Ordinary differential equations on infinite dimensional manifolds

M. Aghasi and A. Suri

Abstract. By using the Tangent bundle of a finite or infinite dimensional manifold we provide an alternative way to study the first and second order ordinary differential equations on M. We apply the vector fields with their integral curves, autoparallel curves and a new technique in the sense of [2], [6], [7], to introduce a new way to study ordinary differential equations on Banach manifolds and also a certain type of Fréchet manifolds obtained as projective limits of Banach manifolds. Moreover we extend the concept of completeness for Banach and Fréchet manifolds. In the other words we will prove that if M is a compact Banach (Fréchet) manifold then it is complete ie it's autoparallel curves are defined on the whole of real line $\mathbb{R}$.

M.S.C. 2000: 58B25; 58A05.
Key words: Banach manifolds, Fréchet manifolds, vector fields, integral curves, autoparallel curves, complete manifold, projective limit.

Introduction.

The study of infinite dimensional manifolds have received an increasing interest by its interaction with the modern differential geometry and theoretical physics. (see for example, [4], [13])
The main problems related to the manifolds modelled on non-Banach spaces and specially Fréchet spaces, are the pathological structure of general linear group $GL(\mathbb{F})$ and the lack of a general solvability for differential equations. (see [9]) These obstacles can be overcomes if we restrict ourselves to the category of Fréchet manifolds obtained as projective limits of Banach manifolds.

In [8] G.N. Galanis successfully generalized a vector bundle structure for TM by replacing the pathological structure of $GL(\mathbb{E})$ by $\mathcal{H}_0(F)$.
In the present work we study the first and second order ordinary differential equations on infinite dimensional manifolds. These equations assigned to vector fields and their integral curves and also autoparallel curves. First we investigate these equations for Banach modelled manifolds and in the further step for a wide class of Fréchet manifolds. Furthermore we develop interesting problems related to the completeness of vector fields from Banach to Fréchet case. Consequently we develop the concept of completeness for Banach manifolds and in the next step for Fréchet manifolds.

Balkan Journal of Geometry and Its Applications, Vol.12, No.1, 2007, pp. 1-8.

1 Preliminaries

In this section we introduce some of the basic notions for the rest of the paper. Let M be a smooth manifold modelled on the Banach space $\mathbb{E}$ with the atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ and let $\{(\pi^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha\in I}$ and $\{(\pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)), \tilde{\Psi}_\alpha)\}_{\alpha\in I}$ be the corresponding trivialization for TM and $T(TM)$ respectively.
A connection on M is a vector bundle morphism

$$\nabla : T(TM) \longrightarrow TM.$$

According to the formalism of [14] the connection ∇ is fully characterized by its family of Christoffel symbols $\{\Gamma_\alpha\}_{\alpha\in I}$ where;

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \to \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{E}))\,; \quad \alpha \in I,$$

and the local expression of ∇, i.e. $\nabla_\alpha = \Psi_\alpha \circ \nabla \circ \tilde{\Psi}_\alpha^{-1}$ is as follows:

$$\begin{array}{rcl}\nabla_\alpha : \psi_\alpha(U_\alpha)\times\mathbb{E}\times\mathbb{E}\times\mathbb{E} & \longrightarrow & \psi_\alpha(U_\alpha)\times\mathbb{E} \\ (y,u,v,w) & \longmapsto & (y, w+\Gamma_\alpha(y)(u,v)).\end{array}$$

2 Ordinary differential equations on Banach manifolds

2.1 Integral curves of vector fields

Let M be a smooth Banach manifold and $\xi \in C^\infty(TM)$. An integral curve of ξ is a smooth curve $\theta : J \longrightarrow M$ such that $T_t\theta(\partial_t) = \xi(\theta(t))$ where J is an open interval of $\mathbb{R}$ containing 0 and ∂_t is the tangent vector of $T_t\mathbb{R}$ produced by $c : \mathbb{R} \longrightarrow \mathbb{R}$, $\dot{c}(0) = 1$.

