On subprojective transformations

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. The aim of this paper is to study subgeodesically related spaces. Using some results of Levi-Civita and Vrâncanu an example of projectively equivalent Riemann metrics is given. ξ-subcharacteristic vector fields are studied for some deformation algebras and it is also illustrated the relation with the concept of ξ-subgeodesically related connexions.

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1 Introduction

Let $M$ be a connected paracompact, smooth manifold of dimension $n \geq 3$. Let $\mathcal{X}(M)$ be the Lie algebra of vector fields on $M$, $T^{(p,q)}(M)$ the $C^\infty(M)$-module of tensor fields of type $(p,q)$ on $M$, $\Lambda^p(M)$ the $C^\infty(M)$-module of $p$-forms on $M$ and $\Delta^p(M)$ the $p$-th de Rham cohomology group of $M$.

Let $\Gamma^i_{jk}$ be the components of an affine symmetric connection $\nabla$ and $\xi^i$ be the components of a vector field $\xi$. One can associate the differential system of equations

$$
\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = a \frac{dx^i}{dt} + b \xi^i,
$$

$a$ and $b$ being functions of $t$, which defines the $\xi$-subgeodesics.

K. Yano introduced the subprojective transformations of connections, which preserve the $\xi$-subgeodesics

$$
\Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_j \omega_k + \delta^i_k \omega_j + \theta_{jk} \xi^i,
$$

where $\omega$ and $\theta_{jk}$ are the components of a 1-form and of a symmetric tensor field of type $(0,2)$, respectively.

Two Riemannian spaces $(M,g)$ and $(M,\mathcal{g})$ are $\xi$-subgeodesically related, the tensor of correspondence $\theta_{jk}$, being $-g_{jk}$, if the Levi-Civita connections associated to $g$ and $\mathcal{g}$ satisfy the Yano formulae (1.2). Therefore there exists a diffeomorphism $f$ between these two spaces which maps $\xi$-subgeodesics onto $\xi$-subgeodesics. $f$ is called the subgeodesic mapping.
If \( \xi^i = 0 \), then the Yano formulae become the Weyl formulae and spaces are geodesically related.

In the present paper subgeodesically and geodesically spaces are considered. The Levi-Civita and Vrăanceanu canonical forms are given for certain projectively equivalent metrics on some Weyl manifolds.

It is also illustrated the close ties that exist between the \( \xi \)-subcharacteristic vector fields and \( \xi \)-subgeodesically related connections.

2 On \( \xi \)-subcharacteristic vector fields

Let \( A \) be a \((1, 2)\)-tensor field on \( M \). The \( \mathcal{C}^\infty(M) \)-algebra if we consider the multiplication rule given by
\[
X \circ Y = A(X, Y), \forall X, Y \in \mathcal{X}(M).
\]
This algebra is denoted by \( \mathcal{U}(M, A) \) and it is called the algebra associated to \( A \). If \( \nabla \) and \( \nabla' \) are two linear connections on \( M \) and \( A = \nabla' - \nabla \), then \( \mathcal{U}(M, A) \) is called the deformation algebra defined by the pair \( (\nabla, \nabla') \).

A vector field \( X \in \mathcal{X}(M) \) is called \( \xi \)-subcharacteristic in the deformation algebra \( \mathcal{U}(M, A) \) if there exists two functions \( \lambda, \mu \in \mathcal{C}^\infty(M) \) such that
\[
(2.1) \quad A(X, X) = \lambda X + \mu \xi.
\]

**Remark 2.1**
1) If \( X \) is a nonvanishing \( \xi \)-subcharacteristic vector field i.e. is a vector field of \( \xi \)-subcharacteristic direction, then (2.1) is equivalent to
\[
A(X, X) \otimes X - X \otimes A(X, X) = \mu(\xi \otimes X - X \otimes \xi).
\]
2) The trajectories of vector fields of \( \xi \)-subcharacteristic directions, called the \( \xi \)-subcharacteristic curves, satisfy the following differential system of equations
\[
(2.2) \quad B_{i j k p q r s}^{l m n o} \frac{dx^k}{dt} \frac{dx^s}{dt} \frac{dx^r}{dt} \frac{dx^h}{dt} = 0,
\]
where 
\[
B_{i j k p q r s}^{l m n o} = (A_{i k}^p \delta_j^q - A_{j k}^i \delta_i^p)(\delta_{k r}^s \xi^p - \delta_{r k}^s \xi^p) - (A_{k p}^i \delta_l^q - A_{q p}^k \delta_i^l)(\delta_{k s}^r \xi^j - \delta_{s k}^r \xi^j).
\]

