Almost Kenmotsu $f$-manifolds

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**Abstract.** A class of manifolds which admit an $f$-structure with $s$-dimensional parallelizable kernel is introduced and studied. Such manifolds are called almost Kenmotsu $f.pk$-manifolds. If $s = 1$, one obtains almost Kenmotsu manifolds and, if $s = 2$, they carry a locally conformal almost Kähler structure. Several foliations canonically associated with an almost Kenmotsu $f.pk$-manifold are studied. Locally conformal almost Kenmotsu $f.pk$-manifolds are characterized. If $s \geq 2$, they set up a class which is disjoint from that of locally conformal almost $C$-manifolds.

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**Introduction**

An $f$-structure on a $C^\infty$ $m$-dimensional manifold $M$ is defined by a non-vanishing tensor field $\varphi$ of type (1,1) which satisfies $\varphi^3 + \varphi = 0$ and has constant rank $r$. It is known that, in this case, $r$ is even, $r = 2n$. Moreover, $TM$ splits into two complementary subbundles $Im \varphi$ and $Ker \varphi$ and the restriction of $\varphi$ to $Im \varphi$ determines a complex structure on such subbundle. It is also known that the existence of an $f$-structure on $M$ is equivalent to a reduction of the structure group to $U(n) \times O(s)$, where $s = m - 2n$ ([2]). An interesting case occurs when the subbundle $Ker \varphi$ is parallelizable, for which the reduced structure group is $U(n) \times \{I_s\}$, and we have an $f$-structure with parallelizable kernel, briefly denoted by $f.pk$-structure, the respective manifold being called an $f.pk$-manifold or a globally framed manifold ([8]). Then, there exists a global frame $\{\xi_i\}$ for the subbundle $Ker \varphi$ with dual 1-forms $\eta^i$, $1 \leq i \leq s$, satisfying $\varphi^2 = -I + \sum_{i=1}^s \eta^i \otimes \xi_i$. It follows that $\varphi \xi_i = 0$, $\eta^i \circ \varphi = 0$. From now on we will omit the sum symbol for repeated indexes varying in $\{1, \ldots, s\}$. It is well known that one can consider compatible Riemannian metrics $g$ on $M$ such that for any tangent vector fields $X, Y$, one has $g(X, Y) = g(\varphi X, \varphi Y) + \eta^i(X)\eta^i(Y)$ and, fixed a compatible metric $g$, $(\varphi, \xi, \eta^i, g)$ is called a metric $f.pk$-structure. Therefore, $T(M)$ splits as complementary orthogonal sum of its subbundles $Im \varphi$ and $Ker \varphi$. We denote their respective differentiable distributions by $D$ and $D^\perp$.
A wide class of $f.p.k$-structures was introduced in [2] by D. Blair according to the following definition. A metric $f.p.k$-structure is said a $K$-structure if the fundamental 2-form $\Phi$, defined usually as $\Phi(\varphi, \varphi Y) = g(X, \varphi Y)$, is closed and the normality condition holds, i.e. $N = [\varphi, \varphi] + 2d\eta^i \otimes \xi_i = 0$, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of $\varphi$. Several subclasses have been studied from different points of view ([2, 3, 4]), also dropping the normality condition and, in this case, the term almost precedes the name of the considered structures or manifolds. If $d\eta^1 = \ldots = d\eta^s = \Phi$, the (almost) $K$-structure is said an (almost) $S$-structure and $M$ an (almost) $S$-manifold. If $d\eta^i = 0$ for all $i \in \{1, \ldots, s\}$, then the (almost) $K$-structure is called an (almost) $C$-structure and $M$ is said an (almost) $C$-manifold.

In [6], we studied normal metric $f.p.k$-structures and then $f.p.k$-manifolds (called Kenmotsu $f.p.k$-manifolds), for which the 2-form $\Phi$ verifies the condition $d\Phi = 2\eta^i \wedge \Phi$, for some $i \in \{1, \ldots, s\}$, also proving that such an index is unique and choosing $i = 1$.

This paper deals with almost Kenmotsu $f.p.k$-manifolds. Firstly, we state general properties involving the coderivative of the $\eta^i$’s with respect to the Levi-Civita connection. Several foliations can be described. In particular, each leaf of the distribution $Im \varphi$ has an almost Kähler structure and we give conditions which are equivalent to the request that $Im \varphi$ has Kähler or, possibly, totally umbilical leaves. Then, we explain a procedure to construct almost Kenmotsu $f.p.k$-manifolds, starting from almost Kähler manifolds. Furthermore, we prove that if the leaves of $Im \varphi$ in an almost Kenmotsu $f.p.k$-manifold $M^{2n+s}$ are totally umbilical, then $M^{2n+s}$ is locally a warped product of an almost Kähler manifold and $\mathbb{R}^s$, with warping function depending on a Euclidean coordinate, only.

In section 3, we study $(2n + s)$-dimensional metric $f.p.k$-manifolds admitting a structure which is locally conformal to an almost Kenmotsu one and prove that, if $s \geq 2$, each of the considered conformal changes is global. We also characterize locally conformal almost $C$-manifolds and prove that an almost Kenmotsu manifold $M^{2n+s}$, $s \geq 2$, cannot be a locally conformal almost $C$-manifold. Note that, when $s = 1$, almost Kenmotsu manifolds set up a subclass of locally conformal almost cosymplectic manifolds ([13]), whereas almost $C$-manifolds coincide with almost cosymplectic manifolds.

We recall that the Levi-Civita connection $\nabla$ of a metric $f.p.k$-manifold satisfies the following formula ([2],[5]):

$$2g((\nabla_X \varphi)Y, Z) = 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + g(N(Y, Z), \varphi X) + N^{(2)}_i(Y, Z)\eta^i(X) + 2d\eta^j(\varphi Y, X)\eta^i(Z) - 2d\eta^j(\varphi Z, X)\eta^i(Y).$$

Each tensor field $N^{(2)}_j$ is defined by $N^{(2)}_j(X, Y) = (L_\varphi X \eta^j)(Y) - (L_\varphi Y \eta^j)(X)$, and can be rewritten as $N^{(2)}_j(X, Y) = 2d\eta^j(\varphi X, Y) - 2d\eta^j(\varphi Y, X)$.

