappa' \tau + \kappa \tau'')p_3 + \kappa R(p_2, p_1)p_1. \]

Now assume that \( \gamma \) is Legendre. Then \( R(p_2, p_1)p_1 = cp_2 \). Hence

\[ \tau_2(\gamma)^+ = (\kappa'' - \kappa^3 - \kappa + c\kappa)\varphi t + 2\kappa' \xi \]

Thus \( \gamma \) is \( \lambda \)-biminimal if and only if

\[ (\kappa'' - \kappa^3 - \kappa + c\kappa)\varphi t + 2\kappa' \xi = 2\lambda \kappa \varphi \ t. \]

From this, we obtain

\[ \kappa = \text{constant}, \quad \kappa^2 = c - 1 - 2\lambda. \]

**Proposition 5.1.** Let \( \gamma \) be a non-geodesic Legendre curve in \( \mathcal{M}^3(c) \). Then \( \gamma \) is \( \lambda \)-biminimal if and only if it is a Legendre helix satisfying \( \kappa^2 = c - 1 - 2\lambda \).
As we obtained in [13], $\gamma$ is biharmonic if and only if its curvature $\kappa$ satisfies $\kappa^2 = c - 1$. Thus we obtain

**Corollary 5.1.** Let $\gamma$ be a Legendre curve in $\mathcal{M}^3(c)$. Then $\gamma$ is biharmonic if and only if it is biminimal.

### 5.3

Let $\mathcal{M}^3(c)$ be a Sasakian 3-space form and $\pi : \mathcal{M} \to \overline{\mathcal{M}}^2(\bar{c} + 3)$ its fibering. Take a curve $\overline{\gamma}$ and denote by $S = S_\gamma = \pi^{-1}\{\gamma\}$ the Hopf cylinder over $\gamma$. The mean curvature vector field of $S$ is $\nabla = Hn$, $H = (\bar{\kappa} \circ \pi)/2$. Here $\bar{\kappa}$ is the signed curvature of $\overline{\gamma}$. Let us denote by $\iota$ the inclusion map of $S$ in $\mathcal{M}^3(c)$. The following formulas were obtained in [13]:

$$\bar{\Delta}_n = \Delta H, \quad \mathcal{R}_n(H) = (c + 1)Hn.$$ 

Hence

$$\tau_2(\iota)^\perp = 2(H'' - 4H^3 + (c - 1)H)n.$$ 

Since $\tau(\iota)^\perp = 2Hn$, $S$ is $\lambda$-biminimal if and only if

$$H'' - 4H^3 + (c - 1)H = \lambda H.$$ 

This is rewritten as

$$(5.5.1) \quad \bar{\kappa}'' - \bar{\kappa}^3 + \{(c - 1) - \lambda\}\bar{\kappa} = 0.$$ 

The equation (5.5.1) implies the following results.

**Theorem 5.1.** A Hopf cylinder $S$ is $(-4)$-biminimal if and only if the base curve is biminimal.

**Theorem 5.2.** A Hopf cylinder $S$ is $c$-biminimal if and only if the base curve is $(c + 4)$-biminimal.

**Corollary 5.2.** A Hopf cylinder $S$ in $S^3$ is $(-4)$-biminimal if and only if the base curve is biminimal in $S^2(4)$.

In [14], the following result is obtained.

**Theorem 5.3.** ([14], Theorem 3.1) Let $\pi : M^3(c) \to \overline{M}^2(\bar{c})$ be a Riemannian submersion with minimal fibers from a space form of constant sectional curvature $c$ to a surface of constant Gaussian curvature $\bar{c}$. Let $\overline{\gamma} : I \subset \mathbb{R} \to \overline{M}^2$ be a curve parameterized by arc-length. Then $S = \pi^{-1}\{\overline{\gamma}\} \subset M^3$ is a biminimal surface if and only if $\overline{\gamma}$ is a $\bar{c}$-biminimal curve.

