Determination of metrics by boundary energy

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Abstract. This paper reformulates a problem of Sharafutdinov [4] and extend the new variant from the single-time context to the multi-time context.

Section 1 is dedicated to the single-time case. It starts with well-known facts of describing geodesics as extremals. Then it is formulated and studied the problem of determination of a metric by the boundary energy. The linearization of this problem leads to the ray transform of a tensor field and to moment problem.

Section 2 extend the single-time case to the multi-time case. It begins with well-known facts about harmonic maps and continues with determining a pair of metrics from boundary energy. Using the linearization, we extend the idea to multi-ray transform of a distinguished tensor field (moment problem).

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1 Single-time Case

1.1 Geodesics

Let \((M, g)\) be a Riemannian manifold, \(\dim M = n\). Consider \((x^1, \ldots, x^n)\) the local coordinates and \(\Gamma^i_{jk}\) the Christoffel symbols of the second type.

**Definition 1.1** Let \(x: [0, 1] \to M, x(t) = (x^1(t), \ldots, x^n(t))\) be a curve on \(M\) joining the points \(x(0) = p\) and \(x(1) = q\) of \(M\). The integral

\[
E_g(x) = \frac{1}{2} \int_0^1 ||\dot{x}(t)||^2 dt = \frac{1}{2} \int_0^1 g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t) dt
\]

is called the energy of the curve \(x\).
Proposition 1.1  A minimum point of the energy functional $E_g$, with the boundary conditions $x(0) = p$ and $x(1) = q$, necessarily verifies the boundary value problem:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = 0, \quad i = 1, n, \quad x(0) = p, \quad x(1) = q,$$

where $L(x^i, \dot{x}^i) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$ is the Lagrangian (kinetic energy) determining the functional. 
Explicitly,

$$\ddot{x}^i + \Gamma_{jk}^i \ddot{x}^j \dot{x}^k = 0, \quad i = 1, n, \quad x(0) = p, \quad x(1) = q.$$

Proof. Let us refer to the second part of the Proposition. We have

$$\frac{\partial L}{\partial x^i} = \frac{1}{2} \frac{\partial}{\partial x^i} \left( g_{jk} \dot{x}^j \dot{x}^k \right) = \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \dot{x}^j \dot{x}^k$$
$$\frac{\partial L}{\partial \dot{x}^i} = \frac{1}{2} \frac{\partial}{\partial \dot{x}^i} \left( g_{jk} \dot{x}^j \dot{x}^k \right) = g_{ij} \dot{x}^j$$
$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = \frac{d}{dt} \left( g_{ij} \dot{x}^j \right) = \frac{\partial g_{ij}}{\partial x^k} \dot{x}^k \dot{x}^j + g_{ij} \ddot{x}^j.$$

Hence, the Euler-Lagrange equations become

$$\frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \dot{x}^j \dot{x}^k - \frac{\partial g_{ij}}{\partial x^k} \dot{x}^j \dot{x}^k - g_{ij} \ddot{x}^j = 0 \Leftrightarrow$$
$$g_{ij} \ddot{x}^j + \frac{1}{2} \left( 2 \frac{\partial g_{ij}}{\partial x^k} \dot{x}^j \dot{x}^k - \frac{\partial g_{jk}}{\partial x^i} \dot{x}^j \dot{x}^k \right) = 0 \Leftrightarrow$$
$$\ddot{x}^p + \frac{1}{2} g^{ip} \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) \dot{x}^j \dot{x}^k = 0 \Leftrightarrow$$
$$\ddot{x}^p + \Gamma_{jk}^p \ddot{x}^j \dot{x}^k = 0, \quad p = 1, n.$$

Remark 1.1 a) We suppose that the problem (1.1) stated in the previous proposition has a unique solution.

b) The equations

$$\ddot{x}^i + \Gamma_{jk}^i \ddot{x}^j \dot{x}^k = 0, \quad i = 1, n$$

of the extremals of the energy functional $E_g$ coincide with the equations of the geodesics of $(M, g)$.

1.2 Determination of a Metric by Boundary Energy

Definition 1.2 Let $(M, g)$ be a compact Riemannian manifold and $\partial M$ be the boundary of $M$. The Riemannian metric $g$ is called simple if any two points $p, q \in \partial M$ can be joined by a unique geodesic

$$x_{pq}: [0, 1] \to M, \quad \overline{M} = M \cup \partial M, \quad x(0) = p, \quad x(1) = q, \quad x(0, 1) \subset M.$$
Definition 1.3 Let \((M, g)\) be a simple Riemannian manifold. The function
\[ E_g: \partial M \times \partial M \to \mathbb{R}, \quad E_g(p, q) = E(x_{pq}), \]
where \(x_{pq}\) is the geodesic joining the points \(p, q\), \((x_{pq} \setminus \{p, q\}) \subset M \setminus \partial M\), is called the boundary energy produced by the metric \(g\).