## 1 Almost Kenmotsu $f.p.k$-manifolds

In [6], a metric $f.p.k$-manifold $M$ of dimension $2n + s$, $s \geq 1$, with $f.p.k$-structure $(\varphi, \xi_i, \eta^i, g)$, is said to be a Kenmotsu $f.p.k$-manifold if it is normal, the $1$-forms $\eta^i$ are closed and $d\Phi = 2\eta^i \wedge \Phi$. 

**Definition 1.1** A metric $f.pk$-manifold $M$ of dimension $2n+s$, $s \geq 1$, with $f.pk$-structure $(\varphi, \xi, \eta', g)$, is said to be an almost Kenmotsu $f.pk$-manifold if the 1-forms $\eta'$ are closed and $d\Phi = 2\eta^1 \wedge \Phi$.

Obviously, a normal almost Kenmotsu $f.pk$-manifold is a Kenmotsu $f.pk$-manifold.

Let $(M^{2n+s}, \varphi, \xi, \eta', g)$ be an almost Kenmotsu $f.pk$-manifold. Since the distribution $D$ is integrable, we have $\mathcal{L}_\xi \eta' = 0$, $[\xi, \xi_j] \in D$ and $[X, \xi_i] \in D$ for any $X \in D$.

Then, the Levi-Civita connection is given by:

\[
(1.1) \quad 2g((\nabla_X \varphi)(Y), Z) = 2g(g(\varphi X, Y)\xi_1 - \eta^1(Y)\varphi(X), Z) + g(N(Y, Z), \varphi X),
\]

for any $X, Y, Z \in \mathcal{X}(M^{2n+s})$. Putting $X = \xi_i$ we obtain $\nabla_{\xi_i} \varphi = 0$ which implies $\nabla_{\xi_i} \xi_j = \nabla_{\xi_j} \xi_i$ since $[\xi_i, \xi_j] = 0$.

For each $i \in \{1, \ldots, s\}$ we put $A_i = -\nabla_{\xi_i}$ and $h_i = \frac{1}{2}\mathcal{L}_{\xi_i} \varphi$.

**Proposition 1.1** For any $i \in \{1, \ldots, s\}$ the tensor field $A_i$ is a symmetric operator such that:

1) $A_i(\xi_j) = 0$, for any $j \in \{1, \ldots, s\}$;

2) $A_i \circ \varphi + \varphi \circ A_i = -2\delta^1_i \varphi$.

**Proof.** $g(A_i X, Y) - g(X, A_i Y) = -2d\eta^1(X, Y) = 0$ implies that $A_i$ is symmetric. For any $i, j, k \in \{1, \ldots, s\}$ deriving $g(\xi_i, \xi_j) = \delta_{ij}$ with respect to $\xi_k$, using $\nabla_{\xi_k} \xi_i = \nabla_{\xi_k} \xi_j$, we get $2g(\xi_i, A_i(\xi_j)) = 0$. Since $\nabla_{\xi_k} \xi_i \in D^\perp$, we conclude $A_i(\xi_j) = 0$. To prove 2), we notice that for any $Z \in \mathcal{X}(M^{2n+s})$ we have $\varphi(N(\xi_i, Z)) = (\mathcal{L}_{\xi_i} \varphi)(Z)$ and, on the other hand, since $\nabla_{\xi_i} \varphi = 0$,

\[
\mathcal{L}_{\xi_i} \varphi = A_i \circ \varphi - \varphi \circ A_i.
\]

Applying (1.1) with $Y = \xi_i$, we have

\[
2g(\varphi A_i X, Z) = -2\eta^1(\xi_i)g(\varphi(X), Z) - g(\varphi(N(\xi_i, Z)), X),
\]

which implies 2). \(\square\)

**Proposition 1.2** For any $i \in \{1, \ldots, s\}$ the tensor field $h_i$ is a symmetric operator and:

1) $h_i(\xi_j) = 0$, for any $j \in \{1, \ldots, s\}$;

2) $h_i \circ \varphi + \varphi \circ h_i = 0$.

**Proof.** Equation 1) is obvious. Suppose $i \geq 2$. Then, from Proposition 1.1 we get $h_1 = A_1 \circ \varphi = -\varphi \circ A_1$ and for any tangent vector fields $X, Y$, $g(h_1(X), Y) = g(\varphi X, A_1 Y) = -g(X, \varphi A_1 Y) = g(X, h_1(Y))$. Now, we consider $i = 1$ and applying Proposition 1.1 we get $h_1 = A_1 \circ \varphi + \varphi = -\varphi \circ A_1 - \varphi$, then $g(h_1(X), Y) = g(\varphi X, A_1(Y)) + g(\varphi X, Y) = g(X, h_1(Y))$. Finally, for $i \geq 2$, $h_i \circ \varphi + \varphi \circ h_i = A_i \circ \varphi^2 - \varphi^2 \circ A_i = 0$ and

\[
h_1 \circ \varphi = -\varphi \circ A_1 \circ \varphi - \varphi^2, \quad \varphi \circ h_1 = \varphi \circ A_1 \circ \varphi + \varphi^2
\]

so $h_1 \circ \varphi + \varphi \circ h_1 = 0$. \(\square\)
Almost Kenmotsu f-manifolds

**Proposition 1.3** Let $M^{2n+s}$ be an almost Kenmotsu f.pk-manifold with structure $(\varphi, \xi, \eta^i, g)$. For any $X \in \mathcal{X}(M^{2n+s})$, we have:

1) $\nabla_X \xi_i = -\varphi h_i X$ for any $i \in \{2, \ldots, s\}$,

2) $\nabla_X \xi_i = -\varphi^2(X) - \varphi h_1 X$,

3) $\nabla \eta^i = g \circ (\varphi \times h_1)$ and $\delta \eta^i = 0$ for any $i \in \{2, \ldots, s\}$,

4) $\nabla \eta^1 = g - \eta^i \otimes \eta^k + g \circ (\varphi \times h_1)$, $\delta \eta^1 = -2n$ and $M^{2n+s}$ cannot be compact.

