Some modified connection structures
associated with the Finslerian gravitational field

Satoshi Ikeda and Mihai Anastasiei

Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. Some modified connection structures of the Finslerian grav-
itational field are considered by modifying the Finslerian independent
variables \((x^i, y^j)\) \((i, j = 1, 2, 3, 4)\) in the following forms: \((x^i, y^j, p_j)\) \((p_j:\)
co-vector dual to \(y^j)\); \((x^i, y^j, z^k)\) \((z^k:\) another vector chosen at one more
microscopic level than the \(y\)--level); \((x, \psi)\) \((\psi:\) spinor).

Mathematics Subject Classification: 53FB40, 83D05.
Key words: Finslerian gravitational fields, connection structures.

1 Introduction

The independent variables of the Finslerian gravitational field are chosen as \((x^i, y^j)\),
where the vector \(y (= y^j; j = 1, 2, 3, 4)\) is attached to each point \(x (= x^i; i = 1, 2, 3, 4)\)
as the independent internal variable. Therefore, the Finslerian gravitational field is
regarded as a unified field between the external \((x)--field spanned by points \(\{x\}\) and
the internal \((y)--field spanned by vectors \(\{y\}\) (cf.
[1]).

From another viewpoint, the Finslerian gravitational field is considered the unified
field over the total space of the vector bundle whose fibre at each point \(x\) is the
\((y)--field and base manifold is the \((x)--field. This vector bundle is not necessarily the
tangent bundle \(TM\) over the base manifold \(M\). It can be any vector bundle case in
which one applies the geometry of vector bundles as it is described in [2]. But the
theory is simpler for the tangent bundle. Thus we consider the tangent bundle as
starting point and we will replace it with certain vector bundles in the next sections.

Then, the adapted frame in the total space is set as follows ([2]):

\[
dX^A = (dx^i, \delta y^i = dy^i + N^i_j dx^j)
\]

\[
\frac{\partial}{\partial X^A} = \left( \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i} \right),
\]

where \(X^A = (x^i, y^j)\) \((A = 1, 2, \ldots, 8)\), and \(N^i_j\) denotes the nonlinear connection repre-
senting physically the interaction between the \((x)--\) and \((y)--fields.

We say that this interaction is holonomic if \( d(\delta y^i) = 0 \) (modulo \( \delta y^i \)). A direct calculation gives \( d(\delta y^i) + \omega^i_j \wedge \delta y^j = \Omega^i \), where \( \omega^i_j = \frac{\partial N^i_k}{\partial y^j} dx^k \) and \( 2\Omega^i = \left( \frac{\delta N^i_k}{\delta x^j} - \frac{\delta N^j_k}{\delta x^i} \right) dx^i \wedge dx^j. \) Thus the said interaction is holonomic if and only if \( \Omega^k = 0 \).

The local vector fields \( \left( \frac{\partial}{\partial y^i} \right) \) span a distribution on \( TM \) called vertical and the local vector fields \( \left( \frac{\delta}{\delta x^i} \right) \) span a distribution that is supplementary to it, called the horizontal distribution. From the equation \[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} = \Omega^k_{ij} \frac{\partial}{\partial y^k} \] it comes out that the interaction between the \((x)\)– and \((y)\)– fields is holonomic if and only if the horizontal distribution is integrable.

On the basis of (1.1), the Finslerian connection structure is stipulated as

\[
\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial X^B} = \Gamma^A_{BC} \frac{\partial}{\partial X^A};
\]

\[
\Gamma^A_{BC} \equiv (L^i_{jk}, C^i_{jk}).
\]

Namely, two kinds of connection coefficients \( L^i_{jk} \) and \( C^i_{jk} \) appear and the covariant derivatives of an arbitrary vector \( V^i \) can be defined by

\[
\begin{align*}
V^i_{|k} &= \frac{\delta V^i}{\delta x^k} + L^i_{jk} V^j,  \\
V^i_{|k} &= \frac{\partial V^i}{\partial y^k} + C^i_{jk} V^j.
\end{align*}
\]

This is the most simplified connection structure and geometrically it corresponds to the Finslerian structure. It preserves by parallelism the vertical and the horizontal distributions and makes covariant constant the almost tangent structure on \( TM \).

By the way, the metrical structure is introduced by

\[
G \equiv G_{AB} dX^A dX^B = g_{ij}(x, y) dx^i \otimes dx^j + g_{ij}(x, y) \delta y^i \otimes \delta y^j.
\]

Here \( g_{ij}(x, y) \) are the gravitational potentials depending also on the \((y)\)–field.

The condition that \( \nabla \) is metrical, that is, \( \nabla G = 0 \) is equivalent with \( g_{ij,k} = 0 \) and \( g_{ij,k} = 0 \). In the other words the tensor field is \( h \)– covariant constant and \( v \)– covariant constant. One can impose only one of these conditions. The most important is \( g_{ij,k} = 0 \) as it can be seen from [2].

