Null 2-type space-like submanifolds of $E_t^5$
with normalized parallel mean curvature vector

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Abstract. The purpose of this article is to classify 3-dimensional null 2-type space-like submanifolds of the pseudo-Euclidean space $E_t^5$ which are constructed from the eigenfunctions of the Laplacian with two eigenvalues 0 and nonzero constant $\lambda$, under certain assumptions.

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Key words: Laplacian, null 2-type submanifold, scalar curvature, mean curvature vector.

1 Introduction

A connected submanifold $M^n$ of a pseudo-Euclidean space $E_t^m$ is called of finite type if its position vector field $x$ can be written as a sum of eigenfunctions of its Laplacian; more precisely, $M^n$ is said to be of finite k-type if its position vector field $x$ admits the following spectral decomposition

$$x = x_0 + x_1 + \cdots + x_k,$$

where $\Delta x_i = \lambda_i x_i, i = 1, 2, \ldots, k, \lambda_1 < \cdots < \lambda_k, x_0$ is a constant vector in $E_t^m$ and $x_1, \ldots, x_k$ are non-constant $E_t^m$-valued maps on $M^n$. If one of the eigenvalues $\lambda_i$ vanishes, then $M^n$ is said to be of null k-type (see [1, 2] for detail). We can choose a coordinate system on $E_t^m$ with $x_0$ as its origin. Then we have the following simple spectral decomposition of $x$ for a null 2-type submanifold $M$:

$$x = x_1 + x_2, \quad \Delta x_1 = 0, \quad \Delta x_2 = \lambda x_2.$$

In [4, 5], B.Y. Chen gave a classification of null 2-type surfaces in the Euclidean space $E^3$ and $E^4$. He proved that circular cylinders and helical cylinders are the only surfaces of null 2-type in $E^3$ and $E^4$, respectively. In [5], he also proved that a surface $M$ in the Euclidean space $E^4$ is of null 2-type with parallel normalized mean curvature vector if and only if $M$ is an open portion of a circular cylinder in a hyperplane of $E^4$. However, in [12], S.J. Li showed that a surface $M$ in $E^m$ with parallel normalized

mean curvature vector is of null 2-type if and only if $M$ is an open portion of a circular cylinder.

Later, in [6], B.Y. Chen and H. Song proved that a space-like surface $M$ in $E^4_t$, $(t = 1, 2)$ is of null 2-type with constant mean curvature if and only if $M$ is an open portion of a helical cylinder of the first kind or a helical cylinder of the second kind in $E^4_t$, $(t = 1, 2)$.

Also, in [11], D.S. Kim and Y.H. Kim gave complete classification theorems on null 2-type surfaces in Minkowski space $E^4$. They proved that a Lorentzian surface $M$ in $E^4_1$, $(t = 1, 2)$ is of null 2-type with constant mean curvature if and only if $M$ is an open portion of a helical cylinder of third kind, a helical cylinder of fourth kind, an extended B-scroll or a cylinder $E^3_1 \times S^1(r)$, $S^1(r) \times E$.

In the case of the classification of hypersurfaces, the constancy of the mean curvature does not provide enough information to obtain a characterization of null 2-type hypersurfaces of Euclidean spaces and Lorentzian spaces. In [9, 10], A. Ferrandez and P. Lucas studied null 2-type hypersurfaces of Euclidean spaces and null 2-type space-like hypersurfaces of Lorentzian spaces with additional assumption of having at most two distinct principal curvatures. They proved that Euclidean hypersurfaces of null 2-type and having at most two distinct principal curvatures are locally isometric to a generalized spherical cylinder, [9], and a space-like hypersurface of the Lorentzian space $E^m_1$ with at most two distinct principal curvatures is of null 2-type if and only if it is locally isometric to a generalized hyperbolic cylinder, [10].

The assumptions on hypersurfaces to be of null 2-type are not enough for submanifolds $M^n$, $n \geq 3$ of the Euclidean spaces $E^m$ and the pseudo-Euclidean spaces $E^m_1$ to be of null 2-type. In [7], the author proved that a 3-dimensional submanifold $M$ of the Euclidean space $E^5$ having two distinct principal curvatures in the parallel mean curvature direction and having a second fundamental form of a constant square length is of null 2-type if and only if $M$ is locally isometric to one of $E \times S^2 \subset E^4 \subset E^5$, $E^2 \times S^1 \subset E^4 \subset E^5$ or $E \times S^1(a) \times S^1(a)$. However, in [8], the author proved that a 3-dimensional submanifold $M$ of the Euclidean space $E^5$ with constant mean curvature and non-parallel mean curvature vector is an open portion of a 3-dimensional helical cylinder if and only if $M$ is flat and of null 2-type.