**Proof.** For $i \geq 2$, since $h_i = -\varphi \circ A_i$, we get $\varphi(\nabla_X \xi_i) = h_i(X)$ and applying $\varphi$, we obtain 1. Now, let $i = 1$. Then $h_1 = -\varphi \circ A_1 - \varphi$ gives $\varphi(\nabla_X \xi_1) = \varphi X + h_1(X)$ and applying $\varphi$ we get 2). Finally, an easy computation gives 3) and 4). □

We obtain immediately the following result.

**Corollary 1.1** All the operators $h_i$ vanish if and only if $\nabla \xi_1 = -\varphi^2$ and $\nabla \xi_i = 0$ for $i \in \{2, \ldots, s\}$. In such a case $\xi_2, \ldots, \xi_s$ are Killing vector fields and $\eta^2, \ldots, \eta^s$ are harmonic 1-forms.

**Proposition 1.4** Let $M^{2n+s}$ be an almost Kenmotsu f.pk-manifold with structure $(\varphi, \xi, \eta^i, g)$. Then for any $X, Y \in \mathcal{X}(M^{2n+s})$, we have:

1) $\varphi(N(X, Y)) + N(\varphi X, Y) = 2\eta^k(X)h_k(Y)$,

2) $(\nabla_X \varphi)Y + (\nabla_Y \varphi)(\varphi X) = -\eta^1(Y)\varphi X - 2g(X, \varphi Y)\xi_1 - \eta^k(Y)h_k(X)$.

**Proof.** The first relation follows by direct computation, using $d\eta^i = 0$ and the definition of the $h_i$'s. In particular, we get

\begin{equation}
(1.2) \quad g(N(\varphi X, Y), \xi_i) = 0, \quad N(Y, \xi_i) = 2\varphi h_i(Y).
\end{equation}

The second relation follows by (1.1) and 1). □

Finally, we consider $(2n + 2)$-dimensional almost Kenmotsu f.pk-manifolds and compare them with locally conformal almost Kähler manifolds with parallel anti-Lee form, considered by Kashiwada in [9]. We recall that an almost Hermitian manifold $(M, J, g)$ is locally conformal almost Kähler if and only if there exists a closed 1-form $\omega$ such that the Kähler 2-form $\Omega$ satisfies $d\Omega = 2\omega \wedge \Omega$. $\omega$ is the Lee form, $\omega = -\omega \circ J$, the anti-Lee form and $B$, $JB$ are the Lee and the anti-Lee vector fields.

We need a result essentially due to Goldberg and Yano ([7, 8]).

**Theorem 1.1** Let $M$ be a $(2n + s)$-dimensional f.pk-manifold with structure $(\varphi, \xi, \eta^i)$, and $s$ even, $s = 2p$. The tensor field $J$ defined by:

\begin{equation}
(1.3) \quad J = \varphi + \sum_{i=1}^{p} (\eta^{2i-1} \otimes \xi_{2i} - \eta^{2i} \otimes \xi_{2i-1})
\end{equation}

is an almost complex structure on $M$ and, if $g$ is a $\varphi$-compatible metric, $(M, J, g)$ is an almost Hermitian manifold with Kähler 2-form.
\(\Omega = \Phi - 2 \sum_{i=1}^{n} \eta_{2i-1}^{2} \wedge \eta_{2i}\). \hfill (1.4)

The previous theorem and Proposition 1.3 easily imply the following result.

**Theorem 1.2** Let \(M^{2n+2}\) be an almost Kenmotsu \textit{f.pk}-manifold with structure \((\varphi, \xi_{1}, \xi_{2}, \eta^{1}, \eta^{2}, g)\) and let \(J\) be the tensor field defined by:

\[ J = \varphi + \eta^{1} \otimes \xi_{2} - \eta^{2} \otimes \xi_{1}. \]

Then, \((M^{2n+2}, J, g)\) is a locally conformal almost Kähler manifold with Lee 1-form \(\eta^{1}\). The anti-Lee 1-form \(\eta^{2} = -\eta^{1} \circ J\) is parallel if and only if \(h_{2} = 0\).

**Theorem 1.3** Let \((M^{2n+2}, J, g)\) be a locally conformal almost Kähler manifold with unit Lee vector field \(B\), anti-Lee vector field \(J(B)\), Lee 1-form \(\omega\) and parallel anti-Lee 1-form \(\bar{\omega}\). Let \(\varphi\) be the tensor field defined by:

\[ \varphi = J - \omega \otimes JB + \bar{\omega} \otimes B. \]

Then \((M^{2n+2}, \varphi, B, JB, \omega, \bar{\omega}, g)\) is an almost Kenmotsu \textit{f.pk}-manifold and the operator \(h_{2}\) vanishes.

**Proof.** Theorem 1.1 ensures that \(g\) is a compatible metric for the \textit{f.pk}-structure \((\varphi, B, JB, \omega, \bar{\omega})\). Note that \(\omega, \bar{\omega}\) are both closed and the fundamental form is given by \(\Phi = \Omega + 2\omega \wedge \bar{\omega}\), so that \(d\Phi = d\Omega = 2\omega \wedge \Omega = 2\omega \wedge \Phi\). Finally, since \(\nabla \bar{\omega} = 0\), we have \(h_{2} = 0\). \(\square\)

## 2 Distributions

We describe some distributions on an almost Kenmotsu \textit{f.pk}-manifold of dimension \(2n + s, s \geq 1\), with structure \((\varphi, \xi_{i}, \eta^{i}, g)\).