Theorem 2.1. *Let M be a smooth manifold modelled on the Banach space $\mathbb{E}$. The existence of an integral curve for $\xi \in C^\infty(TM)$ is equivalent to the existence of solution for a system of differential equations on $\mathbb{E}$.*

Proof: Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ be an atlas for the smooth manifold M. Considering the corresponding atlas for TM, we find out that the local expression of $T_t\theta(\partial_t) = \xi(\theta(t))$ is as follows;

$$\begin{array}{rcl}\Psi_\alpha(T_t(\theta(\partial_t))) & = & \Psi_\alpha([\theta\circ c, \theta(t)]) = ((\psi_\alpha\circ\theta)(t), (\psi_\alpha\circ\theta\circ c)'(0)) \\ & = & ((\psi_\alpha\circ\theta)(t), T_t(\psi_\alpha\circ\theta)(1)) \\ & = & ((\psi_\alpha\circ\theta)(t), (\psi_\alpha\circ\theta)'(t));\ \alpha\in I.\end{array}$$

Hence $(\psi_\alpha \circ \theta)'(t) = \Psi_\alpha^2(\xi(\theta(t)))$, where Ψ_α^2 states the projection of Ψ_α onto its second factor. □

So the existence of integral curves for vector fields generalizes the concept of differential equations on manifolds.

Theorem 2.2. *For any vector field ξ and any $x \in M$, there exists a unique integral curve θ satisfying the initial condition $\theta(0) = x$.*

Recall that a vector field ξ is said to be complete if every integral curve of ξ is defined on the whole of real line $\mathbb{R}$. As it is stated in [10] we have the following criterion for the characterization of manifolds with complete vector fields.

Lemma 2.3. *Suppose that M is a compact manifold modelled on the Banach space $\mathbb{E}$, then every vector field of M is complete.*

2.2 Autoparallel curves

We know that if M is an n-dimensional Riemannian manifold with the Levi-Civita connection, then the existence of geodesics is equivalent to having solution of a system of second order differential equations on the model space i.e. $\mathbb{R}^n$.
In the case of non-Riemannian manifolds we have no metric and hence no geodesic, but still there exist autoparallel curves. Here we characterize the system of differential equations assigned to an autoparallel curve $\gamma : J \longmapsto M$.
According to [14] $\nabla_{T\gamma}T\gamma = \nabla T(T\gamma)$, and the local expression of $\nabla T(T\gamma) = 0$ is as follows;

$$\begin{aligned}
\nabla_{T\gamma}T\gamma &= \Psi_\alpha^{-1} \circ \nabla_\alpha \circ \widetilde{\Psi}_\alpha(T_t(T_t(\gamma(\partial_t)))) \\
&= \Psi_\alpha^{-1} \circ \nabla_\alpha((\psi_\alpha \circ \gamma)(t), (\psi_\alpha \circ \gamma)'(t), (\psi_\alpha \circ \gamma)'(t), (\psi_\alpha \circ \gamma)''(t)) \\
&= \Psi_\alpha^{-1}((\psi_\alpha \circ \gamma)(t), (\psi_\alpha \circ \gamma)''(t) \\
&\quad + \Gamma_\alpha((\psi_\alpha \circ \gamma)(t))[(\psi_\alpha \circ \gamma)'(t), (\psi_\alpha \circ \gamma)'(t)]) \\
&= 0,
\end{aligned}$$

that is $\nabla_{T\gamma}T\gamma = 0$ iff

$$(\psi_\alpha \circ \gamma)''(t) + \Gamma_\alpha((\psi_\alpha \circ \gamma)(t))[(\psi_\alpha \circ \gamma)'(t), (\psi_\alpha \circ \gamma)'(t)] = 0, \ \alpha \in I. \tag{1}$$

So we can state the following theorem.

