Conformal vector fields on tangent bundle of Finsler manifolds

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Abstract. Let \((M, g)\) be a Finsler manifold, \(TM\) its tangent bundle and \(\tilde{g}\) a Riemannian metric on \(TM\) derived from \(g\). Then every complete lift conformal vector field on \(M\) is homothetic.

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Introduction.

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold and \(\phi\) a transformation on \(M\). Then \(\phi\) is called a conformal transformation, if it preserves the angles. Let \(V\) be a vector field on \(M\) and \(\{\varphi_t\}\) be the local one-parameter group of local transformations on \(M\) generated by \(V\). Then \(V\) is called a conformal vector field on \(M\) if each \(\varphi_t\) is a local conformal transformation of \(M\). It is well known that \(V\) is a conformal vector field on \(M\) if and only if there is a scalar function \(\rho\) on \(M\) such that \(\mathcal{L}_V g = 2\rho g\) where \(\mathcal{L}_V\) denotes Lie derivation with respect to the vector field \(V\). Specially \(V\) is called homothetic if \(\rho\) is constant and it is called an isometry or Killing vector field when \(\rho\) vanishes.

There are some lift metrics on \(TM = \bigcup_{x \in M} T_x M\) as follows: complete lift metric or \(g_2\), diagonal lift metric or \(g_1 + g_3\), lift metric \(g_2 + g_3\) and lift metric \(g_1 + g_2\), where \(g_1 := g_{ij} dx^i \otimes dx^j\), \(g_2 := 2g_{ij} dx^i \otimes \delta y^j\) and \(g_3 := g_{ij} \delta y^i \otimes \delta y^j\) are all bilinear differential forms defined globally on \(TM\).

In the study of Finsler geometry the complete lift vector fields have a great significance. More precisely let \(V\) be a vector field on the Finsler manifold \((M, g(x, y))\) and \(X^c\) be the complete lift of \(V\). Then \(V\) is called a conformal vector field of Finsler manifold \((M, g)\) if there is a scalar function\(^1\) \(\Omega\) on \(TM\) which satisfies \(\mathcal{L}_{X^c} g = 2\Omega g\).

\(^1\)By a simple calculation and vertical partial derivative using commutative property of Lie derivative one can show that \(\Omega\) is a function of \(x\) alone [1].
For the complete lift vector fields the following results are well known:

**Theorem A.** [9]: Let $(M, g)$ be a Riemannian manifold, $X$ a vector field on $M$ and $X^C$ complete lifts of $X$ to $TM$. If we consider $TM$ with metric $g_2$ then $X^C$ is a conformal vector field on $TM$ if and only if $X$ is homothetic on $M$.

**Theorem B.** [10]: Let $(M, g)$ be a Riemannian manifold. If we consider $TM$ with metric $g_1 + g_3$ then $X^C$ is a conformal vector field on $TM$ if and only if $X$ is homothetic.

In a recent work we introduced a new Riemannian and pseudo-Riemannian lift metrics on $TM$, $	ilde{g} = a g_1 + b g_2 + c g_3$ where $a$, $b$ and $c$ are certain constant real numbers. That is a combination of diagonal lift, and complete lift metrics, which is in some senses more general than those who are used previously. We have replaced the cited lift metrics in Theorems A and B by $	ilde{g}$. More precisely, we have proved Theorem C in [3] as follows.

**Theorem C.** Let $M$ be an $n$-dimensional Riemannian manifold and let $TM$ be its tangent bundle with metric $\tilde{g}$. Then every complete lift conformal vector field on $TM$ is homothetic.

In the present work we replace the Riemannian metric on $M$ by a Finsler metric endowed with a Cartan connection and prove the following theorem.

**Theorem 1:** Let $(M, g)$ be a $C^\infty$ connected Finsler manifold, $TM$ its tangent bundle and $\tilde{g}$ the Riemannian (or Pseudo-Riemannian) metric on TM derived from $g$. Then every complete lift conformal vector field on $TM$ is homothetic.