The problem of existence of a simple metric \(g\) with the property that a given function \(E: \partial M \times \partial M \to \mathbb{R}\) represents the boundary energy attached to \(g\) cannot have a unique solution. To justify this statement, let \(\varphi: M \to M\) be a diffeomorphism of \(M\) such that \(\varphi|_{\partial M} = \text{id}\) and \(g^1 = \varphi^* g^0\). The diffeomorphism \(\varphi\) transforms the simple metric \(g^0\) into the simple metric \(g^1\). The relation
\[ g^1(x)(\xi, \eta) = g^0((d_x \varphi)\xi, (d_x \varphi)\eta), \]
where \(d_x \varphi: T_x M \to T_{\varphi(x)} M\) is the differential of \(\varphi\), implies that \(g^0\) and \(g^1\) have different families of geodesics, but the energy is the same.

Is the nonuniqueness of the proposed problem settled by the above-mentioned construction?

Problem 1. Let \(g^0\) and \(g^1\) be simple metrics on the manifold \(M\), with the boundary \(\partial M\), such that \(E_{g^0} = E_{g^1}\). Is there a diffeomorphism \(\varphi: M \to M\), such that \(\varphi|_{\partial M} = \text{id}\) and \(g^1 = \varphi^* g^0\)? (the problem of determination of a metric by its boundary energy).

1.3 Linearization of The Problem of Determination of a Metric by its Boundary Energy

Let us linearize the above-mentioned problem.

Let \((g^\tau)\) be a family of simple metrics on \(M\), depending smoothly on the parameter \(\tau \in (-\varepsilon, \varepsilon), \varepsilon > 0\). Let \(x^\tau: [0, 1] \to M\) be a geodesic joining the points \(p = x^\tau(0)\) and \(q = x^\tau(1)\). Consider \(x^\tau(t) = (x^1(t, \tau), \ldots, x^n(t, \tau))\) as the representation of \(x^\tau\) in a local coordinate system. Suppose that \(g^\tau = (g^\tau_{ij})\) and \(x^i(t, \tau), i, j = 1, n\), are \(C^\infty\) functions.

We start with the boundary energy
\[ E_{g^\tau}(p, q) = \frac{1}{2} \int_0^1 g^\tau_{ij}(x^\tau(t)) \dot{x}^i(t, \tau) \dot{x}^j(t, \tau) dt. \]

Differentiating with respect to \(\tau\) and then considering \(\tau = 0\), we have
\[
\left. \frac{\partial}{\partial \tau} \right|_{\tau=0} E_{g^\tau}(p, q) = \int_0^1 f_{ij}(x^0(t)) \dot{x}^i(t, 0) \dot{x}^j(t, 0) dt
+ \frac{1}{2} \int_0^1 \left[ \frac{\partial g^0_{ij}}{\partial x^k}(x^0(t)) \dot{x}^i(t, 0) \dot{x}^j(t, 0) \frac{\partial x^k}{\partial \tau}(t, 0) 
+ 2 g^0_{ij}(x^0(t)) \dot{x}^i(t, 0) \frac{\partial x^j}{\partial \tau}(t, 0) \right] dt,
\]
where

\[ f_{ij} = \frac{1}{2} \frac{\partial}{\partial \tau} \bigg|_{\tau = 0} g^g_{ij}. \]

Using an integration by parts, having in mind that \( \frac{\partial x_i}{\partial \tau}(0, 0) = \frac{\partial x_i}{\partial \tau}(1, 0) = 0 \) and \( x^0 \) is extremal of energy functional, we obtain

\[
\int_0^1 g^g_{ij}(x^0(t)) \dot{x}^i(t, 0) \dot{x}^j(t, 0) dt = \int_0^1 g^g_{ij}(x^0(t)) \dot{x}^i(t, 0) \frac{\partial}{\partial t} \left( \frac{\partial x^j}{\partial \tau} \right) (t, 0) dt
\]

\[ = g^g_{ij}(x^0(t)) \dot{x}^i(t, 0) \frac{\partial x^j}{\partial \tau} (t, 0) \bigg|_0^1 - \int_0^1 \frac{\partial}{\partial t} \left[ g^g_{ij}(x^0(t)) \dot{x}^i(t, 0) \right] \frac{\partial x^j}{\partial \tau} (t, 0) dt
\]

\[ = - \int_0^1 \left[ \frac{\partial g^g_{ij}}{\partial x^k}(x^0(t)) \dot{x}^k(t, 0) \dot{x}^i(t, 0) + g^g_{ij}(x^0(t)) \ddot{x}^i(t, 0) \right] \frac{\partial x^j}{\partial \tau} (t, 0) dt.
\]

The equations (1.2) can be written in the form

\[ \frac{\partial}{\partial \tau} \bigg|_{\tau = 0} \tau \ E^{g^g}_{g^g}(p, q) = \int_0^1 f_{ij}(x^0(t)) \dot{x}^i(t, 0) \dot{x}^j(t, 0) dt.
\]

If we denote

\[ I_f(x_{pq}) = \int_0^1 f_{ij}(x^0(t)) \dot{x}^i(t, 0) \dot{x}^j(t, 0) dt,
\]

the previous relation becomes

\[ \frac{\partial}{\partial \tau} \bigg|_{\tau = 0} \tau \ E^{g^g}_{g^g}(p, q) = I_f(x_{pq}),
\]

\( x_{pq} \) being a geodesic of the metric \( g^0 \).