Theorem 2.4. *Let M be a smooth manifold modelled on the Banach space $\mathbb{E}$. If $x \in M$ and $y \in T_xM$, then there exists a unique autoparallel curve $\gamma : J \longrightarrow M$ such that $\gamma(0) = x$ and $T_t\gamma(\partial_t) = y$.*

Now we probe the relations between autoparallel curves and conjugate connections on a smooth manifold M. For a smooth mapping $g : M \longrightarrow N$ the linear connections ∇_M and ∇_N on M and N (respectively), are g-conjugate if $Tg \circ \nabla_M = \nabla_N \circ T(Tg)$.

Corollary 2.5. *let γ be an autoparallel curve in (M, ∇) and $g : M \longrightarrow M$ be a diffeomorphism. Then $g \circ \gamma$ is an autoparallel curve of (M, ∇') where ∇' is a g-conjugate connection on M.*

Proof : $g \circ \gamma$ is an autoparallel curve in (M, ∇') if and only if:

$$(\psi_\beta \circ g \circ \gamma)''(t) + \Gamma'_\beta((\psi_\beta \circ g \circ \gamma)(t))[(\psi_\beta \circ g \circ \gamma)'(t), (\psi_\beta \circ g \circ \gamma)'(t)] = 0.$$

If $G = \psi_\beta \circ g \circ \psi_\alpha^{-1}$ then;

$$
\begin{aligned}
&(\psi_\beta \circ g \circ \gamma)''(t) + \Gamma'_\beta((\psi_\beta \circ g \circ \gamma)(t))[(\psi_\beta \circ g \circ \gamma)'(t), (\psi_\beta \circ g \circ \gamma)'(t)] \\
= \;& (G \circ \psi_\alpha \circ \gamma)''(t) + DG((\psi_\alpha \circ \gamma)(t))\Big(\Gamma_\alpha((\psi_\alpha \circ \gamma)(t))[(\psi_\alpha \circ \gamma)'(t) \\
&, (\psi_\alpha \circ \gamma)(t)]\Big) - D(DG)((\psi_\alpha \circ \gamma)(t))[(\psi_\alpha \circ \gamma)'(t), (\psi_\alpha \circ \gamma)'(t)] \\
= \;& DG((\psi_\alpha \circ \gamma)(t))((\psi_\alpha \circ \gamma)''(t)) + D(DG)((\psi_\alpha \circ \gamma)(t))[(\psi_\alpha \circ \gamma)'(t) \\
&, (\psi_\alpha \circ \gamma)'(t)] + DG((\psi_\alpha \circ \gamma)(t))\Big(\Gamma_\alpha((\psi_\alpha \circ \gamma)(t))[(\psi_\alpha \circ \gamma)'(t) \\
&, (\psi_\alpha \circ \gamma)'(t)]\Big) - D(DG)((\psi_\alpha \circ \gamma)(t))[(\psi_\alpha \circ \gamma)'(t), (\psi_\alpha \circ \gamma)'(t)] = 0
\end{aligned}
$$

Hence $\nabla'_{Tg\circ\gamma} Tg \circ \gamma = 0$. □

In the following lemma we study about the concept of completeness for Banach manifolds. On the other hand we will investigate that under which conditions the autoparallel curves are defined on the whole of real line $\mathbb{R}$. (or simply M is complete)

In the case that M is a finite dimensional Riemannian manifold we know that if M is complete as a metric space then it is geodetically complete. Also in the case of Finsler-Cartan-Hadamard, the geodesics of a Banach manifold M with spray F are complete iff M is complete as a metric space. (for more details see [12])

Here in spite of this difficulty that we have no metric or even a Finsler manifold, we will obtain an analogous result for the Banach and in the further for Fréchet manifolds.

Lemma 2.6. *Let M be a compact smooth manifold modelled on Banach space $\mathbb{E}$. Then M is complete.*

As it is shown in [2] either in the frame works of tangent bundles and second order tangent bundles (bundle of accelerations), the autoparallel curves are defined with a same system of ordinary differential equations . Thus for the complete proof of the last lemma we refer to [3].