1 Preliminaries.

Let $M$ be a real $n$-dimensional manifold of class $C^\infty$. We denote by $TM \rightarrow M$ the bundle of tangent vectors and by $\pi : TM_0 \rightarrow M$ the fiber bundle of non-zero tangent vectors. A Finsler structure on $M$ is a function $F : TM \rightarrow [0, \infty)$, with the following properties: (I) $F$ is differentiable ($C^\infty$) on $TM_0$; (II) $F$ is positively homogeneous of degree one in $y$, i.e. $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$, where we denote an element of $TM$ by $(x, y)$. (III) The Hessian matrix of $F^2$ is positive definite on $TM_0$.

$\langle g_{ij} \rangle := \left( \frac{1}{2} \left[ \frac{\partial^2}{\partial y_i \partial y_j} F^2 \right] \right)$. A Finsler manifold is a pair of a differentiable manifold $M$ and a Finsler structure $F$ on $M$. The tensor field $g = (g_{ij})$ is called the Fundamental Finsler tensor or Finsler metric tensor. Here, we denote a Finsler manifold by $(M, g)$.

Let $V_v TM = \ker \pi^*_v$ be the set of the vectors tangent to the fiber through $v \in TM_0$. Then a vertical vector bundle on $M$ is defined by $VTM := \bigcup_{v \in TM_0} V_v TM$. A nonlinear connection or a horizontal distribution on $TM_0$ is a complementary distribution $HTM$ for $VTM$ on $TTM_0$. Therefore we have the decomposition

\[ TTM_0 = VTM \oplus HTM. \]
HTM is a vector bundle completely determined by the non-linear differentiable functions $N^j_i(x, y)$ on $TM$, called coefficients of the non-linear connection. Let $HTM$ be a non-linear connection on $TM$ and $\nabla$ a linear connection on $VTM$, then the pair $(HTM, \nabla)$ is called a Finsler connection on the manifold $M$.

Using the local coordinates $(x^i, y^i)$ on $TM$ we have the local field of frames $\{\partial / \partial x^i, \partial / \partial y^i\}$ on $TTM$. It is well known that we can choose a local field of frames $\{\delta / \partial x^i, \partial / \partial y^i\}$ adapted to the above decomposition i.e. $\delta / \partial x^i \in \Gamma(HTM)$ and $\partial / \partial y^i \in \Gamma(VTM)$ set of vector fields on $HTM$ and $VTM$, where

$$ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, $$

and where we use the Einstein summation convention.

Here, in this paper, all manifolds are supposed to be connected.

Let $(M, g(x, y))$ be a Finsler manifold then a Finsler connection is called a metric Finsler connection if $g$ is parallel with respect to $\nabla$. According to the Miron terminology this means that $g$ is both horizontally and vertically metric. The Cartan connection is a metric Finsler connection for which the Deflection, horizontal and vertical torsion tensor fields vanishes.

Let $(M, g(x, y))$ be a Finsler manifold with metric Finsler connection the Curvature tensors of $M$ are defined by

$$ R(X, Y)Z = \{[\nabla X, \nabla Y] - \nabla [X, Y]\} Z, $$

where $X, Y, Z \in \mathcal{X}(TM)$

They are called accordingly to the choice of $X$ and $Y$ in $HTM$ or $VTM$ horizontal or vertical curvature tensors of Finsler manifold.

Let $M$ be a Finsler manifold and $\nabla$ a Finsler connection on $M$, then we have [6]

$$ R^h_{ji} = \delta_i F^h_{j} - \delta_j F^h_{i} + F^m_{j} F^h_{m i} - F^m_{i} F^h_{m j} + C^h_{k m} R^m_{j i}, $$

$$ R^h_{ij} = \delta_j N^h_{i} - \delta_i N^h_{j}, $$

where we have put $\delta_i = \frac{\partial}{\partial x^i}, \delta_i = \frac{\partial}{\partial y^i}, \delta_i = \partial_i - N^m_i \hat{\partial}_m$. If $\nabla$ is a Cartan connection then $N^h_i = y^m F^h_{m i}$.