Given the connection \( \nabla \) we may consider in particular \( y^i_{|k} = L^i_{jk} y^j - N^i_k \) called the \( h \)– deflection tensor and \( y^i_{|k} = \delta^i_k + C^i_{jk} y^j \) called the \( v \)– deflection tensor. If we set \( D_{ij} = g_{ik} y^k_j \) and \( d_{ij} = g_{ik} y^k_i \), we can associate to \( g_{ij} \) two tensor fields \( F_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), f_{ij} = \frac{1}{2}(d_{ij} - d_{ji}) \) that may be regarded as \( h \)– and \( v \)– electromagnetic tensors associated to \( g_{ij} \). They verify the identities that generalize the Maxwell equations (see [2]).

As is understood from (1.2) or (1.3), the connection structure depends essentially on the adapted frame (1.1), which is also constructed by the independent variables \((x^i, y^j)\). Therefore, if the independent variables are modified, then the adapted frame...
is also modified accordingly and then, the Finslerian connection structure (1.2) is modified correspondingly.

Along this line, in the following, we shall take up the following cases: the independent variables \((x^i, y^j)\) are generalized to \((x^i, y^j, p_j)\), \(p_j\) being a covector dual to \(y_j^i\); \((x^i, y^j, z^k)\), \(z^k\) being a vector chosen at one more microscopic level than the \(y\)−level; \((x, \psi)\), \(\psi\) being a spinor.

### 2 Modified Connection Structures - I

First, we shall consider the case where the independent variables become \((x^i, y^j, p_j)\), \(p_j\) being a covector dual to \(y_j^i\). In fact we replace the tangent bundle \(TM\) with the vector bundle \(TM \times T^*M\) over \(M\), where \(T^*M\) is the cotangent bundle of \(M\). Then, the adapted frame is set as follows (cf. [5]):

\[
\begin{align*}
&dX^A \equiv (dx^i, \delta y^i = dy^i + N^i_j dx^j, \delta p_i = dp_i - M_{ji} dx^j), \\
&\frac{\partial}{\partial X^A} = \left( \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j} + M_{ij} \frac{\partial}{\partial p_j}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i} \right),
\end{align*}
\]

where two kinds of nonlinear connections \(N^i_j\) and \(M_{ij}\) appear.

The interaction between the \((x)\)− \((y)\)− and \((p)\)− fields is holonomic if \(d(\delta y^i) = 0\) (modulo \(\delta y^i\)) and \(d(\delta p_i) = 0\) (modulo \(\delta p_i\)). We have

\[
d(\delta p_i) = - \frac{\partial M_{ji}}{\partial p_k} \delta p_k - \Theta_{jki} dx^k \wedge dx^j
\]

for

\[
2\Theta_{jki} = \frac{\partial M_{ji}}{\partial x^k} - \frac{\partial M_{ki}}{\partial x^j} + M_{jk} \frac{\partial M_{ki}}{\partial p_h} - M_{kh} \frac{\partial M_{ji}}{\partial p_h}.
\]

Thus the said interaction is holonomic if and only if \(\Omega^i_{jk} = 0, \Theta_{jki} = 0\). The last equation is equivalent with the integrability of the distribution spanned by \((\delta p_i)\).

Now, the connection structure is given by

\[
\Gamma^A_{BC} = (L^i_{jk}, C^i_{jk}, H^i_{jk}),
\]

by which the covariant derivatives can be defined as follows: e.g.,

\[
\begin{align*}
V^i_{|k} &= \frac{\delta V^i}{\delta x^k} + L^i_{jk} V^j, \\
V^i|_k &= \frac{\partial V^i}{\partial y^k} + C^i_{jk} V^j, \\
V^i\parallel_k &= \frac{\partial V^i}{\partial p_k} + H^i_{jk} V^j,
\end{align*}
\]

In particular, we have

\[
\begin{align*}
p^i_{ij} &= M_{ij} - L^i_{jk} p_k \\
p^i|_j &= \delta_{ij} - C^i_{jk} \\
p^i\parallel_k &= \delta^i_k + H^i_{jk} p_j,
\end{align*}
\]
Thus new types of electromagnetic tensors could be considered. The metrical structure is in this case introduced by

\[ G \equiv g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j + g^{ij}\delta p_i \otimes \delta p_j. \]

A new condition of metrizability of \( \nabla \) appears in the form \( g^{ij}\|_k = 0 \).

Next, if the independent variables are chosen as \((x^i, y^j, z^k)\), \( z^k \) being a vector chosen at one more microscopic level than the \( y \)-level, then the adapted frame is set as follows:

\[ dX^A \equiv (dx^i, \delta y^i = dy^i + N^i_j dx^j, \delta z^k = dz^k + A^k_j dy^j + B^k_j dx^j), \]

\[ \frac{\partial}{\partial X^A} \equiv \left( \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j}, \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - A^i_j \frac{\partial}{\partial z^j}, \frac{\delta}{\partial z^i} = \frac{\partial}{\partial z^i} \right), \]

where three kinds of nonlinear connections \( N^i_j, A^i_j, B^i_j \) appear. In fact here we replace the bundle \( TM \) with the product \( TM \times TM \) over \( M \). The meaning of the assertion that \( z^k \) is a vector chosen at one more microscopic level than the \( y \)-level is provided by the expression \( \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - A^i_j \frac{\partial}{\partial z^j} \). In some sense the variable \( (y^i) \) become more external than the variables \( (z^k) \).