3 Fréchet case

Here we focus on those Fréchet manifolds which can be obtained as projective limits of Banach manifolds.
Let $\{M^i, \varphi^{ji}\}_{i,j\in\mathbb{N}}$ be a projective system of manifolds modelled on the projective system of Banach spaces $\{\mathbb{E}^i, \rho^{ji}\}_{i,j\in\mathbb{N}}$ respectively. Furthermore suppose that for $x = (x^i)_{i\in\mathbb{N}} \in M = \varprojlim M^i$ there exists a projective system of local charts $\{(U^i, \psi^i)\}_{i\in\mathbb{N}}$ such that $x^i \in U^i$ and $\varprojlim U^i$ is open in M. Then we can endow $M = \varprojlim M^i$ with a Fréchet manifold structure modelled on the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$.
Furthermore suppose that for each $i \in \mathbb{N}$, ∇^i be a linear connection on M^i such that $\nabla = \varprojlim \nabla^i$ exists.

The intrinsic problems related to pathological structure of $GL(\mathbb{E})$ overcome if we replace it with the generalized Lie group;

$$\mathcal{H}_0(\mathbb{F}) = \{(l^i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} GL(\mathbb{E}^i) : \; \rho^{ji} \circ l^j = l^k \circ \rho^{jk}; \; for \; k \le j \le i\}.$$

In fact $\mathcal{H}_0(\mathbb{F})$ is isomorphic to the projective limit of Banach Lie groups

$$\mathcal{H}_0^i(\mathbb{F}) = \{(l^1, l^2, ..., l^i) \in \prod_{k=1}^{i} GL(\mathbb{E}^k) : \ \rho^{jk} \circ l^j = l^k \circ \rho^{jk}; \ for \ k \leq j \leq i\}.$$

By these notations we can endow the tangent bundle of $M = \varprojlim M^i$ with a vector bundle structure with fibres of type $\mathbb{F} = \varprojlim \mathbb{E}^i$ and the structure group $\mathcal{H}_0(\mathbb{F})$. (for more details see [8]).

$TM = \varprojlim TM^i$ provides an alternative way to study the first and second order ordinary differential equations on Fréchet modelled manifolds. Moreover we can overcome the difficulties related to the existence and uniqueness of solutions of differential equations on Fréchet manifolds which obtain as projective limits of Banach manifolds. Our results about the existence and uniqueness of solutions will be compatible with the Banach case.

Theorem 3.1. *Let ξ be a vector field on the Fréchet manifold $M = \varprojlim M^i$ which can be considered as the projective limit of vector fields $\{\xi^i\}_{i\in\mathbb{N}}$ of $\{M^i\}_{i\in\mathbb{N}}$ respectively. Then ξ admits locally a unique integral curve θ, satisfying an initial condition of the form $\theta(0) = x$ for $x \in M$.*

Proof : Since for each $i \in \mathbb{N}$, ξ^i is a vector field of the Banach manifold M^i, by theorem 2.1 there exists a unique integral curve θ^i such that

$$(\psi_\alpha^i \circ \theta^i)' = \Psi_\alpha^{2,i}(\xi^i(\theta^i(t))); \ \alpha \in I,$$

and $\theta^i(0) = x^i = \varphi^i(x)$ where $\varphi^i : M \longrightarrow M^i$ is the canonical projection. we claim that $\theta = \varprojlim \theta^i$ exists and satisfies the conditions of the theorem. For this aim we have to prove that $\varphi^{ji} \circ \theta^j = \theta^i$ for $j \geq i$. We prove that $\varphi^{ji} \circ \theta^j$ is also an integral curve of ξ^i.

$$\begin{aligned}
(\psi_\alpha^i \circ (\varphi^{ji} \circ \theta^j))'(t) &= (\rho^{ji} \circ (\psi_\alpha^j \circ \theta^j))'(t) = \rho^{ji}(\psi_\alpha^j \circ \theta^j)'(t) \\
&= \rho^{ji}(\Psi_\alpha^{2,j}(\xi^j(\theta^j(t)))) = \Psi_\alpha^{2,j}(T\varphi^{ji}(\xi^j(\theta^j(t))) \\
&= \Psi_\alpha^{2,j}(\xi^j(\varphi^{ji} \circ \theta^j(t))).
\end{aligned}$$