In the particular case when the energy \( E^{g^g}_{g^g} \) does not depend on \( \tau \), the left side of the equality (1.6) is null.

From \( \varphi|_{\partial M} = \text{id} \) we obtain \( v|_{\partial M} = 0 \), so we have the following linearization of the problem 1: to what extent the family of integrals

\[ I_f(x_{pq}) = \int_{x_{pq}} f_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t) dt,
\]

counted after \( p, q \in \partial M \), determine the tensor \( f = (f_{ij}) \) over a Riemannian manifold \((M, g^0)\)?

The existence of the solutions of the stated problem for the family \((g^\tau)\) implies the existence of an one-parameter group of diffeomorphisms \( \varphi: M \to M \), such that \( \varphi^\tau|_{\partial M} = \text{id} \) and \( g^\tau = (\varphi^\tau)^* g^0 \), that is

\[ g^\tau_{ij} = (g^0_{kl} \circ \varphi^\tau) \frac{\partial x^k}{\partial x^i}(x, \tau) \frac{\partial x^l}{\partial x^j}(x, \tau),
\]
where \( \varphi^{\tau}(x) = (\varphi^{1}(x, \tau), \ldots, \varphi^{n}(x, \tau)) \) and \( x' = \varphi^{\tau}(x) \).

If we differentiate with respect to \( \tau \) and we make \( \tau = 0 \), we obtain

**Theorem 1.1** The relation (1.7) implies

\[
(1.8) \quad f_{ij} = \frac{1}{2}(v_{ij} + v_{ji}),
\]

\( v = (v^{i}) \) being the covariant vector field that generates the one-parameter group \( (\varphi^{\tau}) \) and \( v_{ij} \) is the covariant derivative of the covariant vector field \( (v_{i} = g^{0}_{ij}v^{j}) \) in the metric \( g^{0} \).

**Proof.** The covariant derivative of \( v \) in the metric \( g^{0} \) is

\[
v_{ij} = \frac{\partial v_{i}}{\partial x^{j}} - \Gamma^{k}_{ij}v_{k},
\]

where

\[
v_{i} = g^{0}_{ij}v^{j},
\]

\[
\Gamma^{k}_{ij} = \frac{1}{2}g^{kp}\left( \frac{\partial g^{0}_{ij}}{\partial x^{p}} + \frac{\partial g^{0}_{jp}}{\partial x^{i}} - \frac{\partial g^{0}_{ip}}{\partial x^{j}} \right),
\]

\[
v^{k}(x) = \frac{\partial}{\partial \tau} \bigg|_{\tau=0} \varphi^{k}(x, \tau), \quad i, j, k = 1, n.
\]

Differentiating the relation (1.7) with respect to \( \tau \) and then considering \( \tau = 0 \), we find

\[
2f_{ij} = \frac{\partial}{\partial \tau} \bigg|_{\tau=0} g^{\tau}_{ij} = \frac{\partial g^{\tau}_{kl}}{\partial x^{m}} \left( \frac{\partial}{\partial \tau} \bigg|_{\tau=0} \varphi^{m}(x, \tau) \right) \frac{\partial x^{\ell}}{\partial x^{k}}(x, 0) \frac{\partial x^{\ell}}{\partial x^{j}}(x, 0)
\]

\[
+ \left( g^{0}_{kl} \circ \varphi^{0} \right) \frac{\partial}{\partial \tau} \bigg|_{\tau=0} \frac{\partial x^{\ell}}{\partial x^{k}}(x, 0) \frac{\partial x^{\ell}}{\partial x^{j}}(x, 0)
\]

\[
+ \left( g^{0}_{kl} \circ \varphi^{0} \right) \frac{\partial x^{\ell}}{\partial x^{k}}(x, 0) \frac{\partial}{\partial \tau} \bigg|_{\tau=0} \frac{\partial x^{\ell}}{\partial x^{j}}(x, 0) = \frac{\partial g^{0}_{kl}}{\partial x^{m}}v^{m} \frac{\partial x^{\ell}}{\partial x^{k}}(x, 0) \frac{\partial x^{\ell}}{\partial x^{j}}(x, 0)
\]

\[
+ g^{0}_{kl} \frac{\partial}{\partial x^{m}} \left( \frac{\partial}{\partial \tau} \bigg|_{\tau=0} x^{k}(x, \tau) \right) \frac{\partial x^{\ell}}{\partial x^{j}}
\]

\[
+ g^{0}_{kl} \frac{\partial x^{k}}{\partial x^{j}} \cdot \frac{\partial}{\partial \tau} \bigg|_{\tau=0} x^{\ell}(x, \tau) = \frac{\partial g^{0}_{kl}}{\partial x^{m}}v^{m} \frac{\partial x^{\ell}}{\partial x^{k}}(x, 0) \frac{\partial x^{\ell}}{\partial x^{j}}(x, 0)
\]

\[
+ g^{0}_{kl} \frac{\partial v^{k}}{\partial x^{j}} + g^{0}_{kl} \frac{\partial v^{k}}{\partial x^{j}} + g^{0}_{kl} \frac{\partial v^{k}}{\partial x^{j}}.
\]