The geometric interpretation of vector fields of \( \xi \)-subcharacteristic direction is given by the following result

**Proposition A**[8] Let \( \nabla \) and \( \nabla' \) be two symmetric linear connections on \( M \) and \( \xi \in \mathcal{X}(M) \). Let \( X \in \mathcal{X}(M), X_p \neq 0, \forall p \in M \) such that \( X \) and \( \xi \) are either independent \( \forall p \in M \) or dependent \( \forall p \in M \). The following assertions are equivalent:
1) \( X \) is a vector field of \( \xi \)-subcharacteristic direction in the deformation algebra \( \mathcal{U}(M, \nabla' - \nabla) \).
2) Let any \( p \in M \). If \( c \) is a \((\xi, \nabla')\)-subgeodesic verifying
\[
c(t_0) = p, \quad \frac{dc}{dt} \bigg|_{t_0} = aX_p, a \in \mathbb{R}^*,
\]
then the point \( p \) is \((\xi, \nabla')\)-subgeodesic i.e. \( \xi_p \) belongs to the osculating plane of the curve \( c \) at \( p \).
The following result illustrates the relation between the $\xi$-subcharacteristic vector fields and the $\xi$-subgeodesically related connections:

**Proposition B** [8] Let $\nabla$ and $\nabla'$ be two symmetric linear connections on $M$ and $\xi \in \mathcal{X}(M)$. The following assertions are equivalent:

1) All the elements of the algebra $\mathcal{U}(M, \nabla' - \nabla)$ are $\xi$-subcharacteristic vector fields.

2) In every point $p \in M$ there exists a curve $\xi$-subcharacteristic tangent to a given direction.

3) There exists a symmetric $(0,2)$-tensor field $\theta$ and a 1-form $\omega$ on $M$ such that

$$\nabla' X Y - \nabla X Y = \omega(X)Y + \omega(Y)X + \theta(X,Y)\xi, \forall X,Y \in \mathcal{X}(M).$$

4) $\nabla'$ and $\nabla$ have the same $\xi$-subgeodesics.

3 On geodesically and subgeodesically related Riemann spaces

Let $g$ be a Riemannian metric on $M$. A Weyl manifold is a triple $(M, \hat{g}, W)$, where $\hat{g} = \{e^u g \mid u \in C^\infty(M)\}$ is the conformal class defined by $g$ and $W : \hat{g} \rightarrow \Lambda^1(M)$ is a Weyl structure on the conformal manifold $(M, \hat{g})$, hence

$$W(e^u g) = W(g) - du, \forall u \in C^\infty(M).$$

A linear connection $\nabla$ on $M$ is compatible with the Weyl structure $W$ if

$$\nabla g + W(g) \otimes g = 0.$$ (3.2)

There exists a unique torsion free linear connection $\nabla^W$, verifying (3.2), given by the formula:

$$2g(\nabla X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) +$$ $$(3.3) + W(g)(X)g(Y,Z) + W(g)(Y)g(X,Z) - W(g)(Z)g(X,Y) +$$ $+ g([X,Y],Z) + g([Z,X],Y) - g([Z,Y],X), \forall X,Y,Z \in \mathcal{X}(M).$$

$\nabla^W$ is called the Weyl conformal connection. This connection is invariant under a "gauge transformation" $g \rightarrow e^u g$. So, the 1-form $W(g)$ is required to change by (1.1).

Weyl introduced a 2-form $\psi(W)$ on $M$ by setting $\psi(W) = dW(g)$, $g \in \hat{g}$, and called it the distance curvature. This is a gauge invariant. If $\psi(W) = 0$, then by (1.1), the cohomology class $[W(g)] \in H^1(M)$ of the closed form $W(g)$ does not depend on the choice of a metric in $\hat{g}$. For simplicity, we write $ch(W) = [W(g)]$.