Furthermore $\varphi^{ji} \circ \theta^j(x^j) = x^i$, hence by theorem 2.1 $\varphi^{ji} \circ \theta^j = \theta^i$ i.e. $\{\theta^i\}_{i\in\mathbb{N}}$ is a projective system of curves and $\theta = \varprojlim \theta^i$ exists. θ fulfils the conditions of the theorem, in fact

$$(\psi_\alpha \circ \theta)'(t) = ((\psi_\alpha^i \circ \theta^i)'(t))_{i\in\mathbb{N}} = (\Psi_\alpha^{2,i}(\xi^i(\theta^i(t))))_{i\in\mathbb{N}} = \Psi_\alpha^2(\xi(\theta(t))).$$

For the uniqueness of θ, suppose that γ is another curve satisfying the conditions of the theorem, then $\varphi^i \circ \gamma$ is an integral curve of ξ^i which $\varphi^i \circ \gamma(0) = x^i$. By the theorem 3, $\varphi^i \circ \gamma = \theta^i$ and consequently $\theta = \gamma$. □

Here we generalize lemma 2.3 for projective limit manifolds.

Lemma 3.2. *Let $M = \varprojlim M^i$ be a compact manifold modelled on the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$. If M is compact, then every vector field of M obtained as projective limit of Banach vector fields $\{\xi^i\}_{i\in\mathbb{N}}$ is complete.*

Proof : The proof is a direct result of this note that for any $i \in \mathbb{N}$, the projection map $\varphi : M \longrightarrow M^i$ is continuous and surjective. So M^i is compact and by applying lemma 2.3 for any $i \in \mathbb{N}$, ξ^i is complete. Hence $\xi = \varprojlim \xi^i$ is a complete vector field.□

Theorem 3.3. *Let $M = \varprojlim M^i$ be a smooth manifold modelled on the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$. For $x \in M$ and $y \in T_x M$ there exists a unique autoparallel curve $\gamma : J \longrightarrow M$ such that $\gamma(0) = x$ and $T_t\gamma(\partial_t) = y$.*

Proof : Let $\nabla = \varprojlim \nabla^i$ be a connection on $M = \varprojlim M^i$, then by the theorem 2.4 for each $i \in \mathbb{N}$, there exists a unique autoparallel curve γ^i such that $\gamma^i(0) = \varphi^i(x) = x^i$ and $T_t\gamma^i(\partial_t) = T\varphi^i(y) = y^i$.
We claim that $\gamma = \varprojlim \gamma^i$ exists and fulfils the equations (1). We show that for $j \geq i$, $\varphi^{ji} \circ \theta^j$ is also an autoparallel curve of M^i with respect to ∇^i. In fact;

$$\begin{aligned}
& (\psi^i_\alpha \circ (\varphi^{ji} \circ \theta^j))''(t) + \Gamma^i_\alpha((\psi^i_\alpha \circ (\varphi^{ji} \circ \theta^j))(t)) \\
& \qquad [(\psi^i_\alpha \circ (\varphi^{ji} \circ \theta^j))'(t), (\psi^i_\alpha \circ (\varphi^{ji} \circ \theta^j))''(t)] \\
= \ & (\rho^{ji} \circ (\psi^j_\alpha \circ \theta^j))''(t) + \Gamma^i_\alpha((\rho^{ji} \circ (\psi^j_\alpha \circ \theta^j))(t)) \\
& \qquad [(\rho^{ji} \circ (\psi^j_\alpha \circ \theta^j))'(t), (\rho^{ji} \circ (\psi^j_\alpha \circ \theta^j))'(t) \\
= \ & \rho^{ji}((\psi^j_\alpha \circ \theta^j))''(t) + \Gamma^i_\alpha((\rho^{ji} \circ (\psi^j_\alpha \circ \theta^j))(t)) \\
& \qquad [(\rho^{ji} \circ (\psi^j_\alpha \circ \theta^j))'(t), (\rho^{ji} \circ (\psi^j_\alpha \circ \theta^j))'(t)] \\
= \ & \rho^{ji} \circ \{(\psi^j_\alpha \circ \theta^j)''(t) + \Gamma^j_\alpha((\psi^j_\alpha \circ \theta^j)(t))[(\psi^j_\alpha \circ \theta^j)'(t), (\psi^j_\alpha \circ \theta^j)'(t)]\} \\
= \ & 0.
\end{aligned}$$