On the other hand

\[
v_{ij} + v_{ji} = \frac{\partial v_{i}}{\partial x^{j}} - \Gamma^{m}_{ij}v_{m} + \frac{\partial v_{j}}{\partial x^{i}} - \Gamma^{m}_{ji}v_{m} = \frac{\partial g^{0}_{im}}{\partial x^{j}}v^{m} + g^{0}_{ij} \frac{\partial v^{m}}{\partial x^{i}} + \frac{\partial g^{0}_{jm}}{\partial x^{i}}v^{m}
\]

\[
+ g^{0}_{jm} \frac{\partial v^{m}}{\partial x^{i}} - g^{0}_{im} \left( \frac{\partial g^{0}_{ip}}{\partial x^{j}} + \frac{\partial g^{0}_{jp}}{\partial x^{i}} - \frac{\partial g^{0}_{ij}}{\partial x^{p}} \right) g^{0}_{m,s}v^{s} = \left( \frac{\partial g^{0}_{im}}{\partial x^{j}} + \frac{\partial g^{0}_{jm}}{\partial x^{i}} \right) v^{m}
\]
\[\begin{align*}
+ \left( g^0_{im} \frac{\partial v^m}{\partial x^j} + g^0_{jm} \frac{\partial v^m}{\partial x^i} \right) - \left( \frac{\partial g^0_{jp}}{\partial x^i} + \frac{\partial g^0_{ip}}{\partial x^j} - \frac{\partial g^0_{ij}}{\partial x^p} \right) v^p \\
= \frac{\partial g^0_{ij}}{\partial x^p} v^p + g^0_{jp} \frac{\partial v^p}{\partial x^i} + g^0_{ip} \frac{\partial v^p}{\partial x^j}.
\end{align*}\]

Hence, we obtained the equality

\[f_{ij} = \frac{1}{2}(v_{ij} + v_{ji}).\]

### 1.4 Ray Transform of a Tensor field

Let us generalize this problem from covariant tensors of second order to covariant tensors of superior order.

Let \((M, g)\) be a simple Riemannian manifold and \(\tau_M = (TM, p, M)\), \(\tau'_M = (T'M, p', M)\) the tangent and the cotangent bundle of \(M\), respectively. Let \(S^m\tau'_M\) be the set of the symmetric tensor fields on \(\tau'_M\) and \(C^\infty(S^m\tau'_M)\) the space of the sections of this bundle. Consider \(\nabla\) the covariant derivative and \(\sigma\) the symmetrization.

**Problem 2.** Let \((M, g)\) be a simple Riemannian manifold. Do integrals

\[I_f(x_{pq}) = \int_{x_{pq}} f_{i_1...i_m}(x(t)) \dot{x}^{i_1}(t) \cdots \dot{x}^{i_m}(t) dt,
\]

\(p, q \in \partial M\), determine a symmetric tensor field \(f \in C^\infty(S^m\tau'_M)\) (\(x_{pq}\) is the geodesic joining the endpoints \(p, q\) and \(dt\) is the geodesic arc length)? Particularly, the equality \(I_f(x_{pq}) = 0\) allows us to state the existence of a field \(v \in C^\infty(S^{m-1}\tau'_M)\), such that \(v|_{\partial M} = 0\) and \(\sigma(\nabla v) = f^?\)

The function \(I_f\), determined by the equality (1.9) on the set of the geodesics joining the points situated on the boundary of \(M\), is called _single-ray transform of the tensor field \(f_\)_.

**Remark 1.2** According [4], there are some known results on problem 1.

R. Michel obtained a positive answer to problem 1 in the two-dimensional case when \(g^0\) has constant Gauss curvature.

R. G. Mukhometov, J. W. Bernstein and M. L. Gerver found a solution to the linear problem 2 for simple metrics, in the case \(m = 0\). When \(m = 1\), Yu. E. Anikonov and V. G. Romanov solved problem 2. R. G. Mukhometov generalised these results to metrics whose geodesics form a typical caustics.

### 2 Multi-time Case

#### 2.1 Harmonic maps

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n > 1\), with the boundary \(\partial M\). Let \((x^1, \ldots, x^n)\) be the local coordinates, \(\Gamma_{ij,k}\), respectively \(\Gamma'_{jk}\) the Christoffel symbols of the first type, respectively the second type.
Definition 2.1 The pair of metrics \((h, g)\) is called simple if for any \(\sigma \in \partial M\) there is a unique minimal submanifold \(N\) represented by \(x: T \cup \partial T \to M \cup \partial M, x\big|_{\partial T} = \sigma\). Let \((N, h)\) be a minimal Riemannian submanifold of \(M\), \(\dim N = p, 2 \leq p \leq n\), fixed by a closed border \(\sigma\) of dimension \(p - 1\), included in \(\partial M\). Suppose that \(\partial M\) is foliated by submanifolds of type \(\sigma\). Let \((t^1, \ldots, t^p)\) be the local coordinates in \(N\).

