Nonlinear connections on dual Lie algebroids

Dragoş Hrimiuc and Liviu Popescu

Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. In this paper we start developing the so-called Klein’s formalism on dual Lie algebroids. The nonlinear connection associated to a regular section is naturally obtained. Particularly, this connection is found for the Hamiltonian case.

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1 Preliminaries on Lie algebroids

The notion of Lie algebroid is a generalization of the concepts of Lie algebra and integrable distribution. In [8] A. Weinstein gives a generalized theory of Lagrangian on Lie algebroids and obtains the Euler-Lagrange equations. The same equations were later obtained by E. Martinez [5] using the symplectic formalism and the notion of prolongation of Lie algebra over a mapping introduced by P.J. Higgins and K. Mackenzie [3]. In this paper Klein’s formalism in the case of dual Lie algebroids is investigated. It should be mentioned that our approach is new even for the particular case of the cotangent bundle.

Let \(M\) be a differentiable, \(n\)-dimensional manifold and \((TM, \pi_M, M)\) its tangent bundle. A Lie algebroid over the manifold \(M\) is the triple \((E, [\cdot, \cdot], \sigma)\) where \(\pi: E \to M\) is a vector bundle of rank \(m\) over \(M\), whose \(C^\infty(M)\)-module of sections \(\text{Sec}(E)\) is equipped with a Lie algebra structure \([\cdot, \cdot]\) and \(\sigma: E \to TM\) is a vector bundle homomorphism (called the anchor) which induces a Lie algebra homomorphism (also denoted \(\sigma\)) from \(\text{Sec}(E)\) to \(\chi(M)\), satisfying the compatibility conditions

\[
[s_1, f s_2] = f[s_1, s_2] + (\sigma(s_1)f)s_2
\]

for every \(f \in C^\infty(M)\) and \(s_1, s_2 \in \text{Sec}(E)\). From the above definition we easily get

\[
[\sigma(s_1), \sigma(s_2)] = \sigma(s_1, s_2), \quad [s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0.
\]

For \(f \in C^\infty(M)\) the differential \(df(x) \in E_x^*\) is defined by \(\langle df(x), u \rangle = \sigma(u)f\), for every \(u \in E_x\) and for differentiable \(k\)-form \(\omega \in \bigwedge^k(E) = \text{Sec}((E^*)^k \to M)\), \(k > 0\) its exterior derivative \(d\omega \in \bigwedge^{k+1}(E)\) is defined as follows:

Also, for $\xi \in \text{Sec}(E)$ on can define the Lie derivative with respect to $\xi$ by $L_\xi = i_\xi \circ d + d \circ i_\xi$, where $i_\xi$ is the contraction with $\xi$. If we take the local coordinates $(x^i)$ on an open $U \subset M$, a local basis $\{ s_\alpha \}$ of sections of the bundle $\pi^{-1}(U) \to U$ generates local coordinates $(x^i, y^\alpha)$ on $E$. The local functions $\sigma^i_\alpha(x), L^\gamma_\alpha_\beta(x)$ on $M$ given by

$$
\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta] = L^\gamma_\alpha_\beta s_\gamma, \quad i, j = 1, \ldots, n, \quad \alpha, \beta = 1, \ldots, m,
$$

capture the properties which define the Lie algebroid structure over $M$ in so called structure equations:

$$
\sigma^i_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_\alpha_\beta, \quad \sum_{(\alpha, \beta, \gamma)} \left( \sigma^i_\alpha \frac{\partial L^\delta_\beta_\gamma}{\partial x^i} + L^\delta_\alpha_\eta L^\eta_\beta_\gamma \right) = 0.
$$

Locally, if $f \in C^\infty(M)$ then $df = \frac{\partial f}{\partial x^i} \sigma^i_\alpha s^\alpha$ and if $\theta \in \text{Sec}(E^*)$, $\theta = \theta_\alpha s^\alpha$ then

$$
d\theta = (\frac{\partial \theta_\beta}{\partial x^i} \sigma^i_\alpha - \frac{1}{2} \theta_\gamma L^\gamma_\alpha_\beta) s^\alpha \wedge s^\beta,
$$

where $\{ s^\alpha \}$ is the dual basis of $\{ s_\alpha \}$. Particularly, we have $dx^i = \sigma^i_\alpha s^\alpha$ and $ds^\alpha = -\frac{1}{2} L^\gamma_\alpha_\beta s^\beta \wedge s^\gamma$.