In this work we study the classification of null 2-type space-like submanifolds of the pseudo-Euclidean spaces. We mainly prove that a 3-dimensional space-like submanifold $M$ of the pseudo-Euclidean space $E^5_1$ with parallel normalized non-null mean curvature vector is of null 2-type having two distinct principal curvatures in the mean curvature direction and having a constant scalar curvature $\tau$ if and only if $M$ is locally isometric to one of the following:

1. $S^1(a) \times E^2 \subset E^4 \subset E^5_1$ or $S^2(a) \times E \subset E^4 \subset E^5_1$ when $H$ is space-like,

2. $H^1(a) \times E^2 \subset E^4_1 \subset E^5_1$ or $H^2(a) \times E \subset E^4_1 \subset E^5_1$ when $H$ is time-like, or

3. $H^1(a) \times E^2 \subset E^4_1 \subset E^5_2$, $H^2(a) \times E \subset E^4_1 \subset E^5_2$, or $H^1(a) \times H^1(a) \times E \subset E^5_2$.

The cases (1) and (2) imply that there is no such a submanifold that lies fully in $E^5_1$. 
2 Preliminaries

Let $E_t^m$ be an $m$-dimensional pseudo-Euclidean space with metric tensor given by

$$g = -\sum_{i=1}^{t} (dx_i)^2 + \sum_{i=t+1}^{m} (dx_i)^2$$

where $(x_1,\ldots,x_m)$ is a rectangular coordinate system of $E_t^m$. So $(E_t^m, g)$ is a flat pseudo-Riemannian manifold with signature $(t, m-t)$. When $t = 1$, $E_1^m$ is called the Lorentzian space. The hyperbolic space $H^m(a)$ is defined by

$$H^m(a) = \{ x \in E_1^{m+1} | \langle x, x \rangle = -a^2 \text{ and } x_1 > 0 \},$$

where $x_1$ is the first coordinate in $E_1^{m+1}$.

Let $M$ be an $n$-dimensional pseudo-Riemannian submanifold of an $m$-dimensional pseudo-Euclidean space $E_t^m$. We denote by $h, A, H, \nabla$ and $\nabla^\perp$, the second fundamental form, the Weingarten map, the mean curvature vector, the Riemannian connection and the normal connection of the submanifold $M$ in $E_t^m$, respectively.

Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ be an adapted local orthonormal frame in $E_t^m$ such that $\langle e_A, e_B \rangle = \varepsilon_B \delta_{AB}$, $(\varepsilon_B = \langle e_B, e_B \rangle = \pm 1)$, $e_1, \ldots, e_n$ are tangent to $M^n$ and $e_{n+1}, \ldots, e_m$ are normal to $M^n$. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq m, \quad 1 \leq i, j, k, \ldots \leq n, \quad n + 1 \leq \beta, \nu, \gamma, \ldots \leq m.$$

Let $\{\omega_A\}$ be the dual 1-forms of $\{e_A\}$ defined by $\omega_A(X) = \langle e_A, X \rangle$, $(\omega_A(e_B) = \langle e_B, e_A \rangle = \varepsilon_B \delta_{AB})$. Also, the connection forms $\omega^B_A$ are defined by

$$de_A = \sum_{B=1}^{m} \omega^B_A e_B, \quad \varepsilon_B \omega^B_A + \varepsilon_A \omega^A_B = 0.$$

For lifting or lowering indices we use $\omega^A = \varepsilon_A \omega_A$, $\omega^B_A = \varepsilon_B \omega_{AB}$. Then the structure equations of $E_t^m$ are obtained as follows

$$d\omega^A = \sum_{B=1}^{m} \omega^B_A \wedge \omega^A_B, \quad d\omega^B_A = \sum_{C=1}^{m} \omega^C_A \wedge \omega^B_C. \quad (2.1)$$

Restricting these forms to $M$ we have

$$\omega^\beta = 0, \quad d\omega^\beta = \sum_{i=1}^{m} \omega^i \wedge \omega^\beta_i = 0, \quad \beta = n + 1, \ldots, m.$$

By Cartan’s Lemma, we can write

$$\omega^\beta_i = \sum_{j=1}^{n} h^\beta_{ij} \omega^j, \quad h^\beta_{ij} = h^\beta_{ji}, \quad (2.2)$$

where $h^\beta_{ij}$ are coefficients of the second fundamental form in the direction $e_\beta$. 
The mean curvature vector $H$ is given by

$$
H = \frac{1}{n} \sum_{\beta=n+1}^{m} \varepsilon_{\beta} \text{tr}(h^\beta) e_\beta
$$

and the scalar curvature $\tau$ is given by

$$
n(n - 1)\tau = n^2 |H|^2 - \|h\|^2
$$

where $\|h\|^2$ denotes the square of the length of the second fundamental form which is defined by

$$
\|h\|^2 = \sum_{\beta} \varepsilon_{\beta} \text{tr}(h^\beta)^2 = \sum_{i,j,\beta} \varepsilon_{\beta} \varepsilon_{i} \varepsilon_{j} (h_{ij}^\beta)^2.
$$