Definition 2.2 Let \(x: T \to M\), \(x(t) = (x^1(t), \ldots, x^n(t)), t = (t^1, \ldots, t^p), x\big|_{\partial T} = \sigma, x \in C^\infty(T, M)\).

The integral

\[
E_{(h, g)}(x) = \frac{1}{2} \int_T h^{\alpha \beta}(t)g_{ij}(x(t))x^i_\alpha(x(t))x^j_\beta(x(t))dv_h,
\]

where \((h^{\alpha \beta}) = (h_{\alpha \beta})^{-1}, h_{\alpha \beta} = h \left( \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right), g_{ij} = g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)\) and \(x^i_\alpha = \frac{\partial x^i}{\partial \sigma^\alpha}\), is called the energy of the application \(x\).

Proposition 2.1 A minimum of the energy functional \(E_{(h, g)}\), with the boundary condition \(x\big|_{\partial T} = \sigma\), necessarily verifies the boundary value problem

\[
\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial \sigma^\alpha} \left( \frac{\partial L}{\partial x^i_\alpha} \right) = 0, \quad i = 1, n,
\]

where \(L(x^i, x^i_\alpha) = \frac{1}{2}\sqrt{h}h^{\alpha \beta}g_{ij}x^i_\alpha x^j_\beta\) is the Lagrangian of the functional, \(h = \det(h_{\alpha \beta})\).

Explicitly, \(\tau(x) = 0, x\big|_{\partial T} = \sigma\), where

\[
\tau(x) = h^{\alpha \beta} \left\{ \frac{\partial^2 x^i}{\partial \sigma^\alpha \partial \sigma^\beta} - \Gamma^i_{\alpha \beta} x^i_\gamma + \Gamma^i_{j\beta} x^j_\gamma \right\} \frac{\partial}{\partial x^i},
\]

is the tension field of the application \(x\).

Proof. We have

\[
\frac{\partial L}{\partial x^i} = \frac{1}{2}\sqrt{h}h^{\beta \gamma} \frac{\partial}{\partial x^i}(g_{jk}x^j_\beta x^k_\gamma) = \frac{1}{2}\sqrt{h}h^{\beta \gamma} \frac{\partial g_{jk}}{\partial x^i} x^j_\beta x^k_\gamma
\]

\[
= \frac{1}{2}\sqrt{h}h^{\beta \gamma}(\Gamma_{ij,k} + \Gamma_{ik,j})x^j_\beta x^k_\gamma = \frac{1}{2}\sqrt{h}h^{\beta \gamma}\Gamma_{ij,k} x^j_\beta x^k_\gamma + \frac{1}{2}\sqrt{h}h^{\beta \gamma}\Gamma_{ik,j} x^j_\beta x^k_\gamma
\]

\[
\frac{\partial L}{\partial x^i_\alpha} = \frac{1}{2}\sqrt{h}h^{\beta \gamma} \frac{\partial}{\partial x^i_\alpha}(g_{jk}x^j_\beta x^k_\gamma) = \sqrt{h}h^{\alpha \beta}g_{ij}x^j_\beta,
\]

\[
\frac{\partial}{\partial \sigma^\alpha} \left( \frac{\partial L}{\partial x^i_\alpha} \right) = \frac{\partial}{\partial \sigma^\alpha} \left( \sqrt{h}h^{\alpha \beta}g_{ij}x^j_\beta \right) = \frac{1}{2}\sqrt{h} \frac{\partial h}{\partial \sigma^\alpha} h^{\alpha \beta} g_{ij}x^j_\beta + \sqrt{h} \frac{\partial h^{\alpha \beta}}{\partial \sigma^\alpha} g_{ij}x^j_\beta.
\]
Remark 2.1

Let $\sigma \in \partial M$ and $E_{(h,g)}(\sigma)$ the energy of the submanifold $N$ that corresponds to the boundary $\sigma$. The function $E_{(h,g)}: \partial M \to \mathbb{R}$ generated by the correspondence $\sigma \to E_{(h,g)}(\sigma)$ is called the boundary energy.

2.2 Determining a Pair of Metrics by Boundary Energy

Finally, we obtain

$$\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial \tau} \left( \frac{\partial L}{\partial x^i} \right) = 0 \iff h^{\alpha \beta} g_{ij} \left( \frac{\partial^2 x^j}{\partial \tau^2 \partial \beta} - \Gamma^j_{\alpha \beta} x^j_\gamma + \Gamma^j_{\gamma \beta} x^j_\alpha \right) = 0 \iff h^{\alpha \beta} g_{ij} \left( \frac{\partial^2 x^j}{\partial \tau^2 \partial \beta} - \Gamma^j_{\alpha \beta} x^j_\gamma + \Gamma^j_{\gamma \beta} x^j_\alpha \right) = 0.$$
2.3 Linearization of the Problem of Determining Metrics
by the Boundary Energy

Let us show that the existence problem of the metrics with the property that $E: \partial M \to \mathbb{R}$ represents the boundary energy cannot have a unique solution.