**Proposition 1.** [5] Let $M$ be an $n$-dimensional Finsler space with a Cartan connection, then we have the following equations

1. $F^h_{ij} = \frac{1}{2} g^{hm} (\delta_i g_{mj} + \delta_j g_{im} - \delta_m g_{ij}).$

2. $C_{ijk} = \hat{\partial}_k g_{ij}$ where $C_{ijk} = C^m_{i k m}.$

3. $y^m C_{mi j} = 0.$

4. $R^h_{ij} = y^m R^m_{hij}$.

The Cartan horizontal and vertical covariant derivative of a tensor field of type $(1)$ are given locally as follows:
\[
\n\nabla_j T^j_k i := T^j_k i j = \delta_j T^j_k i + F^h_j T^m_k i - F^m_j T^h_k i - F^m_j T^h_k m.
\]

\[
\nabla_j T^j_k i := T^j_k i j = \partial_j T^j_k i + C^h_j T^m_k i - C^m_j T^h_k i - C^m_j T^h_k m.
\]

2 Lift metrics and conformal vector fields.

2.1 Complete lift vector fields and Lie derivative.

Let \( V = v^i \frac{\partial}{\partial x^i} \) be a vector field on \( M \). Then \( V \) induces an infinitesimal point transformation on \( M \). This is naturally extended to a point transformation of the tangent bundle \( TM \) which is called extended point transformation. Let \( V \) be a vector field on \( M \) and \( \Phi_t \) be the extended point transformation of \( \Phi_t \) and \( \{ \Phi_t \} \) be the local one-parameter groups of \( TM \). If \( X^c \) is a vector field on \( TM \) induced by \( \{ \tilde{\Phi_t} \} \) it is called the complete lift vector field of \( V \).

It can be shown that the extended point transformation is a transformation induced by the complete lift vector field of \( V \), \( X^c = v^i \delta_j \partial_i + y^j \partial_i \) with respect to the decomposition (1.1.1).

Let \( M \) be an \( n \)-dimensional manifold, \( V \) a vector field on \( M \) and \( \{ \phi_t \} \) a 1-parameter local group of local transformations of \( M \) generated by \( V \). Take any tensor field \( S \) on \( M \), and denote by \( \phi_t^*(S) \) the pulled back of \( S \) by \( \phi_t \). Then the Lie derivation of \( S \) with respect to \( V \) is a tensor field \( \mathcal{L}_V S \) on \( M \) defined by:

\[
\mathcal{L}_V S = \frac{\partial}{\partial t} \phi_t^*(S)|_{t=0} = \frac{\phi_t^*(S) - (S)}{t},
\]

on the domain of \( \phi_t \). The mapping \( \mathcal{L}_V \) which map \( S \) to \( \mathcal{L}_V(S) \) is called the Lie derivation with respect to \( V \).

In Finsler geometry the Lie derivative of an arbitrary tensor, \( T^k_{ij} \) is given locally by [Yan1]:

\[
\mathcal{L}_V T^k_{ij} = v^a \nabla_a T^k_{ij} + v^a \nabla_a v^b \nabla_b T^k_{ij} - T^k_{ia} \nabla_a v^i + T^k_{ai} \nabla_i v^a,
\]

or equivalently,

\[
\mathcal{L}_V T^j_k i = v^a \partial_a T^j_k i + y^a \partial_a v^b \partial_b T^j_k i - T^a_i \partial_a v^j + T^a_i \partial_i v^a.
\]

So we have

\[
\mathcal{L}_V y^i = v^a \partial_a y^i + y^b \partial_b v^i - y^a \partial_a v^i = y^b \partial_b v^i - y^a \partial_a v^i = 0,
\]
We have also this interchanging formula between Cartan covariant derivatives and Lie derivatives.

\[ \nabla_k \mathcal{L}_v g_{ij} - \mathcal{L}_v \nabla_k g_{ij} = g_{aj} \mathcal{L}_v F^a_{ik} + g_{ai} \mathcal{L}_v F^a_{jk}. \] (2.2.4)