The first equation of (2.1) gives

$$
d\omega_i^j = \sum_{j=1}^{n} \omega_j^i \wedge \omega_j^j, \quad \varepsilon_{i} \omega_i^j + \varepsilon_{j} \omega_i^j = 0,
$$

where $\{\omega_i^j\}$ is the connection forms on $M$ and uniquely determined by these equations. However, from the second equation of (2.1) we can have the Gauss and Codazzi equations, respectively, as

$$
d\omega_i^\beta = \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^\beta + \sum_{\beta=n+1}^{m} \omega_i^\beta \wedge \omega_\beta^\beta
$$

and

$$
d\omega_i^\beta = \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^\beta + \sum_{\nu=n+1}^{m} \omega_i^\nu \wedge \omega_\nu^\beta.
$$

Using (2.2) and the connection equations $\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_i^k(e_j)e_k$ we can restate the equations of Gauss (2.7) and Codazzi (2.8) relative to the basis $e_1, \ldots, e_n$, respectively, as follows

$$
e_\ell(\omega_i^j(e_k)) - e_k(\omega_i^\ell(e_\ell)) = \sum_{r=1}^{n} \{\omega_r^\ell(e_\ell)\omega_i^j(e_k) - \omega_r^\ell(e_k)\omega_i^\ell(e_\ell)
$$

$$
+ \omega_i^j(e_\ell)[\omega_k^\ell(e_\ell) - \omega_\ell^k(e_\ell)]\} + \sum_{\nu=n+1}^{m} \varepsilon_{\nu} \varepsilon_{i} \varepsilon_{j} (e_k h_{i}^{\ell} h_{j}^{\ell} - e_\ell h_{i}^{\ell} h_{j}^{\ell}),
$$

$1 \leq i < j \leq n, \quad 1 \leq \ell < k \leq n$ and
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\[ e_j(h^\nu_{ik}) - e_k(h^\nu_{ij}) = \sum_{r=1}^{n} \{ h^\nu_{ir} \omega^r_k(e_j) - \omega^r_j(e_k) \} + h^\nu_{jk} \omega^j_k(e_j) - h^\nu_{ik} \omega^i_j(e_k) \] 

(2.10) 

\[ + \sum_{\beta=n+1}^{m} (h^\beta_{ij} \omega^\nu_j(e_k) - h^\beta_{ik} \omega^\nu_j(e_j)), \]

\[ \nu = n + 1, \ldots, m, \quad i = 1, \ldots, n, \quad 1 \leq j < k \leq n. \]

If the normal space of \( M \) in \( E^m \) is flat, then we can choose a parallel orthonormal normal basis on \( M \). Therefore we have \( \omega^\nu_{ij} = 0 \). Hence the equations of Codazzi become

(2.11) 

\[ e_j(h^\nu_{ii}) = \varepsilon_j \varepsilon_i (h^\nu_{ii} - h^\nu_{jj}) \omega^i_j(e_i), \quad i \neq j. \]

and

(2.12) 

\[ \varepsilon_j(h^\nu_{ii} - h^\nu_{kk}) \omega^i_j(e_j) + \varepsilon_k(h^\nu_{jj} - h^\nu_{ii}) \omega^j_k(e_k) = 0, \quad i \neq j \neq k \neq i. \]

3 Some Basic Lemmas

We need the following some well known formulas and lemmas (for details see [1, 2, 3, 5]).

**Lemma 3.1.** Let \( M \) be an \( n \)-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidean space \( E^m \). Then we have

(3.1) 

\[ \Delta H = \Delta \nabla^\perp H + \sum_{i=1}^{n} \varepsilon_i \{(\nabla e_i A H) e_i + A \nabla^\perp_{e_i} H e_i + h(A H e_i, e_i)\}, \]

where \( \Delta \nabla^\perp = -\sum_{i=1}^{n} \varepsilon_i \{\nabla^\perp e_i, \nabla^\perp e_i - \nabla^\perp e_i e_i\} \) is the Laplacian operator associated with the induced normal connection \( \nabla^\perp \).

**Lemma 3.2.** Let \( M \) be an \( n \)-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidean space \( E^m \). Then we have

(3.2) 

\[ \text{tr}(\nabla A H) = \sum_{i=1}^{n} \varepsilon_i (\nabla e_i A H) e_i = \frac{n}{2} \nabla(H, H) + \text{tr}(A \nabla^\perp H), \]

where \( \nabla(H, H) \) is the gradient of \( (H, H) \) and \( \text{tr}(A \nabla^\perp H) = \sum_{i=1}^{n} \varepsilon_i A \nabla^\perp_{e_i} H e_i \).