Let $\Phi: T \times M \to T \times M$, $\Phi(t^1, \ldots, t^n; x^1, \ldots, x^n) = (\psi(t), \varphi(x))$ be a diffeomorphism with the properties $\psi|_{\partial T} = \text{id}$, $\varphi|_{\partial M} = \text{id}$. The diffeomorphism transforms the simple metrics $h^0$, $g^0$ into the simple metrics $h^1 = \psi^*h^0$ and $g^1 = \varphi^*g^0$, because we have

$$h^1(t)(\mu, \nu) = h^0((d_t\psi)\mu, (d_t\psi)\nu)\psi(t),$$

where $d_t\psi: T_\tau T \to T_{\psi(t)}T$ is the differential of $\psi$, and

$$g^1(x)(\xi, \eta) = g^0((d_x\varphi)\xi, (d_x\varphi)\eta)\varphi(x),$$

d$x\varphi: T_\tau M \to T_{\varphi(x)\tau}M$ is the differential of $\varphi$.

$(h^0, g^0)$ and $(h^1, g^1)$ give different families of minimal submanifolds with the same boundary energy $E$.

**Problem 1’.** Let $(h^0, g^0)$ and $(h^1, g^1)$ be pairs of simple metrics, $h^0$, $h^1$ on $T$, respectively $g^0$, $g^1$ on $M$. The equality $E(h^0, g^0) = E(h^1, g^1)$ implies the existence of a diffeomorphism $\Phi: T \times M \to T \times M$, $\Phi = (\psi, \varphi)$, $\psi|_{\partial T} = \text{id}$, $\varphi|_{\partial M} = \text{id}$, $h^1 = \psi^*h^0$ and $g^1 = \varphi^*g^1$ (the problem of finding a pair metrics by the boundary energy)?

2.3 Linearization of the Problem of Determining Metrics
by the Boundary Energy

Let us linearize the problem 1’. Let $(g^\tau)$ be a family of simple metrics on $M$ which depends smoothly on $\tau \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. Let $\sigma \in \partial M$ and $\alpha = E(\sigma)$, $E: \partial M \to \mathbb{R}$ the given frontier energy. Consider $x^\tau: T \to M$ a minimal submanifold of the metric $g^\tau$, for which $x^\tau|_{\partial T} = \sigma$, $x^1 = x^1(t^n)$, $\alpha = \Gamma_i^p$, $i = \Gamma_i^p$. Let $T = [0, a]^p$ with the induced Riemannian metric $(h^\alpha_{ij})$, $t = (t^1, \ldots, t^p)$.

Let $x^\tau(t) = (x^1(t, \tau), \ldots, x^n(t, \tau))$ be the representation of $x^\tau$ in a coordinate system and $g^\tau = (g^\tau_{ij})$. The energy of the deformation $x^\tau$ is

$$E_{(h^\tau, g^\tau)}(\sigma) = \frac{1}{2} \int_T h^\alpha_{ij}(t) g^\tau_{ij}(x^\tau(t))x^\alpha_i(t, \tau)x^\beta_j(t, \tau)dv_h.$$ 

Differentiating with respect to $\tau$, we obtain

$$\left. \frac{\partial}{\partial \tau} \right|_{\tau=0} E_{(h^\tau, g^\tau)}(\sigma) = \int_T \left[ h^\alpha_{ij}(t) f_{ij}(x^0(t)) + k^\alpha_{ij}(t) g^\tau_{ij}(x^0(t)) \right] x^\alpha_i(t, 0)x^\beta_j(t, 0)dv_h$$

$$+ \frac{1}{2} \int_T \left[ h^\alpha_{ij}(t) \frac{\partial g^\tau_{ij}}{\partial x^k}(x^0(t))x^\alpha_i(t, 0)x^\beta_j(t, 0) \frac{\partial x^k}{\partial \tau}(t, 0) \right] dv_h$$

$$+ 2h^\alpha_{ij}(t) g^\tau_{ij}(x^0(t)) x^\alpha_i(t, 0) \frac{\partial x^\beta_j}{\partial \tau}(t, 0)$$

$$d\tau,$$

(2.1)
where \( f_{ij} = \frac{1}{2} \frac{\partial}{\partial \tau} \bigg|_{\tau=0} g^0_{ij} \) and \( k^{\alpha \beta} = \frac{1}{2} \frac{\partial}{\partial \tau} \bigg|_{\tau=0} h_{\tau}^{\alpha \beta}. \)

Integrating by parts and using the fact that \( \frac{\partial x^i}{\partial \tau} \bigg|_{\partial \tau} = 0 \), we have

\[
\int_T h^0_{ij}(t)g^0_{ij}(x^0(t))x^i_a(t, \tau) \frac{\partial x^j_a}{\partial \tau}(t, \tau) \, dt = \int_T h^0_{ij}g^0_{ij} \frac{\partial}{\partial \tau} \bigg( \frac{\partial x^j}{\partial \tau} \bigg) \mathbb{H} d\tau_1 \wedge \cdots \wedge d\tau^p
\]

\[
= - \int_T \left[ \frac{\partial h^0_{ij}}{\partial \tau^3} g^0_{ij} \frac{\partial x^j_a}{\partial \tau} + h^0_{ij} \frac{\partial g^0_{ij}}{\partial \tau^3} \frac{\partial x^j_a}{\partial \tau} + h^0_{ij} \frac{\partial g^0_{ij}}{\partial \tau^3} \frac{\partial x^j_a}{\partial \tau} \right] \, dt \]