**Proposition 2.1** Let \(M^{2n+s}\) be an almost Kenmotsu \textit{f.pk}-manifold with structure \((\varphi, \xi_{i}, \eta^{i}, g)\). The integral manifolds of \(D\) are almost Kähler manifolds with mean curvature vector field \(H = -\xi_{1}\). They are totally umbilical submanifolds of \(M^{2n+s}\) if and only if all the operators \(h_{i}\)'s vanish.

**Proof.** Let \(M'\) be an integral manifold of \(D\). The tensor fields \(\varphi\) and \(g\) induce an almost complex structure \(J\) and a Hermitian metric \(g'\) on \(M'\). Then, for any \(X, Y \in \mathcal{X}(M)\), we have \(\Omega'(X, Y) = g'(X, JY) = g(X, \varphi Y) = \Phi(X, Y)\) and \(d\Omega' = (d\Phi)|_{M'} = 0\), so \(M'\) is an almost Kähler manifold. Computing the second fundamental form, since the \(A_{i}\)'s are the Weingarten operators in the directions \(\xi_{i}\), we get, via Proposition 1.3,

\[
\alpha(X, Y) = \sum_{i=1}^{s} g(A_{i}X, Y)\xi_{i} = g(\varphi^{2}X + \varphi h_{1}(X), Y)\xi_{1} + \sum_{i=2}^{s} g(\varphi h_{i}(X), Y)\xi_{i} = -g(X, Y)\xi_{1} + \sum_{i=1}^{s} g(\varphi h_{i}(X), Y)\xi_{i}.\]
Fixed a local orthonormal frame \((e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n)\) in \(TM'\), applying Proposition 1.1, we obtain \(trA_i = 0\) for \(i \geq 2\), while \(trA_1 = -2n\). Hence we get

\[
H = \frac{1}{2n} \sum_{i=1}^{s} (trA_i) \xi_i = -\xi_1. 
\]

Finally, \(M'\) is totally umbilical if and only if \(h_i = 0\) for each \(i \in \{1, \ldots, s\}\).

\[\Box\]

**Proposition 2.2** In an almost Kenmotsu \(f, pk\)-manifold \((M^{2n+s}, \varphi, \xi_i, \eta^j, g)\) the distribution \(\mathcal{D}\) has Kähler leaves if and only if, for any \(X, Y \in \mathcal{A}(M^{2n+s})\),

\[
(\nabla_X \varphi)(Y) = \sum_{i=1}^{s} (\eta^i(Y)\varphi A_i(X) - g(\varphi A_i(X), Y)\xi_i). \tag{2.1}
\]

**Proof.** Let \(M'\) be an integral manifold of \(\mathcal{D}\) with the corresponding almost Kähler structure. By the Gauss equation \(\nabla_X Y = \nabla_X Y + \sum_{i=1}^{s} g(A_i(X), Y)\xi_i\), we have

\[
(\nabla_X Y) = (\nabla_X \varphi)Y - \sum_{i=1}^{s} g(A_i(X), \varphi Y)\xi_i, \tag{2.2}
\]

so each integral manifold \(M'\) is Kähler if and only if

\[
(\nabla_X \varphi)Y = \sum_{i=1}^{s} g(A_i(X), \varphi Y)\xi_i,
\]

for any \(X, Y \in \mathcal{D}\). Therefore, if \(\mathcal{D}\) has Kähler leaves, given \(X, Y \in \mathcal{A}(M^{2n+s})\), the vector fields \(X - \eta^i(X)\xi_j\) and \(Y - \eta^i(Y)\xi_j\) belong to \(\mathcal{D}\) and using \(\nabla_{\xi_j} \varphi = 0\), we obtain

\[
(\nabla_X \varphi) Y = \eta^k(Y)(\nabla_X \varphi)(\xi_k) + \sum_{i=1}^{s} g(A_i(X), \varphi Y)\xi_i
= -\eta^k(Y)\varphi(\nabla_X \xi_k) + \sum_{i=1}^{s} g(A_i(X), \varphi Y)\xi_i
= \sum_{i=1}^{s} (\eta^i(Y)\varphi A_i(X) - g(\varphi A_i(X), Y)\xi_i). 
\]

Vice versa (2.1) and (2.2) imply \(\nabla_X^i J = 0\) on each integral manifold i.e. the Kähler condition. \[\Box\]

**Proposition 2.3** Let \(M^{2n+s}\) be an almost Kenmotsu \(f, pk\)-manifold with structure \((\varphi, \xi_i, \eta^i, g)\) such that the integral manifolds of \(\mathcal{D}\) are Kähler. Then \(M^{2n+s}\) is a Kenmotsu \(f, pk\)-manifold if and only if \(\nabla \xi_1 = -\varphi^2\) and \(\nabla \xi_i = 0\) for each \(i \in \{2, \ldots, s\}\).

**Proof.** Assuming that the structure is normal, we have \(\mathcal{L}_{\xi_i} \varphi = 0\) for each \(i \geq 1\), which implies \(A_i \circ \varphi = \varphi \circ A_i\). Combining with Proposition 1.1 we get \(A_i = 0\) and then \(\nabla \xi_i = 0\) for \(i \geq 2\), while \(A_1 \circ \varphi = -\varphi\), so that \(\nabla \xi_1 = -A_1 = -\varphi^2\). Vice versa, we notice that for \(i \geq 2\), \(\nabla \xi_i = 0\) implies \(\mathcal{L}_{\xi_i} \varphi = 0\) and from \(\nabla \xi_i = -\varphi^2\) we get \(A_1 = \varphi^2\) and \(\mathcal{L}_{\xi_i} \varphi = 2A_1 \circ \varphi + 2\varphi = 0\). Hence, for any \(i \in \{1, \ldots, s\}\) and \(Z \in \mathcal{A}(M)\) we obtain \(\varphi(N(\xi, Z)) = 0\) and \(N(\xi, Z) \in \mathcal{D}^i\). Thus \(N(\xi, Z) = 0\), since \(g(N(\xi, Z), \xi_k) = 0\) for each \(k \in \{1, \ldots, s\}\). Finally, \(N(\xi, \xi_1) = 0\) is trivial and for \(X, Y \in \mathcal{D}\), \(N(X, Y) = 0\) since \(N(X, Y) = N_f(X, Y) = 0\), the leaves of \(\mathcal{D}\) being Kähler manifolds. \[\Box\]