## 2 Dual Lie algebroids

Let $\tau : E^* \to M$ be the dual of $\pi : E \to M$ and $(E, [\cdot, \cdot], \sigma)$ a Lie algebroid structure over $M$. One can construct a Lie algebroid structure over $E^*$, by taking the prolongation of $(E, [\cdot, \cdot], \sigma)$ over $E^*$ (see [3],[4],[5]). This structure is given by the following objects:

- The associated vector bundle is $(TE^*, \tau_1, E^*)$ where $TE^* = \cup_{u^* \in E^*} T_{u^*} E^*$ with $T_{u^*} E^* = \{(u_x, v_{u^*}) \in E_x \times T_{u^*} E^* | \sigma(u_x) = T_{u^*} \tau(v_{u^*}), \tau(u^*) = x \in M \}$ and the projection $\tau_1 : TE^* \to E^*, \tau_1(u_x, v_{u^*}) = u^*$.

- The Lie algebra structure $[\cdot, \cdot]$ on $\text{Sec}(\tau_1)$ is defined in the following way: if $\rho_1, \rho_2 \in \text{Sec}(\tau_1)$ are such that $\rho_i(u^*) = (X_i(\tau(u^*)), U_i(u^*))$ where $X_i \in \text{Sec}(\pi), U_i \in \chi(E^*)$ and $\sigma(X_i(\tau(u^*))) = T_{u^*} \tau(U_i(u^*)), \ i = 1, 2$, then $[\rho_1, \rho_2](u^*) = ([X_1, X_2](\tau(u^*)), [U_1, U_2](u^*))$

- The anchor is the projection $\sigma^1 : TE^* \to TE^*, \sigma^1(u, v) = v$. 

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Notice that if $\mathcal{T}t : T E^* \to E, \mathcal{T}t (u, v) = u$ then $(VTE^*, \tau_1|VTE^*, E^*)$ with $VTE^* := \text{Ker} T \tau$ is a subbundle of $(TE^*, \tau_1, E^*)$, called the vertical subbundle. If $(x^i, \mu_\alpha)$ are local coordinates on $E^*$ at $u$ and $\{s_\alpha\}$ is a local basis of sections of $\pi : E \to M$ then a local basis of $\text{Sec}(T E^*)$ is $\{X_\alpha, \mathcal{P}^\alpha\}$ where

$$
(2.1) \quad X_\alpha (u^*) = \left( s_\alpha (\tau (u^*)), \sigma_\alpha \frac{\partial}{\partial x^i} |_{u^*} \right), \quad \mathcal{P}^\alpha (u^*) = \left( 0, \frac{\partial}{\partial \mu_\alpha} |_{u^*} \right).
$$

The Lie brackets on the elements of this basis are:

$$
(2.2) \quad [X_\alpha, X_\beta] = L_\alpha^\gamma X_\gamma, \quad [X_\alpha, \mathcal{P}^\alpha] = 0, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0
$$

and therefore

$$
\begin{align*}
dx^i &= \sigma_i^\alpha X_\alpha, \\
d\mu_\alpha &= \mathcal{P}^\alpha, \\
d\lambda^\gamma &= -\frac{1}{2} L_\alpha^\gamma X_\alpha \wedge X_\beta, \\
d\mathcal{P}^\alpha &= 0
\end{align*}
$$