1-type pseudo-Riemannian submanifold of a pseudo-Euclidean space \( E^m \) were completely classified in [3]. They are minimal submanifolds of \( E^m \), minimal submanifolds of a pseudo-Riemannian sphere in \( E^m \) or minimal submanifolds of a pseudo-hyperbolic space in \( E^m \).

For a null 2-type submanifold \( M \) of \( E^m \), using \( \Delta x = -nH \) the definition (1.2) implies

(3.3) 

\[ \Delta H = \lambda H. \]
Lemma 3.3. Let $M$ be an $n$-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidean space $E^n_m$. Then, there is a constant $\lambda \neq 0$ such that $\Delta H = \lambda H$ holds if and only if $M$ is either of 1-type or of null 2-type.

If the mean curvature vector $H$ is non-null, that is, $\langle H, H \rangle \neq 0$, then there is an orthonormal normal frame $e_{n+1}, \ldots, e_m$ such that $H = \alpha e_{n+1}$, where $\alpha^2 = \varepsilon_{n+1}(H, H)$.

Lemma 3.4. Let $M$ be an $n$-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidean space $E^n_m$. If $M$ is not of 1-type, then $M$ is of null 2-type if and only if

$$\text{tr}(\overline{\nabla} A_H) = \text{tr}(\nabla A_H) + \text{tr}(A_{\nabla^\perp} H) = 0$$

and

$$\Delta^\perp H + \sum_{i=1}^n \varepsilon_i h(A_H e_i, e_i) = \lambda H,$$

for some nonzero constant $\lambda$.

From the definition of $\Delta^\perp H$ we have

$$\Delta^\perp H = (\Delta \alpha + \sum_{\nu=n+2}^m \sum_{i=1}^n \varepsilon_i \varepsilon_\nu \varepsilon_{n+1} \alpha (\omega^\nu_{n+1}(e_i))^2) e_{n+1}$$

$$- \sum_{\nu=n+2}^m \{2\omega^\nu_{n+1}(\nabla \alpha) + \alpha \text{tr}(\nabla \omega^\nu_{n+1}) + \sum_{i=1}^n \sum_{\beta=n+2}^m \alpha \varepsilon_i \omega^\beta_{n+1}(e_i) \omega^\nu_{n+1}(e_i)\} e_\nu,$$

where $\nabla \alpha = \sum_{i=1}^n \varepsilon_i (e_i \alpha) e_i$ and $\text{tr}(\nabla \omega^\beta_{n+1}) = \sum_{i=1}^n \varepsilon_i (\nabla e_i \omega^\beta_{n+1})(e_i)$.

Lemma 3.5. Let $M$ be an $n$-dimensional pseudo-Riemannian submanifold of $E^n_m$. If $M$ is not of 1-type and $H = \alpha e_{n+1}$ is non-null, then $M$ is of null 2-type if and only if we have

$$\text{tr}(\nabla A_H) = \frac{n}{2} \nabla \langle H, H \rangle + 2 \text{tr}(A_{\nabla^\perp} H) = 0$$

$$\Delta \alpha = \lambda \alpha - \alpha \varepsilon_{n+1} \|A_{n+1}\|^2 - \alpha \sum_{i=1}^n \sum_{\nu=n+2}^m \varepsilon_i \varepsilon_\nu \varepsilon_{n+1} (\omega^\nu_{n+1}(e_i))^2,$$

$$\varepsilon_\beta \text{tr}(A_H A_\beta) = 2\omega^\beta_{n+1}(\nabla \alpha) + \alpha \text{tr}(\nabla \omega^\beta_{n+1}) + \alpha \sum_{i=1}^n \sum_{\nu=n+2}^m \varepsilon_i \omega^\nu_{n+1}(e_i) \omega^\beta_{n+1}(e_i),$$

where $\lambda$ is a nonzero constant, $\beta = n+2, \ldots, m$ and $\|A_{n+1}\|^2 = \sum_{i=1}^n \varepsilon_i \langle A_{n+1} e_i, A_{n+1} e_i \rangle$. 


By direct calculation the equation (3.7) becomes
\[
\text{tr}(\bar{\nabla} A_H) = \frac{n}{2} \varepsilon_{n+1} \nabla(\alpha^2) + 2A_{n+1}(\nabla \alpha) + 2\alpha \sum_{i=1}^{n} \sum_{\nu=n+2}^{m} \varepsilon_i \omega_{n+1}(e_i) A_{e_v}(e_i) = 0.
\]

Using this equation we can have the following corollary from Lemma 3.5.