**Proposition 2.4** An almost Kenmotsu \(f, pk\)-manifold \(M^{2+s}\) such that \(\nabla \xi_1 = -\varphi^2\) and \(\nabla \xi_i = 0\) for \(i \geq 2\) is a Kenmotsu \(f, pk\)-manifold.
Proof. When \( n = 1 \), the integral manifolds of the distribution \( \mathcal{D} \) are almost Kähler of dimension two and then they are Kähler. So we apply the previous proposition. □

**Proposition 2.5** The distribution \( \mathcal{D}^\perp = \langle \xi_1, \ldots, \xi_s \rangle \) is integrable, with totally geodesic flat leaves.

**Proof.** Just note that \([\xi_i, \xi_j] = 0 \) and \( \nabla_{\xi_i}\xi_j = 0 \). □

When \( s \geq 2 \), we can consider other distributions.

**Proposition 2.6** The distribution \( \mathcal{D}' = \mathcal{D}^\perp < \xi_1 \rangle \) is integrable. Its leaves are minimal almost Kenmotsu manifolds.

**Proof.** Since \( \mathcal{D}' = \{ X \in \mathcal{X}(M) \mid g(X, \xi_i) = 0, i \geq 2 \} \), and \( df_i = 0 \), the distribution is clearly involutive with \((2n+1)\)-dimensional integral manifolds. Let \( M' \) be an integral manifold, \( \nabla' \) its Levi-Civita connection and \( \varphi' \) the tensor field defined by \( \varphi'(X) = \varphi(X) \) for any \( X \in \mathcal{X}(M') \). It is easy to verify that \( \varphi'^2 = -I + \eta_i \otimes \xi_1 \) and \( df' = 2\eta_i \otimes \Phi' \) so \( M' \) is an almost Kenmotsu manifold. Now, for any \( X, Y \in \mathcal{X}(M') \), \((\nabla'_{X}\varphi')(Y) = (\nabla'_{X}\varphi)(Y) - \alpha(X, \varphi Y) \), \( \alpha \) being the second fundamental form. Then, since for any \( i \geq 2 \) the Weingarten operators are \( A_i = -\varphi \circ h_1 \), the mean curvature vector field is given by

\[
H = \frac{1}{2n+1} \sum_{i=2}^{s} \left( \sum_{k=1}^{n} g(\varphi e_k, h_i e_k) + g(\varphi^2 e_k, h_i \varphi e_k) \right) \xi_i = 0.
\]

□

**Proposition 2.7** For any \( i \in \{1, \ldots, s\} \), let \( D_i = \ker \eta_i \). Then:

1) for each \( i \neq 1 \), the distribution \( D_i = \mathcal{D}^\perp < \xi_1, \ldots, \xi_i, \ldots, \xi_s \rangle \) is integrable and the integral manifolds are minimal almost Kenmotsu \( f.pk. \)-hypersurfaces;

2) the distribution \( D_1 = \mathcal{D}^\perp < \xi_2, \ldots, \xi_s \rangle \) is integrable and its leaves are almost \( C \)-manifolds with mean curvature \( H = -\frac{2n}{2n+s-1} \xi_1 \).

**Proof.** The integrability of the described distributions follows from the condition \( df_i = 0 \), for each \( i \in \{1, \ldots, s\} \). Assume \( i \neq 1 \) and let \( M' \) be an integral manifold of \( D_i \). Then, the unique Weingarten operator is \( A_{i} = A_{i} \), the second fundamental form is given by \( \alpha(X, Y) = g(A_{i}X, Y)\xi_{1} \) and its trace vanishes since \( A_{i} \) anticommutes with \( \varphi \) and \( A_{i}(\xi_{q}) = 0 \) for \( q \neq i \). So \( M' \) is minimal. By restriction, the structure on \( M' \) determines an almost Kenmotsu \( f.pk. \)-structure \((\varphi', \xi_1, \ldots, \xi_i, \ldots, \xi_s, \eta_1, \ldots, \eta_i, \ldots, \eta_s, g') \) on \( M' \). Now, suppose \( i = 1 \). The induced structure on each leaf of \( D_1 \) has closed fundamental form and since \( df_i = 0 \) for \( i \geq 2 \), we obtain an almost \( C \)-manifold. The unique Weingarten operator \( A_{1} \) verifies \( A_{1}(X) = \varphi^2(X) + \varphi h_1(X) \) for any \( X \in D_1 \). Hence \( \alpha(X, Y) = -g(\varphi X, \varphi Y)\xi_{1} - g(h_1 X, \varphi Y)\xi_1 \) and \( H = -\frac{2n}{2n+s-1} \xi_1 \). □