where $\{X_\alpha, \mathcal{P}^\alpha\}$ is the dual basis of $\{X_\alpha, \mathcal{P}^\alpha\}$. Also if $\rho = \rho^\alpha X_\alpha + \rho_\alpha \mathcal{P}^\alpha$ is a section of $TE^*$, then $\frac{\partial}{\partial x^i} (\rho) = \frac{\partial}{\partial x^i} \rho^\alpha + \rho_\alpha \frac{\partial}{\partial x^i}. The canonical symplectic structure of a Lie algebroid $TE^*$ is given by $\omega = -d\theta$ where $\theta = \mu_\alpha X_\alpha$ is the Liouville form. In local coordinates we get

$$
(2.3) \quad \omega = X^\alpha \wedge \mathcal{P}^\alpha + \frac{1}{2} \mu_\alpha L_\alpha^\beta X_\beta \wedge X_\gamma.
$$

We remark that $VTE^*$ is Lagrangian for $\omega$, i.e. $\omega (\rho_1, \rho_2) = 0$, for every vertical sections $\rho_1, \rho_2$.

3 Nonlinear connection on $TE^*$

**Definition 1.** A nonlinear connection (or connection) on $TE^*$ is an almost product structure $\mathcal{N}$ on $\tau_1 : TE^* \to E^*$(i.e. a bundle morphism $\mathcal{N} : TE^* \to TE^*$, such that $\mathcal{N}^2 = id$ ) smooth on $TE^* \setminus \{0\}$ such that $VTE^* = \text{Ker} (id + \mathcal{N})$.

(i) If $\mathcal{N}$ is a connection on $TE^*$ then $HTE^* = \text{Ker} (id - \mathcal{N})$ is the horizontal subbundle associated to $\mathcal{N}$ and $TE^* = VTE^* \oplus HTE^*$. Each $\rho \in \text{Sec}(\tau_1)$ can be written as $\rho = \rho^h + \rho^v$ where $\rho^h, \rho^v$ are sections in the horizontal and respective vertical subbundles. If $\rho^h = 0$ then $\rho$ is called vertical and if $\rho^v = 0$ then $\rho$ is called horizontal. The section $\mathcal{C}$ given locally by $\mathcal{C} = \mu_\alpha \mathcal{P}^\alpha$ defines a global vertical section that is called Liouville section.

(ii) A connection $\mathcal{N}$ on $E^*$ induces two projectors $h, v : TE^* \to TE^*$ such that $h(\rho) = \rho^h$ and $v(\rho) = \rho^v$ for every $\rho \in \text{Sec}(\tau_1)$. We have

$$
(3.1) \quad h = \frac{1}{2} (id + \mathcal{N}), \quad v = \frac{1}{2} (id - \mathcal{N}),
$$

$$
(3.2) \quad \ker h = \text{Im} v = VTE^*, \quad \text{Im} h = \ker v = HTE^*.
$$

(iii) Locally a connection can be expressed as

$$
(3.3) \quad \mathcal{N}(X_\alpha) = X_\alpha + 2N_{\alpha \beta} \mathcal{P}^\beta, \quad \mathcal{N}(\mathcal{P}^\alpha) = -\mathcal{P}^\alpha;
$$
where $N_{\alpha\beta} = N_{\alpha\beta}(x, \mu)$ are the local coefficients of $N$. The vector fields
\begin{equation}
\delta^*_\alpha = h(\mathcal{X}_\alpha) = \mathcal{X}_\alpha + N_{\alpha\beta}\mathcal{P}^\beta
\end{equation}
generate a basis of $HTE^*$. The frame $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ is a local basis of $TE^*$ called adapted. The dual adapted basis is $\{\mathcal{X}^\alpha, \delta P_\alpha\}$ where $\delta P_\alpha = \mathcal{P}_\alpha - N_{\alpha\beta}\mathcal{X}^\beta$.