**Corollary 3.1.** Let \( M \) be an \( n \)-dimensional pseudo-Riemannian submanifold of \( E^{n+2}_t \). If \( M \) is not of \( 1 \)-type, \( H = \alpha e_{n+1} \) is non-null and the normalized mean curvature vector, \( e_{n+1} \), is parallel, then \( M \) is of null 2-type if and only if we have
\[
A_{n+1}(\nabla \alpha^2) + \frac{n\alpha \varepsilon_{n+1}}{2} \nabla(\alpha^2) = 0,
\]
\[
\Delta \alpha = \lambda \alpha - \alpha \varepsilon_{n+1} \| A_{n+1} \|^2,
\]
\[
\text{tr}(A_{n+1}A_{n+2}) = 0,
\]
where \( \lambda \) is a nonzero constant.

In [9, 10], the following theorems are given on null 2-type hypersurfaces of Euclidean spaces and null 2-type space-like hypersurfaces of Lorentzian space.

**Theorem 3.1.** ([9]) Let \( M \) be a Euclidean hypersurface with at most two distinct principal curvature. Then, \( M \) is of null 2-type if and only if it is locally isometric to \( E^p \times S^{n-p}(a) \).

**Theorem 3.2.** ([10]) Let \( M^n \) be a space-like hypersurface of the Lorentzian spaces \( E^1_t \) with at most two distinct principal curvature. Then, \( M^n \) is of null 2-type if and only if it is locally isometric to \( E^p \times H^{n-p}(a) \).

## 4 Null 2-type space-like submanifolds of \( E^5_t \)

We prove the followings.

**Proposition 4.1.** Let \( M \) be a 3-dimensional space-like submanifold of the pseudo-Euclidean space \( E^5_t \) with parallel normalized mean curvature vector such that \( M \) is not of 1-type. If \( M \) is of null 2-type with the Weingarten map in the direction of the mean curvature vector \( H \) has two distinct eigenvalues, then the mean curvature \( \alpha \) is constant on \( M \).

**Proof.** As the codimension is 2 and the normalized mean curvature vector, \( e_4 = H/\alpha, \alpha^2 = \varepsilon_4(H,H) \), is parallel, then the unit normal vector \( e_5 \) is also parallel. Therefore the normal space is flat, i.e., \( \omega^4_i = 0 \) on \( M \). Hence we can have the diagonalized Weingarten maps in the direction \( e_4 \) and \( e_5 \). Since \( A_4 \) has two distinct eigenvalues, say, \( h_{11}^4 \neq h_{22}^4 = h_{33}^4 \). We can write
\[
A_4 = \text{diag}(h_{11}^4, h_{44}^4, h_{33}^4) \quad \text{and} \quad A_5 = \text{diag}(h_{11}^5, h_{22}^5, h_{33}^5).
\]
with \( h_{11}^5 + h_{22}^5 + h_{33}^5 = 0 \). However, from (3.12) we get
\[
\text{tr}(A_4 A_5) = (h_{11}^4 - h_{22}^4) h_{11}^5 = 0
\]
because of \( h_{11}^5 + h_{22}^5 + h_{33}^5 = 0 \). As \( h_{11}^4 - h_{22}^4 \neq 0 \) we have \( h_{11}^5 = 0 \) and \( h_{22}^5 = -h_{33}^5 \).

Assume that \( \alpha \) is not constant. Let \( V = \{ p \in M : \nabla \alpha^2(p) \neq 0 \} \) which is open in \( M \). From (3.10) it is seen that the vector \( \nabla \alpha^2 \) is an eigenvector of \( A_4 \) corresponding to the eigenvalue \( -\frac{3 \alpha \epsilon_4^4}{2} \). Then we may say that \( \nabla \alpha^2 \) is parallel to \( e_1 \) or \( e_3 \) (the same as \( e_2 \)). For the last case it could also be proved that the mean curvature \( \alpha \) is constant by using the same way as in the first case. Thus \( h_{11}^4 = -\frac{3 \alpha \epsilon_4^4}{2} \) and \( h_{22}^4 = h_{33}^4 = \frac{9 \alpha \epsilon_4^4}{4} \) because of \( 3 \alpha \epsilon_4 = h_{11}^4 + 2 h_{22}^4 \). Then we have
\[
\omega_1^4 = -\frac{3 \alpha \epsilon_4^4}{2} \omega_1, \quad \omega_2^4 = \frac{9 \alpha \epsilon_4^4}{4} \omega_2, \quad \omega_3^4 = \frac{9 \alpha \epsilon_4^4}{4} \omega_3.
\]
Since \( \nabla \alpha^2 \) is parallel to \( e_1 \) we can have \( e_2(\alpha) = e_3(\alpha) = 0 \), that is, \( e_2(h_{11}^4) = e_3(h_{11}^4) = 0 \), and
\[
d\alpha = e_1(\alpha) \omega^1.
\]
However, by using the equation of Codazzi (2.11) for \( \nu = 4 \) if \( i = 1 \) we have
\[
\omega_1^2(e_1) = \omega_1^3(e_1) = 0,
\]
because of \( h_{11}^4 - h_{22}^4 \neq 0 \), and if \( j = 1 \), considering \( h_{22}^4 = h_{33}^4 = \frac{9 \alpha \epsilon_4^4}{4} \), then we obtain
\[
\omega_2^1(e_3) = \omega_3^1(e_3) = \frac{3 \epsilon_1(\alpha)}{5 \alpha}.
\]
Also, the equation of Codazzi (2.12) for \( \nu = 4 \) and \( j = 1 \) implies that
\[
\omega_3^2(e_3) = \omega_3^3(e_2) = 0.
\]
Applying the structure equations and using (4.6), it can be shown that \( d\omega^1 = 0 \).