The second integral of (2.1) becomes

\[
\int_T h^0_{ij} \frac{\partial g^0_{ij}}{\partial x^i} x^i_a x^j_a - \frac{\partial h^0_{ij}}{\partial \tau^3} g^0_{ij} x^i_a x^j_a - \frac{\partial h^0_{ij}}{\partial \tau^3} g^0_{ij} x^i_a x^j_a - \frac{\partial h^0_{ij}}{\partial \tau^3} g^0_{ij} x^i_a x^j_a - \frac{\partial h^0_{ij}}{\partial \tau^3} g^0_{ij} x^i_a x^j_a - \frac{\partial h^0_{ij}}{\partial \tau^3} g^0_{ij} x^i_a x^j_a
\]
\[-2h_0^{\alpha\beta} g_{ij}^0 x_i^\alpha \Gamma_\gamma^\beta \partial x_j^\beta \nu \partial \nu.\]

Since \(x_0\) is an extremal of the energy, we have

\[h_0^{\alpha\beta} g_{ij}^0 \left(-\Gamma_\mu^p x_i^\alpha x_j^\beta - \frac{\partial^2 x^p}{\partial \xi^\alpha\partial \xi^\beta} \right) = -2g_{ij}^0 h^{\alpha\beta} \Gamma_\alpha^\gamma x_\gamma^p\]

and the previous integral becomes

\[
\int_T \left[-2g_{ij}^0 h_0^{\alpha\beta} \Gamma_\gamma^\alpha x_\gamma^p - 2 \frac{\partial h_0^{\alpha\beta}}{\partial \xi^\rho} g_{ij}^0 x_i^\alpha - 2h_0^{\alpha\beta} g_{ij}^0 x_\gamma^p \Gamma_\gamma^\beta \right] \partial x_j^\beta \nu \partial \nu \partial \nu h.
\]

\[
= -2 \int_T g_{ij}^0 x_i^\alpha \left(h_0^{\alpha\beta} \Gamma_\mu^\alpha + \frac{\partial h_0^{\alpha\beta}}{\partial \xi^\gamma} + h_0^{\alpha\beta} \Gamma_\gamma^\beta \right) \partial x_j^\beta \nu \partial \nu h.
\]

\[
= -2 \int_T g_{ij}^0 x_i^\alpha \left(h_0^{\alpha\beta} \Gamma_\gamma^\alpha - h_0^{\mu\nu} \Gamma_\mu^\beta - h_0^{\gamma\beta} \Gamma_\nu^\alpha + h_0^{\alpha\beta} \Gamma_\gamma^\nu \right) \partial x_j^\beta \nu \partial \nu h = 0.
\]

Denoting \(F_{ij}^{\alpha\beta} = h_0^{\alpha\beta} f_{ij} + k^{\alpha\beta} g_{ij}^0\), we have the equality

\[
\frac{\partial}{\partial \tau} \bigg|_{\tau=0} E_{(h^0, g^0)}(\sigma) = \int_T F_{ij}^{\alpha\beta} x_i^\alpha(t, 0) x_j^\beta(t, 0) \partial \nu h.
\]

Using the functional \(I_F(x_0) = \int_T F_{ij}^{\alpha\beta} (x(t)) x_i^\alpha(t) x_j^\beta(t) \partial \nu h\), the previous relation becomes

\[\frac{\partial}{\partial \tau} \bigg|_{\tau=0} E_{(h^0, g^0)}(\sigma) = I_F(x^0),\]

where \(x^0\) is a minimal submanifold of the metric \(g^0\).

The existence of solutions of this problem for the family \((g^\tau)\) implies the existence of an one-parameter group of diffeomorphisms \(\Phi^\tau(t, x) = (\psi^\tau(t), \varphi^\tau(x))\), such that \(g^\tau = (\varphi^\tau)^* g^0\) and \(h^\tau = (\psi^\tau)^* h^0\). Explicitly

\[h^\tau_{\alpha\beta} = \left(h_0^{\mu\nu} \circ \psi^\tau\right) \frac{\partial h^\mu_{\alpha\beta}}{\partial \xi^\tau} (t, \tau) \frac{\partial h^\nu_{\alpha\beta}}{\partial \xi^\tau} (t, \tau),\]

where \(\psi^\tau(t) = (\psi^1(t, \tau), \ldots, \psi^n(t, \tau))\), \(t' = \psi^\tau(t)\),

\[g_{ij}^\tau = \left(g_{k\ell}^0 \circ \varphi^\tau\right) \frac{\partial x^k_{\alpha\beta}}{\partial \xi^\tau} (x, \tau) \frac{\partial x^\ell_{\alpha\beta}}{\partial \xi^\tau} (x, \tau),\]

where \(\varphi^\tau(x) = (\varphi^1(x, \tau), \ldots, \varphi^n(x, \tau))\), \(x' = \varphi^\tau(x)\).