**Definition 2.** A connection $N$ on $TE^*$ is called symmetric if $HTE^*$ is Lagrangian for $\omega$.

**Proposition 1.** $N$ is symmetric iff locally
\begin{equation}
N_{\alpha\beta} - N_{\beta\alpha} = \mu_\gamma L^\gamma_{\alpha\beta}.
\end{equation}

**Proposition 2.** The Lie brackets of the adapted basis $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ are
\begin{equation}
[\delta^*_\alpha, \delta^*_\beta] = L^\gamma_{\alpha\beta}\delta^*_\gamma + R_{\alpha\beta\gamma}\mathcal{P}^\gamma, \quad [\delta^*_\alpha, \mathcal{P}^\beta] = -\frac{\partial N_{\alpha\gamma}}{\partial \mu_\beta}\mathcal{P}^\gamma, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0,
\end{equation}
where
\begin{equation}
R_{\alpha\beta\gamma} = \sigma^\gamma_\alpha \frac{\partial N_{\beta\gamma}}{\partial x^\delta} - \sigma^\gamma_\beta \frac{\partial N_{\alpha\gamma}}{\partial x^\delta} + N_{\alpha\delta} \frac{\partial N_{\beta\gamma}}{\partial \mu_\delta} - N_{\beta\delta} \frac{\partial N_{\alpha\gamma}}{\partial \mu_\delta} + L^\gamma_{\alpha\beta}\mathcal{L}^\gamma_{\alpha\beta}.
\end{equation}

**Definition 3.** The curvature of a connection $N$ on $TE^*$ is given by $\Omega = Nh$ where $h$ is defined by (3.1), and $N_h = -\frac{1}{2}[h, h]$ is the Nijenhuis tensor of $h$.

In the local coordinates
\begin{equation}
\Omega = -\frac{1}{2}R_{\alpha\beta\gamma}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{P}^\gamma
\end{equation}
where $R_{\alpha\beta\gamma}$ is given by (3.7) and is called the curvature tensor of $N$.

The curvature is an obstruction to the integrability of $HTE^*$. We have

**Proposition 3.** $HTE^*$ is integrable if and only if the curvature vanishes.

**Remark 1.** Two connections $N$ on $TE^*$ and $N'$ on $TE^*$ are called $\sigma^1$-related if $N \circ \sigma^1 = \sigma^1 \circ N'$. In this case $N(\sigma^1(\delta^*_\alpha)) = \sigma^1(\delta^*_\alpha)$ from which we easily obtain
\begin{equation}
\sigma^1(\delta^*_\alpha) = \sigma^1_\alpha \delta^*_\alpha, \quad N\sigma^1 = \sigma^1_\alpha N_{\alpha\beta},
\end{equation}
where $N_{\alpha\beta}$ are the coefficients of $N$ and $\delta_i = \frac{\partial}{\partial x^i} + N_{\alpha\beta} \frac{\partial}{\partial \mu_\beta}$ is a local adapted frame of the horizontal subbundle $HTE^*$. Also for the curvature tensors of two $\sigma^1$-related connections we have:
\begin{equation}
R_{\alpha\beta\gamma} = \sigma^1_\alpha \sigma^1_\beta R_{ij\gamma},
\end{equation}
where $R_{ij\gamma} = \delta_i(N_{j\gamma}) - \delta_j(N_{i\gamma})$.

### 4 Almost tangent structures and connections

**Definition 4.** An almost tangent structure $J$ on $TE^*$ is a bundle morphism $J : TE^* \rightarrow TE^*$ of $\tau_J : TE^* \rightarrow E^*$, of rank $m$, such that $J^2 = 0$. An almost tangent structure $J$ on $TE^*$ is called adapted if $\text{Im}J = \text{Ker}J = VTE^*$.

Locally, an adapted almost tangent structure is given by $J = \iota_{\alpha\beta}\mathcal{X}^\alpha \otimes \mathcal{P}^\beta$ where
the coefficients matrix \((t_{\alpha\beta}(x, \mu))\) has rank \(m\).