Then, the connection structures is prescribed by

\[ \Gamma^A_{BC} \equiv (L^i_{jk}, C^i_{jk}, E^i_{jk}), \]

by which the covariant derivatives can be defined as follows: e.g.,

\[ V^i_k = \frac{\delta V^i}{\delta x^k} + L^i_{jk} V^j, \]

\[ V^i|k = \frac{\delta V^i}{\delta y^k} + C^i_{jk} V^j, \]

\[ V^i\parallel k = \frac{\partial V^i}{\partial z^k} + E^i_{jk} V^j, \]

etc. The metrical structure is given by

\[ G \equiv g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j + g_{ij}\delta z^i \otimes \delta z^j. \]

### 3 Modified Connection Structures - II

In this section, we shall consider the case where the independent variables become \((x, \psi)\), \( \psi \) being a spinor. Then, if we put the adapted frame in the form (cf. [4])

\[ dX^A \equiv (dx^i, \delta \psi = d\psi + \Theta_i \psi dx^i + \Xi \psi d\psi = P \delta), \]

\[ \frac{\partial}{\partial X^A} \equiv \left( \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial \psi}, \frac{\delta}{\partial \psi} = \frac{\partial}{\partial \psi} \right), \]

where \( \Theta_i \) and \( \Xi \) denote the spin gauge fields and \( \delta \psi = d\psi + N_i dx^i, P = I + \Xi \psi \) and \( PN_i \equiv \Theta_i \psi \), then the connection structure can be introduced by
Some modified connection structures

\[ \Gamma^A_{BC} \equiv (L^i_{jk}, L_k, C^i_j, C), \]

by which the following covariant derivatives can be defined: e.g.,

\[ V^i_k = \frac{\delta V^i}{\delta x^k} + L^i_{jk} V^j, \]

\[ V^i| = \frac{\partial V^i}{\partial \psi} + C^i_j V^j, \]

etc. Further, for a spinorial field \( \Phi(x, \psi) \), its covariant derivatives are given by

\[ D\Phi = d\Phi - \Theta_i \Theta dx^i - \Xi \Phi dx\psi = (\Phi|_k) dx^k + (\Phi|) \psi, \]

\[ \Phi|_k = \frac{\delta \Phi}{\delta x^k} - \overline{\Theta}_k \Phi, \]

\[ \Phi| = \frac{\partial \Phi}{\partial \psi} - \Xi \Phi, \]

where \( \overline{\Theta}_k = \Theta_k - \Xi N_k \) and \( \Xi = P^{-1} \Xi. \)

4 Other Comments

In this Section, we shall show some other modified connection structures.

First, if the adapted frame (1.1) is changed to

\[ dX^A \equiv (dx^i, \delta y^i = P^i_j dy^j + Q^i_j dx^j), \]

\[ \frac{\partial}{\partial X^A} \equiv \left( \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j}, \frac{\delta}{\delta y^i} = (P^{-1})_j^i \frac{\partial}{\partial y^j} \right), \]

where \( N^i_j = (P^{-1})_j^i Q^i_j \), then the connection structure is given by

\[ \Gamma^A_{BC} \equiv (L^i_{jk}, (P^{-1})_j^i C^i_j + (P^{-1})_j^i C^i_j). \]

Second, if the independent variables are chosen as \( (x^i, p_j) \) and the adapted frame in this case is given by (cf. [5])

\[ dX^A \equiv (dx^i = dx^i + \Pi^i_j \delta p_j, \delta p_i = dp_i - M^i_j dx^j), \]

\[ \frac{\partial}{\partial X^A} \equiv \left( \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + M^i_j \frac{\partial}{\partial p_j}, \frac{\delta}{\partial p_i} = \frac{\partial}{\partial p_j} - M^i_j \frac{\delta}{\delta x^j} \right), \]

then the connection structure is given by (see (2.2))

\[ \Gamma^A_{BC} \equiv (L^i_{jk}, H^i_j - \Pi^i_l L^l_{ji}). \]

Finally, if the nonlinear connection \( N^i_j \) in (1.1) is changed as \( \overline{N}^i_j = N^i_j - F^i_j \), then the connection structure becomes (cf. [2])

\[ \Gamma^A_{BC} \equiv (\overline{L}^i_{jk}, \overline{H}^i_j - \Pi^i_l \overline{L}^l_{ji}). \]

Thus, we can consider many interesting modified connection structures by modifying the independent variables or the adapted frame in various ways.
References


Authors’ addresses:

Satoshi Ikeda
Department of Mechanical Engineering, Faculty of Science and Technology
Tokyo University of Science, Noda, Chiba 278-8510, Japan.

Mihai Anastasiei
Faculty of Mathematics, University "AL.I.Cuza" Iași, 700506, Iași, Romania

Mathematics Institute "O.Mayer" Iași, Iași Branch of the Romanian Academy, Iași, Romania.

email: anastas@uaic.ro