Furthermore $\varphi^{ji} \circ \gamma^j(0) = \varphi^{ji}(x^j) = x^i$ and $T_t\varphi^{ji} \circ \gamma^j(\partial_t) = T\varphi^{ji}(y^j) = y^i$, hence $\varphi^{ji} \circ \gamma^j = \gamma^i$ i.e. $\gamma = \varprojlim \gamma^i$ exists. On the other hand;

$$\begin{aligned}
& (\psi_\alpha \circ \gamma)''(t) + \Gamma_\alpha((\psi_\alpha \circ \gamma)(t))[(\psi_\alpha \circ \gamma)'(t), (\psi_\alpha \circ \gamma)'(t)] \\
& = ((\psi^i_\alpha \circ \gamma^i)''(t) + \gamma^i_\alpha((\psi^i_\alpha \circ \gamma^i)(t))[(\psi^i_\alpha \circ \gamma^i)'(t), (\psi^i_\alpha \circ \gamma^i)'(t)])_{i\in\mathbb{N}} \\
& = 0.
\end{aligned}$$

Let $\bar{\gamma}$ be another autoparallel curve such that $\bar{\gamma}(0) = x$ and $T_t\bar{\gamma}(0) = y$, then because of the following equations, $\varphi^i \circ \bar{\gamma}$ is an autoparallel of M^i with respect to ∇^i.

$$\begin{aligned}
& (\psi^i_\alpha \circ \varphi^i \circ \bar{\gamma})''(t) + \Gamma^i_\alpha((\psi^i_\alpha \circ \varphi^i \circ \bar{\gamma})(t))[(\psi^i_\alpha \circ \varphi^i \circ \bar{\gamma})'(t), (\psi^i_\alpha \circ \varphi^i \circ \bar{\gamma})'(t)] \\
= \ & (\rho^i \circ \psi_\alpha \circ \bar{\gamma})''(t) + \Gamma^i_\alpha((\rho^i \circ \psi_\alpha \circ \bar{\gamma})(t))[(\rho^i \circ \psi_\alpha \circ \bar{\gamma})'(t), (\rho^i \circ \psi_\alpha \circ \bar{\gamma})'(t)] \\
= \ & \rho^i((\psi_\alpha \circ \bar{\gamma})''(t) + \Gamma_\alpha((\psi_\alpha \circ \bar{\gamma})(t))[(\psi_\alpha \circ \bar{\gamma})'(t), (\psi_\alpha \circ \bar{\gamma})'(t)]) \\
= \ & 0.
\end{aligned}$$

Moreover $\varphi^i \circ \bar{\gamma}(0) = \varphi^i(x) = x^i$ and $T_t\varphi^i \circ \bar{\gamma}(\partial_t) = y^i$. Hence according to theorem 2.4 $\varphi^i \circ \bar{\gamma} = \theta^i$ i.e. θ is unique. □

At the sequel we state the following lemma which is an extension of lemma 2.6.

Lemma 3.4. *Suppose that $M = \varprojlim M^i$ is a compact Fréchet manifold modelled on the Fréchet space $\mathbb{F} = \varprojlim \mathbb{F}^i$. Then M is complete.*

Proof: For any $i \in \mathbb{N}$ the projection $\varphi^i : M \longrightarrow M^i$; $(x^i)_{i\in\mathbb{N}} \longmapsto x^i$ is a continuous surjection. So the compactness of M implies that for any $i \in \mathbb{N}$, M^i is a compact Banach manifold modelled on the Banach space $\mathbb{E}^i$. Let γ be an autoparallel on M then $\varphi^i \circ \gamma$ is an autoparallel curve on M^i. Using lemma 2.6 yields that γ^i is defined on the whole of real line $\mathbb{R}$ and consequently $\gamma = \varprojlim \gamma^i$. □