**Proposition 4.** \(J\) is an integrable if and only if

\[
\frac{\partial t_{\alpha\gamma}}{\partial \mu_\beta} = \frac{\partial t_{\beta\gamma}}{\partial \mu_\alpha}
\]

\((4.1)\)

Proof. \(J\) is an integrable if and only if the Nijenhuis tensor \(N_J(\rho, \upsilon) = [J\rho, J\upsilon] - J[J\rho, \upsilon] - J[\rho, J\upsilon] = 0\). This is locally equivalent to

\[
N_J(\mathcal{X}_\alpha, \mathcal{X}_\beta) = \left(t_{\alpha\gamma} \frac{\partial t_{\beta\epsilon}}{\partial \mu_\gamma} - t_{\beta\gamma} \frac{\partial t_{\alpha\epsilon}}{\partial \mu_\gamma}\right) \mathcal{P}^\epsilon, \quad N_J(\mathcal{X}_\alpha, \mathcal{P}^\beta) = N_J(\mathcal{P}^\alpha, \mathcal{P}^\beta) = 0.
\]

Therefore \(J\) is integrable iff \(t_{\alpha\gamma} \frac{\partial t_{\beta\epsilon}}{\partial \mu_\gamma} = t_{\beta\gamma} \frac{\partial t_{\alpha\epsilon}}{\partial \mu_\gamma}\) that is equivalent to \((4.1)\). \(\square\)

**Remark 2.** (i) An adapted almost tangent structure \(J\) on \(TE^*\) is called symmetric if \(\omega(J\rho, \upsilon) = \omega(J\upsilon, \rho)\). Locally, this requires the symmetry of the tensor \(t_{\alpha\beta}\).

(ii) If \(g\) is a pseudo-Riemannian metric on the vertical bundle \(VT E^*\) then there exists an unique symmetric adapted almost tangent structure on \(TE^*\) such that

\[
g(J\rho, J\upsilon) = -\omega(J\rho, \upsilon)
\]

\((4.2)\)

In this case we say that \(J\) is induced by the metric \(g\).

Locally if \(g(x, \mu) = g^{\alpha\beta} \mathcal{P}_\alpha \otimes \mathcal{P}_\beta\) then \(t^{\alpha\beta} = g^{\alpha\beta}\). In particular, any regular Hamiltonian \(H : E^* \to \mathbb{R}\) on \(E^*\) induces a pseudo-Riemannian metric on \(VT E^*\) (the metric tensor is \(g^{\alpha\beta} = \frac{\partial^2 H}{\partial \mu_\alpha \partial \mu_\beta}\)) therefore, it induces an unique symmetric adapted almost tangent structure (denoted \(J_H\)) such that \((4.2)\) is verified. Moreover, this is a tangent structure i.e., \(J_H\) is integrable.

(iii) Any symmetric adapted almost tangent structure \(J\) on \(TE^*\) induces a pseudo-Riemannian metric on the vertical bundle \(VT E^*\) as defined by \((4.2)\).

**Definition 5.** The torsion of a connection \(\mathcal{N}\) is the vector valued two form \(T := [J, h]\) where \(h\) is given by \((3.1)\) and \([J, h]\) is the Frolicher-Nijenhuis bracket.

**Remark 3.** \(T\) is a semibasic vector-valued form. Its local expression is:

\[
T = \frac{1}{2} \left( t_{\alpha\gamma} \frac{\partial N_{\beta\epsilon}}{\partial \mu_\gamma} - t_{\beta\gamma} \frac{\partial N_{\alpha\epsilon}}{\partial \mu_\gamma} + \delta^\alpha_{\beta} (t_{\gamma\epsilon}) - \delta_{\alpha\beta} (t_{\gamma\epsilon}) - L^\epsilon_{\alpha\beta} t_{\gamma\epsilon}\right) \mathcal{X}_\alpha \wedge \mathcal{X}_\beta \otimes \mathcal{P}^\gamma.
\]

**Proposition 5.** Let \(\mathcal{N}\) be a bundle morphism of \(\tau_1 : TE^* \to E^*\), smooth on \(TE^* \setminus \{0\}\). Then \(\mathcal{N}\) is a connection on \(TE^*\) if and only if there exists an adapted almost tangent structure \(J\) on \(TE^*\) such that

\[
J\mathcal{N} = \mathcal{J}, \quad \mathcal{N} J = -\mathcal{J}.
\]

**Definition 6.** Let \(J\) be an adapted almost tangent structure on \(TE^*\). A section \(\rho\) of \(TE^*\) is called \(J\)-regular if

\[
J[\rho, J\upsilon] = -J\upsilon,
\]

for every section \(\upsilon\) of \(TE^*\).