In particular, the base curves of biminimal Hopf cylinders in $S^3$ are 4-biminimal curves in $S^2(4)$. This result for $S^3$ can be generalized to Sasakian space forms as follows:

**Theorem 5.4.** Let $\mathcal{M}^3(c)$ be a Sasakian 3-space form and $\pi : \mathcal{M}^3(c) \to \overline{\mathcal{M}}^2(\bar{c})$ ($\bar{c} = c + 3$), its associated fibering. Let $\overline{\gamma} : I \subset \mathbb{R} \to \overline{M}^2$ be a curve parameterized by arc-length. Then the Hopf cylinder $S = \pi^{-1}\{\overline{\gamma}\}$ is a biminimal surface if and only if $\overline{\gamma}$ is a $\bar{c}$-biminimal curve.
Proof. A Hopf cylinder $S_\gamma$ in $\mathcal{M}^3(c)$ is biminimal if and only if $\kappa'' - \kappa^3 = 0$. This is equivalent to

$$\kappa'' - \kappa^3 + (c + 3)\kappa = (c + 3)\lambda.$$ 

Namely, $\bar{\gamma}$ is $(c + 3)$-biminimal in the base space form. □

Remark 1. The $\lambda$-biminimality is different from $\Delta H = \lambda H$ or $\Delta^\perp H = \lambda H$. In fact, the following results are known.

Proposition 5.2. ([13], Theorem 2.1) A Hopf cylinder satisfies $\Delta H = \lambda H$ if and only if the base curve is a geodesic ($\lambda = 0$) or a Riemannian circle ($\lambda \neq 0$). In the latter case, $\lambda = \bar{\kappa}^2 + 2 > 2$.

Proposition 5.3. ([13], Theorem 2.3, Corollary 2.2) A Hopf cylinder satisfies $\Delta^\perp H = \lambda H$ if and only if the base curve is

1. $\lambda = 0$: geodesic, Riemannian circle or a Riemannian clothoid,
2. $\lambda > 0$: $\bar{\kappa}(s) = a \cos(\sqrt{\lambda}s) + b \sin(\sqrt{\lambda}s)$,
3. $\lambda < 0$: $\bar{\kappa}(s) = a \cosh(\sqrt{-\lambda}s) + b \sinh(\sqrt{-\lambda}s)$.

Add in Proof:

1. A simply connected Sasakian 3-space form $\mathcal{M}^3(c)$ is isomorphic to one of the following model spaces:
   - the special unitary group $SU(2)$ if $c > 1$ or $-3 < c < 1$,
   - the unit 3-sphere $S^3$ if $c = 1$,
   - the Heisenberg group $Nil$ if $c = -3$,
   - the universal covering group $\tilde{SL}_2\mathbb{R}$ of the special linear group $SL_2\mathbb{R}$ if $c < -3$.

Theorem 5.4 for $Nil$ and $\tilde{SL}_2\mathbb{R}$ is obtained independently by Loubeau and Montaldo [15].

2. In Example 4.1, we showed that the only biharmonic Legendre curves in Sol are Legendre geodesics. Y.-L. Ou and Z.-P. Wang studied biharmonic curves in Sol. In particular, they showed the nonexistence of proper biharmonic helices in Sol [17]. More generally, Caddeo, Montalod, Oniciuc and Piu [6] showed the non-existence of proper biharmonic curves in Sol parametrised by arclength.


Acknowledgement The author would like to thank Eric Loubeau, Stefano Montaldo, Ye-Lin Ou and Tooru Sasahara for their useful comments.
References


http://www.mat.unb.br/~ma/cont/


Author’s address:

Jun-ichi Inoguchi
Department of Mathematics Education, Faculty of Education,
Utsunomiya University, Minemachi 350, Utsunomiya, 321-8505, Japan.
e-mail: inoguchi@cc.utsunomiya-u.ac.jp