### 2.2 A lift metric on tangent bundle.

Let \((M, g)\) be a Finsler manifold. In this section we define a new Riemannian or Pseudo-Riemannian metric on \(TM\) derived from the Finsler metric. This metric is in some senses more general than the other lift metrics defined previously on \(TM\). By mean of the dual basis \(\{dx^i, \delta y^i\}\) analogously to the Riemannian geometry the tensors; \(g_1 := g_{ij} dx^i \otimes dx^j\) \(g_2 := 2g_{ij} dx^i \otimes \delta y^j \) \(g_3 := g_{ij} \delta y^i \otimes \delta y^j\) are all quadratic differential tensors defined globally on \(TM\), see [9]. Now let’s consider the Finsler metric tensor \(\bar{g}\) with the components \(g_{ij}(x, y)\) defined on \(TM\). The tensor field \(\bar{g} = \alpha g_1 + \beta g_2 + \gamma g_3\) on \(TM\), where the coefficient \(\alpha, \beta, \gamma\) are real numbers, has the components

\[
\begin{pmatrix}
\alpha g & \beta g \\
\beta g & \gamma g
\end{pmatrix}
\]

with respect to the dual basis of \(TM\). From the linear algebra we have \(\det \bar{g} = (\alpha \gamma - \beta^2)^n \det g^2\). Therefore \(\bar{g}\) is nonsingular if \(\alpha \gamma - \beta^2 \neq 0\) and it is positive definite if \(\alpha, \gamma\) are positive and \(\alpha \gamma - \beta^2 > 0\). Indeed \(\bar{g}\) define respectively a Pseudo-Riemannian or a Riemannian lift metric on \(TM\).

**Definition 1.** Let \((M, g)\) be a Finsler manifold. Consider tensor field \(\bar{g} = \alpha g_1 + \beta g_2 + \gamma g_3\) on \(TM\), where the coefficient \(\alpha, \beta, \gamma\) are real numbers. If \(\alpha \gamma - \beta^2 \neq 0\) then \(\bar{g}\) is non-singular and it can be regarded as a Pseudo-Riemannian metric on \(TM\). If \(\alpha\) and \(\gamma\) are positive such that \(\alpha \gamma - \beta^2 > 0\) then \(\bar{g}\) is positive definite and consequently can be regarded as a Riemannian metric on \(TM\). \(\bar{g}\) is called the lift metric of \(g\) on \(TM\).

### 2.3 Conformal vector fields.

Let \((TM, G(x, y))\) be a Riemannian (or pseudo-Riemannian) manifold. A vector field \(\bar{X} \in \mathcal{X}(TM)\) on \(TM\) is called a conformal vector field on \(TM\) if there is a scalar function \(\Omega\) on \(TM\) such that

\[ \mathcal{L}_{\bar{X}} G = 2\Omega G. \]

If \(\Omega\) is constant then the vector field \(\bar{X}\) is called homothetic and if \(\Omega\) is zero then its called an isometric or a Killing vector field.

Now let we consider \((TM, \bar{g}(x, y))\) with the complete lift vector field \(X^c\) of an arbitrary vector field \(V\) on \(M\). Then by above definition we call \(X^c\) a conformal vector field on \(TM\) if

\[ \mathcal{L}_{X^c} \bar{g} = 2\bar{\Omega}. \]
3 Main results

Analogous to the Riemannian geometry [7], by straight forward calculation we have the following lemmas in Finsler geometry.

**Lemma 1.** Let \((M, g)\) be a Finsler manifold with Cartan connection, then we have;

1. \([X_i, X_j] = R^h_{ij}X_T\)
2. \([X_i, X_T^j] = \partial_jN^hX_T\)
3. \([X_T^i, X_T^j] = 0\).

where we put \(X_i = \delta_i\) and \(X_T = \partial_t\) for simplicity.