Locally \(\rho = \rho^\alpha \mathcal{X}_\alpha + \rho_\beta \mathcal{P}^\beta\) is \(J\)-regular iff
Example 1. Let $H$ be a regular Hamiltonian on $E^*$. One can associate to $H$ a remarkable $\mathcal{J}_H$-regular section $\rho \in \text{Sec}(\tau_1)$, locally given by

$$\rho = \frac{\partial H}{\partial \mu^\alpha} X^\alpha + \rho_\alpha P^\alpha,$$

which will be called a semi-Hamiltonian section. Moreover, the equation

$$i_{\rho H} \omega = dH,$$

defines an unique $\mathcal{J}_H$-regular section $\rho_H \in \text{Sec}(\tau_1)$ (see [4]) locally given by

$$\rho_H = \frac{\partial H}{\partial \mu^\alpha} X^\alpha - (\sigma^i_\alpha \frac{\partial H}{\partial x^i} + \mu_{\gamma} L_{\alpha\beta}^\gamma \frac{\partial H}{\partial \mu^\beta}) P^\alpha,$$

called the Hamilton section.

Theorem 1. Let $\mathcal{J}$ be an adapted almost tangent structure on $TE^*$. If $\rho$ is a $\mathcal{J}$-regular section of $TE^*$ then

$$N = -\mathcal{L}_\rho \mathcal{J},$$

is a connection on $TE^*$.

Proof. Since $N(v) = -\mathcal{L}_\rho \mathcal{J}(v) = -[\rho, \mathcal{J} v] + \mathcal{J} [\rho, v]$ then $\mathcal{J} N(v) = -\mathcal{J} [\rho, \mathcal{J} v] + \mathcal{J}^2 [\rho, v] = \mathcal{J} \nu$ and $N \mathcal{J}(v) = -[\rho, \mathcal{J}^2 v] + \mathcal{J} [\rho, \mathcal{J} v] = -\mathcal{J} \nu$. By using Proposition 5 we get the proof of the theorem.

Remark 4. The connection (4.7) is induced by $\mathcal{J}$ and $\rho$. Its local coefficients are given by

$$N_{\alpha\beta} = \frac{1}{2} \left( t_{\alpha\gamma} \frac{\partial \rho^\beta}{\partial \mu^\gamma} - \sigma^i_\alpha t_{\gamma\beta} \frac{\partial \rho^\gamma}{\partial x^i} - \rho t_{\alpha\beta} + \rho \rho_{\lambda\beta} \gamma_{\lambda\alpha} \right).$$

Proposition 6. The torsion of the connection (4.7) vanishes.

Proof. We have $T = [\mathcal{J}, h] = \frac{1}{2} (\mathcal{J}, [\mathcal{J}, id] + [\mathcal{J}, -[\rho, \mathcal{J}]]) = \frac{1}{2} [\mathcal{J}, [\mathcal{J}, \rho]]$. Using Jacobi identity we obtain that $T = 0$.

Proposition 7. The connection $N = -\mathcal{L}_{\mu_H} \mathcal{J}_H$ is symmetric.

Proof. Use (4.6) and (4.8) after some computations we get (3.5).

5 Homogeneous connections

Definition 7. An adapted almost tangent structure on $TE^*$ is called homogeneous if $\mathcal{L}_{\mathcal{C}} \mathcal{J} = -\mathcal{J}$.

Notice that $\mathcal{J}$ is homogeneous if the local components $t_{\alpha\beta}(x, \mu)$ are 0-homogeneous with respect to $\mu$.