**Example 1** Let \((N^{2n}, J, \tilde{g}), n \geq 2 \), be a strictly almost Kähler manifold and consider \( \mathbb{R}^s \times N^{2n} \), with coordinates \( t^1, \ldots, t^s \) on \( \mathbb{R}^s \). For any \( i \in \{1, \ldots, s\} \), we put \( \xi_i = \frac{\partial}{\partial t^i}, \eta_i = dt^i \) and define the tensor field \( \varphi \) on \( \mathbb{R}^s \times N^{2n} \) such that \( \varphi X = JX \), if \( X \) is a vector field on \( N^{2n} \) and \( \varphi X = 0 \) if \( X \) is tangent to \( \mathbb{R}^s \).
Furthermore, we consider the metric \( g = g_0 + c e^{2t_1} \tilde{g} \), where \( g_0 \) denotes the Euclidean metric on \( \mathbb{R}^s \) and \( c \in \mathbb{R}^+ \). Then, the warped product \( \mathbb{R}^s \times_f \mathbb{R}^{2n} \), \( f^2 = ce^{2t_1} \), with the structure \((\varphi, \xi_i, \eta^i, g)\), is a strictly almost Kenmotsu \( f.pk \)-manifold. Namely, it is easy to verify that the 1-forms \( \eta^i \)'s are dual of the \( \xi_i \)'s with respect to \( g \), \( \varphi^2 = -I + \eta^i \otimes \xi_i \) and \( g \) is a compatible metric. Furthermore, we get \( \Phi = ce^{2t_1} p^2 \), where \( p_2 \) is the projection on \( \mathbb{R}^{2n} \) and \( \Omega \) is the fundamental form of \( \mathbb{R}^{2n} \). Then, since \( d\Phi = 0 \), \( d\Phi = 2ce^{2t_1} dt^1 \wedge p^2 = 2dt^1 \wedge \Phi \). Finally, since the torsion \( N_f \) does not vanish, \( N^{2n} \) being strictly almost Kähler, we obtain that the \( f.pk \)-structure is not normal.

**Remark 1** In [14], Oguro and Sekigawa describe a strictly almost Kähler structure on the Riemannian product \( \mathbb{H}^3 \times \mathbb{R} \). Thus the warped product \( \mathbb{R}^s \times_f (\mathbb{H}^3 \times \mathbb{R}) \), \( f^2 = ce^{2t_1} \), is a \((4+s)\)-dimensional strictly almost Kenmotsu \( f.pk \)-manifold.

**Theorem 2.1** Let \( (M^{2n+s}, \varphi, \xi_i, \eta^i, g) \) be an almost Kenmotsu \( f.pk \)-manifold. Assume that \( h_i = 0 \) for all \( i \in \{1, \ldots, s\} \). Then, \( M^{2n+s} \) is locally a warped product \( B^s \times_f \mathbb{R}^{2n} \), where \( \mathbb{R}^{2n} \) is an almost Kähler manifold, \( B^s \) is a flat manifold with coordinates \((t^1, \ldots, t^s)\) and \( f^2 = ce^{2t_1}, c \) a positive constant.

Proof. We know that \( T(M^{2n+s}) = \text{Ker} \varphi \oplus \text{Im} \varphi \) and the corresponding distributions \( \langle \xi_1, \ldots, \xi_s \rangle \) and \( D \) are both integrable. Their integral manifolds are totally geodesic flat manifolds and totally umbilical almost Kähler manifolds with second fundamental form \( \alpha = -g \otimes \xi_1 \), mean curvature \( H = -\xi_1 \), respectively. Thus, as a manifold, \( M^{2n+s} \) is locally a product \( B \times F \) with \( T(B) = \langle \xi_1, \ldots, \xi_s \rangle \) and \( F \) is almost Kähler. We can choose a neighborhood with coordinates \((t^1, \ldots, t^s, x^1, \ldots, x^{2n})\) such that \( \pi_*(\xi_i) = \frac{\partial}{\partial t^i}, \pi \) denoting the projection onto \( B \). Then \( \pi : B \times F \to B \) is a \( C^\infty \)-submersion with vertical distribution \( V = T(F) \) and horizontal distribution \( H = T(B) \). Moreover, the splitting \( V \oplus H \) is orthogonal with respect to the metric \( g \) and, since, for any \( p \in B \times F \), \( g_p(\xi_i, \xi_j) = \delta_{ij} = g_\pi(p)(\pi_*(\xi_i), \pi_*(\xi_j)) \), \( \pi \) is a Riemannian submersion. The horizontal distribution is integrable, so the O'Neill tensor \( A \) vanishes. Moreover \( N = 2nH = -2n\xi_1 \) is a basic vector field. Now, computing the trace-free part \( T^0 \) of the O'Neill tensor \( T \), for any \( U, V \) vector fields, we get:

\[
T^0_U V = h(\nabla_U V) - \frac{1}{\pi_0} g(U, V) N = \alpha(U, V) + g(U, V) \xi_1 = 0;
\]

\[
T^0_U \xi_1 = T_U \xi_1 + \frac{1}{\pi_0} g(N, \xi_1) U = v(\nabla_U \xi_1) - g(\xi_1, \xi_1) U = U - U = 0;
\]

\[
T^0_U \xi_i = v(\nabla_U \xi_i) - g(\xi_i, \xi_i) U = 0, \quad i \geq 2.
\]

Thus \( T^0 = 0 \) and \( B \times F \), and then \( M^{2n+s} \), is locally a warped product and \( N = -2n\xi_1 \) is \( \pi \)-related to \(-\frac{2n}{\pi} \text{grad}_{\pi_0} f, g_0 \) being the flat metric on \( B ([1], 9.104) \). It follows that \( \frac{1}{\pi} \text{grad} f = \frac{\partial}{\partial t^1} \), which implies \( f = kt^1 \) and \( f^2 = ce^{2t_1} \), with \( c \) a positive constant. Finally, the warped metric is locally given by \( \sum_{t=1} dt^t \otimes dt^t + ce^{2t_1} \tilde{g}, \tilde{g} \) being an almost Kähler metric. \( \Box \)

### 3 Conformal changes

Let \( M \) be an \( f.pk \)-manifold of dimension \( 2n + s \) with structure \((\varphi, \xi_i, \eta^i, g)\). A local conformal change of the structure is given by a family \((U_\alpha, \sigma_\alpha)_{\alpha \in A} \) where \((U_\alpha)_{\alpha \in A} \) is
an open covering of $M$ and, for any $\alpha \in A$, $\sigma_\alpha \in \mathcal{F}(U_\alpha)$. Putting

$$\varphi_\alpha = \varphi|_{U_\alpha}, \xi_\alpha^i = e^{\sigma_\alpha} \xi_i|_{U_\alpha}, \eta_\alpha^i = e^{-\sigma_\alpha} \eta_i|_{U_\alpha}, g_\alpha = e^{-2\sigma_\alpha} g|_{U_\alpha},$$

$(U_\alpha, \varphi_\alpha, \xi_\alpha^i, \eta_\alpha^i, g_\alpha)$ is an $f.pk$-manifold. Note that for $s = 1$ this is the concept of conformal change of an almost contact metric structure.