Let’s denote by \(\mathcal{L}_{X^c}\) the lie derivative with respect to the complete lift vector field \(X^c\). Then we obtain the following lemma:

**Lemma 2.** Let \((M, g)\) be a Finsler manifold with Cartan connection, then we have;

1. \(\mathcal{L}_{X^c}X_i = [X^c, X_i] = [v^hX_i + y^m\n^h|_mX_T^i, X_i] = v^h[X_i, X_i] - X_i(v^h)X_i + y^m\n^h|_m[X_T^i, X_i] - X_i(y^mv^h|_m)X_T^i = -\partial_i v^hX_i - \mathcal{L}_{X^c}N^hX_T^i\).

Thus we get (1). We can prove (2) by the same way as the proof of (1). Next we prove (3). Since \((dx^h, \delta y^h)\) is the dual basis of \((X_h, X_T^i)\), if we put

\(\mathcal{L}_{X^c}dx^h = \alpha_m^hdx^m + \beta_m^h\delta y^m\).

Then we have

\[0 = \mathcal{L}_{X^c}(dx^h(X_i)) = (\mathcal{L}_{X^c}dx^h)X_i + dx^h(\mathcal{L}_{X^c}X_i) = \alpha_i^h - \partial_i v^h,\]

and

\[0 = \mathcal{L}_{X^c}(dx^h(X_T^i)) = (\mathcal{L}_{X^c}dx^h)X_T^i + dx^h(\mathcal{L}_{X^c}X_T^i) = \beta_i^h.\]

Thus we get (3). By the same way as the proof of (3), we can prove (4).

**Lemma 3.** Let \((M, g)\) be a Finsler manifold with Cartan connection, then we have;

1. \(\mathcal{L}_{X^c}(g_{ij}dx^i dx^j) = (\mathcal{L}_{X^c}g_{ij})dx^i dx^j\)
2. \(\mathcal{L}_{X^c}(g_{ij}dx^i \delta y^j) = g_{mi}(\mathcal{L}_{X^c}N^m_j)dx^i dx^j + (\mathcal{L}_{X^c}g_{ij})dx^i \delta y^j\)
3. \(\mathcal{L}_{X^c}(g_{ij}dx^i \delta y^j) = 2(g_{mi}(\mathcal{L}_{X^c}N^m_j)dx^i \delta y^j + (\mathcal{L}_{X^c}g_{ij})\delta y^i\delta y^j\).

Proof. By mean of above lemma, we get

\(\mathcal{L}_{X^c}(g_{ij}dx^i dx^j) = X^c(g_{ij})dx^i dx^j + 2g_{ij}(\mathcal{L}_{X^c}dx^i)dx^j = (v^hX_i + y^m\n^h|_mX_T^i)(g_{ij})dx^i dx^j + 2g_{ij}(\partial_i v^h dx^m)dx^j = (\mathcal{L}_{X^c}g_{ij})dx^i dx^j\). Thus we have (1), (2) and (3) are easily proof by the same way as the proof of (1).
Theorem 1. Let \((M,g)\) be a \(C^\infty\) connected Finsler manifold, \(TM\) its tangent bundle and \(\tilde{g}\) the Riemannian (or Pseudo-Riemannian) metric on \(TM\) derived from \(g\). Then every complete lift conformal vector field on \(TM\) is homothetic.

Proof. Let \(V\) be a vector field on \(M\), \(X^c\) the complete lift vector field of \(V\) which is conformal and \(\tilde{g}\) be a Pseudo-Riemannian metric on \(TM\) derived from \(g\). We have by definition \(\mathcal{L}_{X^c}\tilde{g} = 2\Omega\tilde{g}\). The Lie derivative of \(\tilde{g}\) gives

\[
\mathcal{L}_{X^c}\tilde{g} = \alpha (\mathcal{L}_V g)_{ij} dx^i dx^j + 2\beta (\mathcal{L}_V g)_{ij} \delta y^i \delta y^j + 2\gamma (\mathcal{L}_V N^a_i N^a_j) dx^i dx^j \\
+ \gamma (\mathcal{L}_V g)_{ij} \delta y^i \delta y^j + 2\gamma g_{aj} (\mathcal{L}_V N^a_i) dx^i \delta y^j.
\]

(3.1)