Proposition 8. Let $\mathcal{J}$ be a homogeneous adapted tangent structure. A section $\rho \in \text{Sec}(\tau_1)$ is $\mathcal{J}$-regular if and only if $\mathcal{J} \rho = \mathcal{C}$.

Proof. If $\rho$ is $\mathcal{J}$-regular then $t^{\alpha\beta} = \frac{\partial \rho^\beta}{\partial \mu^\alpha}$, hence $\rho^\beta$ must be 1-homogeneous with
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respect to \( \mu \), therefore \( \mu_{\alpha} t^{\alpha \beta} = \rho^{\beta} \), that is equivalent to \( J \rho = C \). Vice versa, if \( J \rho = C \) then \( \rho^{\alpha} = \mu_{\alpha} t^{\alpha \beta} \) and thus \( \partial \rho^{\alpha} / \partial \mu_{\alpha} = t^{\alpha \alpha} + \mu_{\beta} \partial t^{\alpha \beta} / \partial \mu_{\alpha} = t^{\alpha \alpha} \).

**Remark 5.** (i) Based on the above result, the local expression for a \( J \)-regular section with \( J \) a homogeneous adapted tangent structure is

\[
\rho = \mu_{\alpha} t^{\alpha \beta} X_{\beta} + \rho_{\gamma} P^{\gamma}.
\]

(ii) \( J \) is homogeneous iff \( CH = 2H \) (i.e. \( H \) is 2-homogeneous on the fibres) and therefore \( H = \frac{1}{2} g^{\alpha \beta} \mu_{\alpha} \mu_{\beta} \). Accordingly \( \rho_{H} = \mu_{\alpha} g^{\alpha \beta} X_{\beta} + \rho_{\gamma} P^{\gamma} \).

(iii) The coefficients of the connection (4.7) generated by \( \rho \) from (5.1) can be written in the following form:

\[
N_{\alpha \beta} = \frac{1}{2} \left( \mu_{\epsilon} t^{\gamma} \left( \sigma_{\gamma}^{\alpha} \frac{\partial t_{\beta}}{\partial x^{\epsilon}} - \delta_{\gamma}^{\alpha} \frac{\partial t_{\alpha \beta}}{\partial x^{\epsilon}} \right) + t_{\alpha \gamma} \frac{\partial \rho_{\beta}}{\partial \mu_{\gamma}} - \mu_{\epsilon} \frac{\partial t_{\alpha \beta}}{\partial \mu_{\epsilon}} + \mu_{\epsilon} t^{\gamma} t_{\lambda \beta} L_{\lambda}^{\gamma} \right).
\]

(iv) If \( \rho \in \text{Sec}(\tau_{1}) \) is given by (5.1) and \( N \) is any connection on \( TE^{\ast} \) then \( \xi = h(\rho) \) is a \( J \)-regular section of \( \tau_{1} \) and is independent of \( \rho \). We call this section associated to \( N \). The local expression of \( \xi \) is

\[
\xi = \mu_{\alpha} t^{\alpha \beta} X_{\beta} + \mu_{\alpha} t^{\alpha \beta} N_{\gamma \beta} P^{\gamma}.
\]

For example the \( J \)-regular section associated to \( N = -L_{\rho} J \) is \( \xi = \frac{1}{2}(\rho + [C, \rho]) \) or locally

\[
\xi = \mu_{\alpha} t^{\alpha \beta} X_{\beta} + \frac{1}{2} \mu_{\alpha} \frac{\partial \rho_{\gamma}}{\partial \mu_{\alpha}} P^{\gamma}.
\]

**References**


Authors’ addresses:

Dragoș Hrimiuc
University of Alberta, Department of Mathematics
T6G 2G1 CAB, Edmonton, Canada
email: hrimiuc@math.ualberta.ca

Liviu Popescu
University of Craiova, Faculty of Economic Sciences
13, Al. I.Cuza st., Craiova, Romania
email: liviu.popescu@central.ucv.ro