The 2-form $\psi(W)$ and the class $ch(W)$ are the obstructions for a Weyl structure to be a Riemannian structure. Indeed:

**Proposition C** [2] Let $(M, \hat{g}, W)$ be a Weyl manifold and $\nabla^W$ be the Weyl conformal connection. Then the following two conditions are equivalent:

1) $\psi(W) = 0$ and $ch(W) = 0;$
2) There is a Riemann metric in \( \hat{\mathcal{g}} \) such that \( \nabla^W g = 0 \).

Let \( \pi \) be a 1-form on \( M \). We denote by \( \nabla^L \) the connection compatible with the Weyl structure \( W \), which is \( \pi \)-semi-symmetric i.e.
the torsion tensor is required to be \( L (X, Y) = \pi(Y)X - \pi(X)Y, \forall X, Y \in \mathcal{X}(M) \) and

\[
2g(\nabla^Y X, Z) = X( g(Y, Z)) + Y( g(X, Z)) - Z( g(X, Y)) + W(g)(X, Y, Z) - \frac{1}{2} W(g)(X, Y, Z) - \frac{1}{2} g(X, Y) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X)
\]

holds. The relation between these two connections is given by

\[
\nabla^L X = \nabla^W X + \pi(X)X - g(X, Y)\pi^X,
\]

where \( g(Z, \pi^X) = \pi(Z), \forall Z \in \mathcal{X}(M) \).

We denote by \( \nabla^\ast \) the transposed connection of \( \nabla^L \) i.e.

\[
\nabla^\ast X = \nabla^W X + [X, Y].
\]

The relations (3.5) and (3.6) lead to

\[
\nabla^L X = \nabla^W X + \pi(X)X - g(X, Y)\pi^X.
\]

Let us denote by \( \nabla^\sigma \) the symmetric connection associated to \( \nabla^L \) i.e.

\[
\nabla^\sigma = \frac{1}{2}(\nabla^W + \nabla^L).
\]

Hence

\[
\nabla^\sigma X = \nabla^W X + \frac{1}{2} \pi(X)X + \frac{1}{2} g(X, Y)\pi^X.
\]

Let \( (M, g) \) be a Riemannian manifold. Let \( (M, \hat{\mathcal{g}}, W) \) be a Weyl manifold and \( \pi \in \wedge^1(M) \). Let \( \nabla^\sigma \) be the Levi-Civita connection associated to \( g \). From (3.3) one gets

\[
\nabla^\sigma X = \nabla^W X + \phi(X)Y + \phi(Y)X - g(X, Y)\phi^X
\]

where \( 2\phi = W(g) \) and \( g(\phi^X, X) = \phi(X), \forall X \in \mathcal{X}(M) \). The relation (3.8) leads to

\[
\nabla^\sigma X = \nabla^W X + (\phi + \frac{1}{2}\pi)(X)Y + (\phi + \frac{1}{2}\pi)(Y)X - g(X, Y)(\pi + \phi)^X.
\]

Let us suppose that \( \nabla^\sigma \) is the Levi-Civita connection associated to another Riemannian metric \( \hat{\mathcal{g}} \) on \( M \). Let \( g_{ij}, \hat{g}_{ij}, \phi_i, \pi_i \) be the local components of \( g, \hat{g}, \phi \) and \( \pi \) respectively, in a local system of coordinates \( (x^1, \ldots, x^n) \). We denote with \( \tilde{\Gamma}^{ik}_{jk} \) the Christoffel symbols of the metrics
The relation (3.10) becomes

\[ ds^2 = g_{ij} dx^i dx^j, \]
\[ d\tilde{s}^2 = \tilde{g}_{ij} dx^i dx^j. \]

The relation (3.10) becomes

\[ (3.10)' \quad \begin{vmatrix} i \\
\end{vmatrix} jk = \begin{vmatrix} i \\
\end{vmatrix} jk + \delta_j^i (\varphi_k + \frac{1}{2} \pi_k) + \delta_k^i (\varphi_j + \frac{1}{2} \pi_j) - g_{jk} (\pi^i + \varphi^i), \]

where \( \pi^i = g^{ij} \pi_j, \varphi^i = g^{ij} \varphi_j. \) Considering \( i = j \) in (3.10)' and summing, one gets

\[ (3.10)'' \quad n \varphi_k + \frac{n-1}{2} \pi_k = \begin{vmatrix} i \\
\end{vmatrix} ik - \begin{vmatrix} i \\
\end{vmatrix} ik = \frac{\partial}{\partial x^k} \left( \ln \left( \frac{\det (\tilde{g}_{ij})}{\det (g_{ij})} \right) \right). \]