Hence we have locally
\[
\omega^1 = du,
\]
where \( u \) is a local coordinate on \( U \). From (4.3) and (4.7) we have \( d\alpha \wedge du = 0 \). This shows that \( \alpha \) is a function of \( u \), i.e., \( \alpha = \alpha(u) \) and \( d\alpha = \alpha'(u) du \). Thus, by (4.5) we have
\[
\omega_2^1(e_2) = \omega_3^1(e_3) = \frac{3 \alpha'}{5 \alpha}.
\]
Considering (4.4) and (4.6), from the equation of Gauss (2.9) for \( i = \ell = 1, j = k = 2 \) we get
\[
e_1(\omega_2^1(e_2)) = (\omega_2^1(e_2))^2 + \epsilon_4 h_{11}^4 h_{22}^4.
\]
Using (4.8), the equation (4.9) turns into
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(4.10) \[ 40\alpha'' - 64(\alpha')^2 + 225\varepsilon_4\alpha^4 = 0. \]

Let \( y = (\alpha')^2 \). Then the above equation can be reduced to the following first order differential equation:

(4.11) \[ 2\alpha y' - 64y + 225\varepsilon_4\alpha^4 = 0, \]

where \( y' \) denotes the first derivative of \( y \) with respect to \( \alpha \). For this equation we obtain the solution

(4.12) \[ (\alpha')^2 = C\alpha^{16/5} - \varepsilon_4 \left( \frac{225}{16} \right)^2 \alpha^4, \]

where \( C \) is a constant.

When we use the definition of \( \Delta \alpha \), the fact that \( \nabla \alpha^2 \) is parallel to \( e_1 \) and the equation (4.8) we obtain

(4.13) \[ \Delta \alpha = 6(\alpha')^2 - \alpha''. \]

Also, since \( \|A_4\|^2 = \frac{99\alpha^2}{8} \), considering (4.13) and the second equation (3.11) of Corollary 3.1 we get

(4.14) \[ 40\alpha\alpha'' - 48(\alpha')^2 + 40\lambda\alpha^2 - 495\varepsilon_4\alpha^4 = 0. \]

Combining (4.10) and (4.14) we obtain

(4.15) \[ (\alpha')^2 = 45\varepsilon_4\alpha^4 - \frac{5}{2}\lambda\alpha^2. \]

As a result, using (4.12) and (4.15) we deduce that \( \alpha \) is locally constant on \( V \) which is a contradiction with the definition of \( M \). Therefore \( \alpha \) is constant on \( M \). \( \square \)

Let \( H^1(a) \times H^1(a) \times E = \{(x_1, x_2, \ldots, x_5) : -x_1^2 + x_3^2 = -a^2, -x_2^2 + x_4^2 = -a^2\} \). For later use we need the connection forms \( \omega^B_A \) of \( H^1(a) \times H^1(a) \times E \in \mathbb{E}_5^\lambda \). By a suitable choice of the Euclidean coordinates, its equation takes the following form

\[ x(u_1, u_2, u_3) = (a \cosh u_2, a \cosh u_3, a \sinh u_2, a \sinh u_3, u_1), \]

where \( a \) is a nonzero constant. If we put

\[ e_1 = \frac{\partial}{\partial u_1} = (0, 0, 0, 0, 1), \quad e_2 = \frac{1}{a} \frac{\partial}{\partial u_2} = (\sinh u_2, 0, \cosh u_2, 0, 0), \]

\[ e_3 = \frac{1}{a} \frac{\partial}{\partial u_2} = (0, \sinh u_3, 0, \cosh u_3, 0), \]

\[ e_4 = \frac{1}{\sqrt{2}}(\cosh u_2, \cosh u_3, \sinh u_2, \sinh u_3, 0), \]

\[ e_5 = \frac{1}{\sqrt{2}}(\cosh u_2, -\cosh u_3, \sinh u_2, -\sinh u_3, 0), \]
then, by a straightforward calculation we obtain
\[ \omega^1 = du_1, \quad \omega^2 = adu_2, \quad \omega^3 = adu_3, \quad \omega^4 = \omega^1 = \omega^2 = \omega^3 = \omega^4 = \omega^5 = 0, \]
(4.16) \[ \omega^2_2 = -\frac{1}{a\sqrt{2}}\omega^2, \quad \omega^3_3 = -\frac{1}{a\sqrt{2}}\omega^3, \quad \omega^5_5 = \frac{1}{a\sqrt{2}}\omega^3. \]