Instead of (2.3), we need

\[h_{\tau}^{\alpha\beta} = \left(h_0^{\mu\nu} \circ \psi^{-\tau}\right) \frac{\partial h^\mu_{\alpha\beta}}{\partial \xi^\tau} (t, \tau) \frac{\partial h^\nu_{\alpha\beta}}{\partial \xi^\tau} (t, \tau).
\]

**Theorem 2.1** The relations (2.4) and (2.5) imply

\[f_{ij} = \frac{1}{2} (\psi_{ij} + \psi_{ji}),\]

\[\psi_{ij} = \frac{1}{2} (\psi_{ij} + \psi_{ji}).\]
where \( v^k(x) = \frac{\partial}{\partial \tau} (x^k(x, \tau)) \), \( v_i = \delta^0_{ij} v^j \) and \( v_{ij} \) is the covariant derivative of \( (v_i) \) and
\[
(2.7) \quad k^\alpha\beta = \frac{1}{2} (u^\alpha;\beta + u^\beta;\alpha),
\]
where \( \frac{\partial}{\partial \tau} \bigg|_{\tau=0} (\psi^\alpha)(t, \tau) = u^\alpha \), \( u^\alpha;\beta = h^\alpha_{\mu \beta} \) and \( u^\nu \) is the covariant derivative of \( (u^\alpha) \).

Proof. The relation (2.6) is similar to relation (1.8). Differentiating the relation (2.5) with respect to \( \tau \), we have
\[
2k^\alpha\beta = \frac{\partial h^\mu_0}{\partial \tau} \left( \frac{\partial}{\partial t} \bigg|_{t=0} \psi^\alpha \right) + \frac{\partial h^\beta_0}{\partial \tau} \left( \frac{\partial}{\partial t} \bigg|_{t=0} \psi^\alpha \right) + \frac{\partial h^\mu_0}{\partial \tau} \frac{\partial}{\partial \tau} \left( \frac{\partial}{\partial t} \right)(t, 0) \frac{\partial h^\beta_0}{\partial t}(t, 0)
\]
\[
+ (h^\mu_0 \circ \psi^\alpha) \frac{\partial}{\partial \tau} \left( \frac{\partial}{\partial t} \bigg|_{t=0} \right) \frac{\partial h^\beta_0}{\partial t}(t, 0) = - \frac{\partial h^\mu_0}{\partial \tau} \mu^\gamma \partial^\gamma_\nu \delta^\nu_\beta
\]
\[
+ h^\mu_0 \frac{\partial}{\partial \tau} \left( \frac{\partial}{\partial t} \bigg|_{t=0} \right) \delta^\beta_\nu + h^\mu_0 \delta^\beta_\nu \frac{\partial h^\alpha_0}{\partial t}(t, 0)
\]
\[
= \frac{\partial h^\alpha_0}{\partial \tau} - u^\alpha + h^\mu_0 \frac{\partial h^\beta_0}{\partial \tau} + h^\mu_0 \frac{\partial h^\beta_0}{\partial t}(t, 0).
\]

On the other hand
\[
u^\alpha;\beta + u^\beta;\alpha = u^\alpha_{\mu \beta} h^\beta_0 + u^\beta_{\mu \alpha} h^\beta_0 = h^\beta_0 \left( \frac{\partial u^\alpha}{\partial \tau} + \Gamma^\alpha_{\mu \nu} u^\nu \right) + h^\beta_0 \left( \frac{\partial u^\beta}{\partial \tau} + \Gamma^\beta_{\mu \nu} u^\nu \right)
\]
\[
= h^\mu_0 \frac{\partial u^\alpha}{\partial \tau} + h^\mu_0 \frac{\partial u^\beta}{\partial \tau} + \left( h^\mu_0 \Gamma^\alpha_{\mu \nu} + h^\mu_0 \Gamma^\beta_{\mu \nu} \right) u^\nu
\]
\[
= h^\mu_0 \frac{\partial u^\alpha}{\partial \tau} + h^\mu_0 \frac{\partial u^\beta}{\partial \tau} - \frac{\partial h^\alpha_0}{\partial \tau} u^\mu.
\]

The equality (2.7) was proved.

We have the following linearization of the problem 1': do integrals (2.2) determine the tensor \( \left( F_{ij}^{\alpha \beta} \right) \)?

### 2.4 Multi-ray Transform of a Distinguished Tensor Field

Problem 2'. Generalizing the problem to tensor fields of any rank, the following question appears: to what extent the integrals
\[
(2.8) \quad I_F(x) = \int_T F_{i_1 \cdots i_m}^{a_1 \cdots a_m}(x(t)) x_{a_1}^1(t) \cdots x_{a_m}^m(t) dv_h
\]
determine a symmetric tensor field \( F \)?

The function \( I_F \), determined by the equality (2.8) on the set of submanifolds \( \sigma \in \partial M \), is called multi-ray transform of the tensor \( F \).
References


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