Example 3.5. *Consider $\mathbb{E}$ as a smooth manifold with the total chart $(\mathbb{E}, id_{\mathbb{E}})$. If we endow $M = \mathbb{E}$ with the canonical flat connection i.e. the connection on M such that its Christoffel symbol vanishes everywhere, then the equations related to integral curves of vector fields will reduce to;*

$$\theta'(t) = \Psi^2_\alpha(\xi(\theta(t))).$$

Also the equation (1) takes the form

$$\theta''(t) = 0,$$

i.e. $\theta(t) = at + b$. (Note that in the Riemannian case autoparallel curves coincide with geodesics.)

As we mentioned earlier, the assigned equations are ordinary differential equations on the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$ where the first one can be solved if we assume that the vector field ξ is a projective limit vector field, and for the second equation if $\nabla = \varprojlim \nabla^i$.

Acknowledgement. The authors acknowledge financial support provided by the Azad University, Shahreza branch. This paper is extracted from the project titled " Autopapralle sections of infinite dimensional manifolds" which is partially supported by Azad University, Shahreza branch.

References

[1] M.C. Abbati, A. Mania, *On differential structure for projective limits of manifolds*, J. Geom. Phys. 29, 1-2 (1999), 35-63.

[2] M. Aghasi, C.T.J. Dodson, G.N. Galanis and A. Suri, *Infinite dimensional second order ordinary differential equations via T^2M*, Nonlinear analysis (2006), doi:10.1016/j.na.2006.9.043

[3] M. Aghasi, C.T.J. Dodson, G.N. Galanis and A. Suri, *Conjuagate connections and differential equations on infinite dimensional manifolds*, MIMS EPrint, 2006.8.

[4] D.E. Blair, *Spaces of metrics and curvature functionals*, in Handbook of differential geometry, Vol. I, North Holland, Amsterdam 2000, 153-185.

[5] C.T.J. Dodson, G.N. Galanis and E. Vassiliou, *Isomorphism classes for Banach vector bundle structures of second tangents*, to appear in Math. Proc. Camb. Phil. Soc.

[6] G.N. Galanis, *On a type of linear differential equations in Fréchet spaces*, Annali della Scuola Normale Superiore di Pisa, 4, 24 (1997), 501-510.

[7] G.N. Galanis and P.K. Palamides, *Nonlinear differential equations in Fréchet spaces and continuum-cross-sections*, Anal. St. Univ. 'Al.I. Cuza', 51 (2005), 41-54.

[8] G.N. Galanis, *Differential and geometric structure for the tangent bundle of a projective limit manifold*, Rendiconti del Seminario Matematico di Padova, 112 (2004), 1-12.

[9] R.S. Hamilton, *The inverse functions theorem of Nash and Moser,* Bull. of Amer. Math. Soc. 7 (1982), 65-222.

[10] J.M. Lee, *Differential and physical geometry*, Addison-Wesley, Reading Massachusetts, 1972.

[11] S. Lang, *Differential manifolds*, Addison-Wesley, Reading Massachusetts, 1972.

[12] K.-H. Neeb, *A Cartan-Hadamard theorem for Banach-Finsler manifolds*, Geometriae Dedicata 95 (2002), 115-156.

[13] A. Sergeev, $Diff_+(S^1)/S^1$ *as a space of complex structure on loop spaces of compact Lie groups. Stochastic processes, physics and geometry: new interplay, II (Leipzig), 1999), 573-588,* CMS Conf. Proc. 29, Amer. Math. Soc., Providence, RI, 2000.

[14] J. Vilms, *Connections on tangent bundles*, J. Diff. Geom. 41 (1967), 235-243.

Authors' address:

M. Aghasi and A. Suri
Department of Mathematical Sciences,
Isfahan University of Technology,
Isfahan 84156, Iran.
e-mail: m.aghasi@cc.iut.ac.ir, a.suri@math.iut.ac.ir