**Theorem 4.1.** Let \( M \) be a 3-dimensional space-like submanifold of the pseudo-Euclidean space \( E^7 \) with parallel normalized non-null mean curvature vector such that \( M \) is not of 1-type. Then \( M \) is of null 2-type having two distinct principal curvatures in the mean curvature direction and having a constant scalar curvature \( \tau \) if and only if \( M \) is locally isometric to one of the following:

1. \( S^1(a) \times E^2 \subset E^4 \subset E^5_1 \) or \( S^2(a) \times E \subset E^4 \subset E^5_1 \) when \( H \) is space-like,
2. \( H^1(a) \times E^2 \subset E^4_1 \subset E^5_2 \) or \( H^2(a) \times E \subset E^4_1 \subset E^5_2 \) when \( H \) is time-like, or
3. \( H^1(a) \times E^2 \subset E^4_2 \), \( H^2(a) \times E \subset E^4_1 \subset E^5_2 \), or \( H^1(a) \times H^1(a) \times E \subset E^5_2 \).

**Proof.** As the codimension is 2 and the normalized mean curvature vector, \( e_4 = H/\alpha \), is parallel, then the normal space is flat. Let \( M \) be of null 2-type and let the Weingarten map in the direction \( H \) has two distinct principal curvatures. Then the mean curvature \( \alpha \) on \( M \) is constant by Proposition 4.1. However, as in the proof of Proposition 4.1 we can have
\[ A_4 = \text{diag}(h^4_{11}, h^4_{22}, h^4_{22}) \quad \text{and} \quad A_5 = \text{diag}(0, h^5_{22}, -h^5_{22}). \]

By using (3.11) we have \( \|A_4\|^2 = (h^4_{11})^2 + 2(h^4_{11})^2 = \lambda \) which is constant. Hence, as \( \alpha \) is constant, it is easily seen that the eigenvalues \( h^4_{11} \) and \( h^4_{22} \) of \( A_4 \) are constant. Since the scalar curvature and the eigenvalues of \( A_4 \) are constant, by using (2.4) and (2.5) we obtain \( h^2_{22} = \text{const}. \)

Using the fact that \( h^4_{11} \neq h^4_{22} = h^4_{33}, \) \( h^5_{11} = 0, \) \( h^5_{22} = -h^5_{33} \) and all \( h^\nu_\ell \)'s are constant, from the equations of Codazzi (2.11) and (2.12) for \( \nu = 4 \) we obtain
\[ \omega^1_j(e_i) = 0, \quad i = 1, 2, 3, \quad j = 2, 3 \]
(4.17)
and for \( \nu = 5 \) from (2.12) we get
\[ h^5_{22} \omega^2_2(e_i) = 0, \quad i = 1, 2, 3. \]
(4.18)
However, by using the equations of Gauss (2.9), for \( i = \ell = 1, j = k = 2 \) and for \( i = \ell = 2, j = k = 3 \), we obtain, respectively,
\[ h^4_{11} h^4_{22} = 0, \]
(4.19)
and
\[ e_2(\omega^2_2(e_3)) - e_3(\omega^2_2(e_2)) = (\omega^2_2(e_2))^2 + (\omega^2_2(e_3))^2 + \varepsilon_4(h^4_{22})^2 - \varepsilon_5(h^5_{33})^2. \]
(4.20)
Since \( A_4 \) has two distinct eigenvalues, one of \( h^4_{11} \) and \( h^4_{22} \) is a non-zero constant. Considering the equations (4.18), (4.19) and (4.20) we have the following classifications.
Let $t = 1$, that is, $\varepsilon_5 = -1$.

**Case 1.** $h_{11} = 0$ and $h_{22} = 0$. Then, by (4.18) we get $h_{32} = 0$ or $\omega_3(e_i) = 0$, $i = 1, 2, 3$. Using the second part, the equation (4.20) implies that $h_{32} = 0$. Thus, $A_5$ vanishes. Since the normal space is flat and $A_5 \equiv 0$, then $M$ is contained in a hyperplane $P$ of $E_5^0$.

If $H$ is space-like, then $P$ is a space-like hyperplane of $E_1^5$. Therefore, by Theorem 3.1 $M$ is locally isometric to the circular cylinder $S^1(a) \times E^2 \subset E^4 \subset E_1^5$.