So we have

\[
\mathcal{L}_{X^c}\tilde{g} = \left[\alpha (\mathcal{L}_V g)_{ij} + 2\beta g_{ai} (\mathcal{L}_V N^a_j) \right] dx^i dx^j \\
+ \left[\beta (\mathcal{L}_V g)_{ij} + 2\gamma g_{aj} (\mathcal{L}_V N^a_i) \right] dx^i \delta y^j \\
+ \left[\gamma (\mathcal{L}_V g)_{ij} \right] \delta y^i \delta y^j \\
= 2\Omega\tilde{g}.
\]

Comparing with the definition of \(\tilde{g}\), we find;

\[
\alpha (\mathcal{L}_V g)_{ij} + \beta g_{ai} (\mathcal{L}_V N^a_j) = 2\alpha \Omega g_{ij}.
\]

(3.2)

\[
\beta (\mathcal{L}_V g)_{ij} + \gamma g_{aj} (\mathcal{L}_V N^a_i) = 2\beta \Omega g_{ij}.
\]

(3.3)

\[
\gamma (\mathcal{L}_V g)_{ij} = 2\gamma \Omega g_{ij}.
\]

(3.4)

I) If \(\gamma \neq 0\) then from (3.4) we have

\[
(\mathcal{L}_V g)_{ij} = 2\Omega g_{ij},
\]

and from (3.3) we have

\[
\mathcal{L}_V N^a_i = 0.
\]

Using this and \(N^b_i = y^m F^b_{m, i}\) we get

\[
0 = \mathcal{L}_V N^b_i = \mathcal{L}_V (y^m F^b_{m, i}) = y^m \mathcal{L}_V F^b_{m, i}.
\]

(3.5)

Where the last equality holds from equation (2.2.2).

II) If \(\gamma = 0\) since \(\alpha \gamma - \beta^2 \neq 0\) we have \(\beta \neq 0\) so from (3.3) we have

\[
(\mathcal{L}_V g)_{ij} = 2\Omega g_{ij},
\]

and from (3.2) we have

\[
g_{ai} (\mathcal{L}_V N^a_j) + g_{aj} (\mathcal{L}_V N^a_i) = 0.
\]

Using this and equation (2.2.2) and \(N^a_i = y^k F^a_{k, i}\), we have

\[
y^k (g_{ai} (\mathcal{L}_V F^a_{k, j}) + g_{aj} (\mathcal{L}_V F^a_{k, i}) = 0.
\]

(3.6)

In each case I) and II) we have
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\( \mathcal{L}_v g_{ij} = 2\Omega g_{ij} \),

or from equation (1.6)

\[
v^m \partial_m g_{ij} + g_{mj} \partial_i v^m + g_{im} \partial_j v^m + y^a \partial_a v^m \partial_m g_{ij} = 2\Omega g_{ij}.
\]

Applying \( \partial_k \) to the both side of the above equation, we find;

\[
2v^m \partial_m C_{ijk} + 2C_{mjk} \partial_i v^m + 2C_{imk} \partial_j v^m + 2\partial_k v^m C_{ijm} + 2y^a \partial_a v^m \partial_k C_{ijm} = 2g_{ij} \partial_k \Omega + 4\Omega C_{ijk}.
\]

By using \( y^i C_{ijk} = 0 \), we obtain \( \partial_k \Omega = 0 \). Therefore \( \Omega \) is a function of \( x \) alone.

From (2.2.4) we have

\[
y^k (\nabla_k \mathcal{L}_v g_{ij} - \mathcal{L}_v \nabla_k g_{ij}) = y^k (g_{ai} \mathcal{L}_v F_{jk}^a + g_{aj} \mathcal{L}_v F_{ik}^a).
\]

By using (3.5), (3.6) and (3.7) in each case I) and II) we find that

\( y^k \nabla_k \Omega = 0 \).

Since \( \Omega \) is a function of \( x \) alone, we obtain \( \partial_i \Omega = 0 \). This together with connectedness of \( M \), shows that \( \Omega \) is constant.

References


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