**Definition 3.1** An $f.pk$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ is said to be a locally conformal almost Kenmotsu $f.pk$-manifold if there exists a local conformal change $(U_\alpha, \sigma_\alpha)_{\alpha \in A}$ such that for each $\alpha \in A$, $(U_\alpha, \varphi_\alpha, \xi_\alpha^i, \eta_\alpha^i, g_\alpha)$ is an almost Kenmotsu $f.pk$-manifold.

It follows that for any $\alpha \in A$ we have $dh_\alpha^i = 0$ so that there exists a unique $k \in \{1, \ldots, s\}$, which a priori depends on $\alpha$, such that $d\Phi_\alpha = 2h^k_\alpha \wedge \Phi$, where $\Phi_\alpha$ is defined by $\Phi_\alpha(X, Y) = g_\alpha(X, \varphi_\alpha Y) = e^{-2\sigma_\alpha} g(X, \varphi Y)$, for any vector fields $X, Y$ on $U_\alpha$. Moreover, on each $U_\alpha$ we easily obtain

$$2h_\alpha^i = \mathcal{L}_{\xi_i} \varphi_\alpha = 2e^{\sigma_\alpha} h_i^\alpha - (d\sigma_\alpha \circ \varphi_\alpha) \otimes \xi_i.$$

**Definition 3.2** An $f.pk$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ is said to be a globally conformal almost Kenmotsu $f.pk$-manifold if there exists a smooth function $\sigma$ on $M^{2n+s}$ such that, putting

$$\tilde{\varphi} = \varphi, \tilde{\xi}_i = e^{\sigma} \xi_i, \tilde{\eta}_i = e^{-\sigma} \eta_i, \tilde{g} = e^{-2\sigma} g,$$

$(M^{2n+s}, \tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{g})$ is an almost Kenmotsu $f.pk$-manifold.

**Theorem 3.1** Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a locally conformal almost Kenmotsu $f.pk$-manifold and $s \geq 2$. Then, up to a rearrangement of the $\xi_i$’s, there exists a function $\sigma \in \mathcal{F}(M^{2n+s})$ such that

$$d\Phi = 2(d\sigma + e^{-\sigma} \eta^i) \wedge \Phi,$$

$$dh^i = d\sigma \wedge \eta^i, \quad i \in \{1, \ldots, s\}.$$

**Proof.** Firstly we prove that there exists a closed 1-form $\omega$ such that $dh^i = \omega \wedge \eta^i$ for each $i \geq 1$. Namely, considering $\alpha \in A$, since $\eta_\alpha^i = e^{-\sigma_\alpha} \eta_i|_{U_\alpha}$, $dh_\alpha^i = 0$ implies $dh^i|_{U_\alpha} = d\sigma_\alpha \wedge \eta_i|_{U_\alpha}$. Thus, for $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, for any $i \in \{1, \ldots, s\}$ we get $d\sigma_\alpha \wedge \eta^i = d\sigma_\beta \wedge \eta^i$ and so $(d\sigma_\alpha - d\sigma_\beta) \wedge \eta^i = 0$. Therefore, for any vector field $X$ and any $j \in \{1, \ldots, s\}$ we obtain

$$(d\sigma_\alpha - d\sigma_\beta)(X)\eta^j(\xi_j) = (d\sigma_\alpha - d\sigma_\beta)(\xi_j)\eta^j(X)$$

and choosing $X \in \mathcal{D}$ and $j = i$ we get $(d\sigma_\alpha - d\sigma_\beta)(X) = 0$. Furthermore, since $s \geq 2$, we can choose $X = \xi_k$ with $k \neq j$ obtaining $(d\sigma_\alpha - d\sigma_\beta)(\xi_k) = 0$. Hence, the local 1-forms $d\sigma_\alpha$ give rise to the required global 1-form $\omega$.

Now, for any $\alpha \in A$, we have $d\Phi_\alpha = 2h_\alpha^k \wedge \Phi_\alpha$, and, denoting by $\nabla^\alpha$ the Levi-Civita connection on $(U_\alpha, g_\alpha)$, we have $\nabla^\alpha \xi_\alpha^i = -\delta_i^j \varphi^2 - \varphi \circ h_\alpha^i$. Let $\beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$. Then, $d\Phi_\beta = 2h_\beta^k \wedge \Phi_\beta$ and, in the intersection, $\nabla^\alpha \xi_\alpha^i = \nabla^\beta \xi_\beta^i$ implies $\delta_i^j \varphi^2 + \varphi \circ h_\alpha^i = \delta_i^j \varphi^2 + \varphi \circ h_\beta^i$. Now, assuming $t \neq k$, choosing $i = t$ and then $i = k$, we get...
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\[ \varphi^2 + \varphi \circ h_i^a = \varphi \circ h_i^b, \quad \varphi \circ h_i^a = \varphi^2 + \varphi \circ h_i^b, \]

which easily imply \( \varphi^2 = 0 \), so obtaining a contradiction. Thus we have \( t = k \) and we can suppose that, up to a rearrangement, \( d\Phi_{\alpha} = 2h_i^a \wedge \Phi_{\alpha} \), for each \( \alpha \in A \). Finally, differentiating \( \Phi_{\alpha} = e^{-2\sigma} \Phi \), we get \( d\Phi = 2(e^{-\sigma} \eta^i + d\sigma) \wedge \Phi \), in \( U_\beta \) and, comparing with the analogous expression in \( U_\beta \), \( \sigma_\alpha \) and \( \sigma_\beta \) coincide in \( U_\alpha \cap U_\beta \). Hence there exists a function \( \sigma \in F(M^{2n+s}) \) such that \( \omega = d\sigma \) and \( d\Phi = 2(e^{-\sigma} \eta^i + d\sigma) \wedge \Phi \).