Let us denote with \( \xi = (\pi + \varphi)^i \). The formula (3.10)' implies

\[ (3.12) \quad \begin{vmatrix} i \\
\end{vmatrix} jk = \begin{vmatrix} i \\
\end{vmatrix} jk + \delta_j^i \omega_k + \delta_k^i \omega_j - g_{jk} \xi^i, \]

where \( \omega_i = \varphi_i + \frac{1}{2} \pi_i, \xi^i = \varphi^i + \pi^i. \) Therefore the metrics (3.11) are \( \xi^i \)-subgeodesically related. There exist differentiable mappings \( u \) and \( h \), with variables \( (x^1, \ldots, x^n) \), such that \( \xi_i = \frac{\partial u}{\partial x^i} \) and \( \omega_i = \frac{\partial h}{\partial x^i} \).

We consider \( \tilde{g} = e^{2\nu} g. \) One has

\[ (3.13) \quad \begin{vmatrix} i \\
\end{vmatrix} jk = \begin{vmatrix} i \\
\end{vmatrix} jk + \delta_j^i \xi_k + \delta_k^i \xi_j - g_{jk} \xi^i, \]

where \( \begin{vmatrix} i \\
\end{vmatrix} jk \) are the Christoffel symbols associated to \( \tilde{g}. \) Therefore one gets

\[ (3.14) \quad \begin{vmatrix} i \\
\end{vmatrix} jk = \begin{vmatrix} i \\
\end{vmatrix} jk + \delta_j^i \sigma_k + \delta_k^i \sigma_j, \]

where \( \sigma_i = \omega_i - \xi_i. \) Hence the metrics

\[ (3.15) \quad d\tilde{s}^2 = \tilde{g}_{ij} dx^i dx^j, \quad d\tilde{s}^2 = \tilde{g}_{ij} dx^i dx^j \]

are geodesically related. So, the metrics (3.15) can be reduced to the canonical forms of Levi-Civita and Vranceanu (according to the fact that the Riemann space \( (M, \tilde{g}) \) is of cathegory \( n \) or cathegory \( m < n \)).

\[ (3.16) \quad dV^2 = a_1 (x^1) f'(x^1)^2 + \ldots + a_n (x^n) f'(x^n)^2, \]
\[ dL^2 = \frac{1}{x_1 \ldots x_n} \left( \frac{a_1 (x^1) f'(x^1)^2}{x_1^2} + \ldots + \frac{a_n (x^n) f'(x^n)^2}{x_n^2} \right), \]

where \( f(x) = (x - x^1) \ldots (x - x^n) \) or

\[ (3.17) \quad dV^2 = a_1 (x^1) f'(x^1)^2 + F^2 c_{\lambda \mu} (x^{m+1}, \ldots, x^p) dx^\lambda dx^\mu + \]
\[ + F(k^2) c_{\alpha \beta} (x^{p+1}, \ldots, x^n) dx^\alpha dx^\beta, \]
\[ dL^2 = \frac{1}{x_1 \ldots x_p} \left( \frac{a_1 (x^1) f'(x^1)^2}{x_1^2} + \frac{F^2}{c^2} c_{\lambda \mu} (x^{m+1}, \ldots, x^p) dx^\lambda dx^\mu + \right. \]
\[ + \left. F(k^2) c_{\alpha \beta} (x^{p+1}, \ldots, x^n) dx^\alpha dx^\beta \right), \]
where \( F(x) = (x - x^1) \cdot \ldots \cdot (x - x^m), 1 \leq i \leq m, m + 1 \leq \lambda, \mu \leq p, p + 1 \leq \alpha', \beta' \leq n \) and \( c^2 \) and \( k^2 \) are nonvanishing constants. Therefore the metrics (3.11) can be reduced to

\[
 ds^2 = e^{-2u(x^1, \ldots, x^n)} dV^2, \quad d\tilde{s}^2 = dL^2.
\]

Hence we obtain:

**Theorem 3.1** Let \( (M, g) \) be a Riemannian space and \( W \) a Weyl structure on the conformal manifold \( (M, \tilde{g}) \). Let \( \pi \) be a 1-form on \( M, \tilde{\nabla} \) be the \( \pi \)-semi-symmetric conformal connection, \( \tilde{\nabla} \) be the symmetric connection associated to \( \tilde{\nabla} \). We suppose that \( \tilde{\nabla} \) is the Levi-Civita connection associated to another Riemannian metric \( \tilde{g} \) on \( M \). Then

i) The 1-forms \( W(g) \) and \( \pi \) are exact.

ii) The metrics (3.11) can be reduced to

\[
 ds^2 = e^{-2u(x^1, \ldots, x^n)} dV^2, \quad d\tilde{s} = dL^2,
\]

where \( dV^2 \) and \( dL^2 \) are the canonical forms of Levi-Civita and Vr˘ anceanu, given by (3.16) or (3.17), according to the case when the equation

\[
 \det(\tilde{g}_{ij} - r^2 g_{ij}) = 0
\]

has distinct roots or has \( m < n \) equal roots.

**Remark 3.1.** Let us consider the first formula (3.17) for \( c = k \). Multiplying all the variables \( x^1, \ldots, x^n \) with the same constant, we can suppose that \( c \) is the unit. Therefore the metric \( dV^2 \) can be written

\[
 (3.18) \quad dV^2 = a_i(x^i)F'(x^i)(dx^i)^2 + F(1)c_{\alpha\beta}dx^\alpha dx^\beta.
\]

One gets the next result, under the same hypothesis of the previous theorem:

**Theorem 3.2.** The metric \( ds^2 = g_{ij}dx^idx^j \) can be written

\[
 ds^2 = e^{-2u(x^1, \ldots, x^n)} dV^2, \quad dV^2 \text{ is given by the first formula of (3.16) or by the expression (3.18),}
\]

if the equation \( \det(\tilde{g}_{ij} - r^2 g_{ij}) = 0 \) has distinct roots or has \( m < n \) equal roots, respectively.

The last result underlines the connection between the concept of \( \xi \) - subcharacteristic vector fields and of those of deformation algebra on Weyl manifolds:

**Theorem 3.3.** Let \( (M, g) \) be a Riemannian space and \( W \) a Weyl structure on the conformal manifold \( (M, \tilde{g}) \). Let \( \pi \) be a 1-form on \( M, \tilde{\nabla} \) be the \( \pi \)-semi-symmetric conformal connection, \( \tilde{\nabla} \) be the symmetric connection associated to \( \tilde{\nabla} \). We suppose that \( \tilde{\nabla} \) is the Levi-Civita connection associated to another Riemannian metric \( \tilde{g} \) on \( M \). Let \( \tilde{\nabla} \) be a connection conformally related to the Levi-Civita connection \( \nabla \).

Then the deformation algebras \( \hat{U}(M, \tilde{\nabla} - \tilde{\nabla}) \) and \( \hat{U}(M, \tilde{\nabla} - \nabla) \) have the same \( \xi \)-subcharacteristic vector fields, where \( \xi = (\pi + 12W(g))^\sharp \).

**Proof.** One considers

\[
 \tilde{A} = \tilde{\nabla} - \tilde{\nabla} \quad \text{and} \quad \hat{A} = \tilde{\nabla} - \nabla.
\]

\( \tilde{\nabla} \) and \( \hat{\nabla} \) being geodesically related, one has

\[
 \hat{A}(X, Y) - \tilde{A}(X, Y) = \frac{1}{2} \pi(X)Y + \frac{1}{2} \pi(Y)X.
\]

Let \( X \in \hat{U}(M, \hat{A}) \) be a \( \xi \)-subcharacteristic vector field. So, there exist \( \lambda, \mu \in \mathcal{C}^\infty(M) \) such that \( \hat{A}(X, X) = \lambda X + \mu \xi \), where \( \lambda = (W(g) + \pi)(X) \) and \( \mu = -g(X, X) \).
Therefore $\tilde{A}(X, X) = \nu X + \mu \xi$, where $\nu = (W(g) + \frac{3}{2}\pi)(X)$ and $X$ is a $\xi$-subcharacteristic vector field of the algebra $\mathcal{U}(M, \tilde{A})$.

The converse inclusion is analogous.

References


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