If $H$ is time-like, then $P$ is a Lorentzian hyperplane of $E_1^5$. Therefore, by Theorem 3.2 $M$ is locally isometric to the hyperbolic cylinder $H^1(a) \times E^2 \subset E_1^4 \subset E_1^5$.

**Case 2.** $h_{11} = 0$ and $h_{33} \neq 0$. Then, by (4.18) we have $h_{32} = 0$ or $\omega_3(e_i) = 0$, $i = 1, 2, 3$. Suppose that $\omega_3(e_i) = 0$ for $i = 1, 3$. Then, from (4.20) we get $\varepsilon_4(h_{32}^2 - 2\varepsilon_3(h_{33}^2) = 0$ which implies that $h_{32} = h_{33} = 0$ as $\varepsilon_4\varepsilon_5 = -1$. This is a contradiction because $h_{32} \neq 0$. Therefore $\omega_3(e_i) \neq 0$ at least for one $i \in \{1, 2, 3\}$ and $h_{33} = 0$, and hence $A_5$ vanishes on $M$. Considering that the normal space is flat, $M$ lies in a hyperplane $P$ of $E_1^5$.

If $H$ is space-like, then $P$ is a space-like hyperplane of $E_1^5$. Therefore, by Theorem 3.1 $M$ is locally isometric to $S^2(a) \times E^1 \subset E_1^4 \subset E_1^5$.

If $H$ is time-like, then $P$ is a Lorentzian hyperplane of $E_1^5$. Therefore, by Theorem 3.2 $M$ is locally isometric to $H^2(a) \times E^1 \subset E_1^4 \subset E_1^5$.

Let $t = 2$, that is, $\varepsilon_4\varepsilon_5 = 1$. Then the normal space is time-like.

**Case 3.** $h_{11} = 0$ and $h_{33} = 0$. Then, by (4.18) we have $h_{32} = 0$ or $\omega_3(e_i) = 0$, $i = 1, 2, 3$. Using the second part, the equation (4.20) implies that $h_{32} = 0$. Therefore $A_5$ vanishes. Since the normal space is flat and $A_5 \equiv 0$, then $M$ is contained in a Lorentzian hyperplane $P$ of $E_2^5$. Therefore, by Theorem 3.2 $M$ is locally isometric to $H^1(a) \times E^2 \subset E_1^4 \subset E_1^5$.

**Case 4.** $h_{11} = 0$ and $h_{33} \neq 0$. Then, by (4.18) we get $h_{32} = 0$ or $\omega_3(e_i) = 0$, $i = 1, 2, 3$.

**Subcase 4-a.** $\omega_3(e_i) \neq 0$ for at least one $i \in \{1, 2, 3\}$ and $h_{33} = 0$. Hence, we have $A_3 = \text{diag}(0, h_{32}, h_{33})$ and $A_5 \equiv 0$. Considering that the normal space is flat, $M$ lies in a Lorentzian hyperplane $P$ of $E_2^5$. Therefore, $M$ is locally isometric to $H^2(a) \times E \subset E_1^4 \subset E_1^5$ by Theorem 3.2.

**Subcase 4-b.** $h_{32} \neq 0$ and $\omega_3(e_i) = 0$, $i = 1, 2, 3$. From (4.20) we get $\varepsilon_4(h_{32}^2 - 2\varepsilon_3(h_{33}^2) = 0$ which implies that $h_{32} = 3\varepsilon_3(h_{33}^2) = 0$ as $\varepsilon_4\varepsilon_5 = 1$. Putting $\mu_0 = h_{32} = -3\varepsilon_3$, we have $A_4 = \text{diag}(0, \mu_0, \mu_0)$ and $A_5 = \text{diag}(0, \mp\mu_0, \pm\mu_0)$. Considering $\omega_3(e_i) = 0$, $i = 1, 2, 3$ (4.17) it is seen that $M$ is flat. Also, we can write

$$
\omega_1^4 = 0, \quad \omega_2^4 = \mu_0 \omega_3, \quad \omega_3^4 = \mu_0 \omega_3, \quad \omega_4^5 = 0, \quad \omega_5^5 = \pm\mu_0 \omega_3
$$

Since $M$ has a flat normal connection it is seen that the connection forms $\omega^4_i$ coincide with the connection forms of $H^1(a) \times H^1(a) \times E$ given in (4.16). Therefore, from the fundamental theorem of submanifolds, $M$ is locally isometric to $H^1(a) \times H^1(a) \times E \subset E_2^5$.

The converses of all these cases are trivial.

**Remark:** The cases (1) and (2) show that in the case $t = 1$ there is no such a submanifold that lies fully in $E_1^5$. □
References


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