**Proposition 3.1** Let \( (M^{2n+s}, \varphi, \xi, \eta^i, g) \), \( s \geq 2 \), be an f.pk-manifold which admits a function \( \sigma \in F(M^{2n+s}) \) such that (3.3) holds. Then, \( M^{2n+s} \) is a globally conformal almost Kenmotsu f.pk-manifold, with function \( \sigma \).

**Proof.** We put \( \tilde{\varphi} = \varphi, \tilde{\xi}_i = e^\sigma \xi_i, \tilde{\eta}^i = e^{-\sigma} \eta^i, \tilde{g} = e^{-2\sigma} g \). Then one easily verifies that \( (M^{2n+s}, \tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}^i, \tilde{g}) \) is an f.pk-manifold with fundamental form \( \tilde{\Phi} = e^{-2\sigma} \Phi \) and \( d\tilde{\Phi} = 2\tilde{\eta}^i \wedge \tilde{\Phi}, \tilde{d}\tilde{\eta}^i = 0 \), for each \( i \in \{1, \ldots, s\} \).

**Remark 2** The previous two results allow to state that an f.pk-manifold \( M^{2n+s} \), with \( s \geq 2 \), is locally conformal almost Kenmotsu if and only if it is globally conformal almost Kenmotsu or, equivalently, if and only if there exists a function \( \sigma \in F(M^{2n+s}) \) such that (3.3) holds. Moreover, assuming that \( M^{2n+s} \) is connected, the function \( \sigma \) is a constant if and only if \( M^{2n+s} \) is homothetic to an almost Kenmotsu f.pk-manifold. Furthermore, since the normality condition is not involved in the previous discussion, the same equivalences hold for locally (globally) conformal Kenmotsu f.pk-manifolds.

We remark that the hypothesis \( s \geq 2 \) is essential in the above results. Namely, when \( s = 1 \), Olszak proved that an almost contact metric manifold \( (M^{2n+1}, \varphi, \xi, \eta, g) \) is locally conformal almost cosymplectic if and only if there exists a closed 1-form \( \omega \) such that \( d\Phi = 2\omega \wedge \Phi \) and \( d\eta = \omega \wedge \eta \). Furthermore, \( M^{2n+1} \) is almost \( \alpha \)-Kenmotsu if and only if it is locally conformal almost cosymplectic with \( \omega = \alpha \eta, \alpha \) being a non-vanishing constant. This means that when \( s = 1 \) the almost \( \alpha \)-Kenmotsu manifolds, with \( \alpha \) constant, set up a subclass of the locally conformal almost cosymplectic manifolds. Now, we investigate the case \( s \geq 2 \) from this point of view.

We need the following characterization of locally conformal almost \( C \)-manifolds.

**Proposition 3.2** Let \( (M^{2n+s}, \varphi, \xi_i, \eta^i, g) \), \( s \geq 2 \), be an f.pk-manifold. Then, \( M^{2n+s} \) is a locally conformal almost \( C \)-manifold if and only if there exists a 1-form \( \omega \) such that

\[ d\omega = 0, \quad d\Phi = 2\omega \wedge \Phi, \quad d\eta^i = \omega \wedge \eta^i, \text{ for each } i \in \{1, \ldots, s\}. \]

**Proof.** Assuming that \( M^{2n+s} \) is a locally conformal almost \( C \)-manifold, we apply the same technique as at the beginning of the proof of Theorem 3.1 and determine a closed 1-form \( \omega \) such that \( d\eta^i = \omega \wedge \eta^i \) for each \( i \in \{1, \ldots, s\} \). The condition \( d\Phi = 2\omega \wedge \Phi \) is achieved since an almost \( C \)-manifold has closed fundamental form. Vice versa, \( \omega \) being locally exact, we consider an open covering \( (U_\alpha)_{\alpha \in A} \) such that, for any \( \alpha \in A, \omega_{|U_\alpha} = d\sigma_\alpha \). Then, putting

\[ \varphi_\alpha = \varphi_{|U_\alpha}, \quad \xi_\alpha = e^{\sigma_\alpha} \xi_{|U_\alpha}, \quad \eta^i_\alpha = e^{-\sigma_\alpha} \eta^i_{|U_\alpha}, \quad g_\alpha = e^{-2\sigma_\alpha} g_{|U_\alpha}, \]

it is easy to check that \( (U_\alpha, \varphi_\alpha, \xi_\alpha^i, \eta^i_\alpha, g_\alpha) \) is an almost \( C \)-manifold. \( \square \)
Proposition 3.3  The class of the almost Kenmotsu $f.pk$-manifolds of dimension $2n + s, s \geq 2,$ is disjoint from the class of the locally conformal almost $C$-manifolds.

Proof. Let $(M^{2n+s}, \varphi, \xi_i, \eta_i, g)$, $s \geq 2$, be an $f.pk$-manifold which is almost Kenmotsu and locally conformal almost $C$-manifold. Then there exists a 1-form $\omega$ such that $d\Phi = 2\omega \wedge \Phi$, $d\eta^i = \omega \wedge \eta^i$ for each $i \in \{1, \ldots, s\}$. Furthermore, one has $d\Phi = 2\eta^1 \wedge \Phi$ and $d\eta^i = 0$. This implies $\omega \wedge \eta^i = 0$ and then, since $s \geq 2$, we get $\omega = 0$ and $\eta^1 \wedge \Phi = 0$. Choosing $X \in \mathcal{D}$, $\|X\| = 1$ and computing $(\eta^1 \wedge \Phi)(\xi_1, X, \varphi X)$ we get $\eta^1(\xi_1) = 0$ which is a contradiction. □

Remark 3  It is also easy to verify that in dimension $2n + s, s \geq 2$, the locally conformal almost $C$-manifolds set up a class which is disjoint from the class of locally conformal almost Kenmotsu $f.pk$-manifolds.

References

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