Some properties of strongly $S$-decomposable operators

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Abstract

In this paper we describe several basic properties of strongly $S$-decomposable operators, namely their behaviour regarding: the direct sums, restrictions, quotients, the Riesz-Dunford functional calculus and the quasinilpotent equivalence.

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Introduction

Let $X$ be a Banach space, $B(X)$ the algebra of all linear bounded operators on $X$, and $\mathbb{C}$ the field of complex numbers. An operator $T \in B(X)$ is said to have the single-valued extension property, if for any analytic function $f : D_f \to X$, $D_f \subset \mathbb{C}$ open, with $(\lambda I - T)f(\lambda) \equiv 0$ implies $f(\lambda) \equiv 0$, ([3], [5]).

For an operator $T \in B(X)$ having the single-valued extension property and for $x \in X$ we can consider the set $\rho_T(x)$ of elements $\lambda_0 \in \mathbb{C}$ such that there exists an analytic function $\lambda \to x(\lambda)$ defined in a neighborhood of $\lambda_0$ with values in $X$ which verifies $(\lambda I - T)x(\lambda) \equiv x$; $x(\lambda)$ is unique, $\rho_T(x)$ is open and $\rho(T) \subset \rho_T(x)$. Take $\sigma_T(x) = C\rho_T(x) = \mathbb{C} \setminus \rho(x)$ and $X_T(F) = \{x \in X|\sigma_T(x) \subset F\}$ where $F \subset \mathbb{C}$ is closed. $\rho_T(x)$ is named the local resolvent set of $x$ with respect to $T$ and $\sigma_T(x)$ the spectrum of $x$ with respect to $T$.

If $T \in B(X)$ and $Y$ is an invariant (closed) subspace of $T$, let us denote by $T|Y$ the restriction of $T$ to $Y$. In what follows by subspace of $X$, we mean a closed linear manifold of $X$. Recall that $Y$ is a spectral maximal space of $T$ if it is an invariant subspace of $T$ such that for any other invariant subspace $Z$ of $T$, the inclusion $\sigma(T|Z) \subset \sigma(T|Y)$ implies $Z \subset Y$, ([3]).

An open set $\Omega \subset \mathbb{C}$ is of analytic uniqueness of $T \in B(X)$ if for any open set $w \subset \Omega$ and any analytic function $f_0 : w \to X$ satisfying the equation $(\lambda I - T)f(\lambda) \equiv 0$, there follows $f_0(\lambda) \equiv 0$ in $w$. For $T \in B(X)$ there exists a unique maximal open $\Omega_T$ of analytic uniqueness ([8], 2.1). We shall denote $S_T = C\Omega_T = \mathbb{C} \setminus \Omega_T$ and call this the analytic residuum of $T$. For $x \in X$, a point $\lambda \in \delta_T(x)$, if in a neighbourhood $V_\lambda$ of $\lambda$ there exists at least an analytic function $f_x$ (called $T$-associated with $x$) such that...
\[(\mu I - T)f_x(\mu) = x\]

for \(\mu \in V_\lambda\). We shall put \(\gamma_T(x) = C\delta_T(x), \rho_T(x) = \delta_T \cap \Omega_T, \sigma_T(x) = C\rho_T(x) = \gamma_T \cup S_T\) and

\[X_T(F) = \{x; x \in X, \sigma_T(x) \subset F\}\]

where \(S_T \subset F \subset C, ([8])\). \(T \in B(X)\) has the single-valued extension property if and only if \(S_T(x) = \delta_T(x)\) a unique analytic function \(x(\lambda)\), \(T\)-associated with \(x\), for any \(x \in X\). We recall that if \(T \in B(X), S_T \neq \emptyset\) and \(X_T(F)\) is closed for \(F \subset C\) closed \((F \supset S_T)\) then \(X_T(F)\) is a spectral maximal space of \(T\) ([8], Proposition 2.4 and 3.4).

**Definition 1.** A family of open sets \(\{G_S \cup \{G_i\}_i^*\}\) is an \(S\)-covering of the closed set \(\sigma \subset C\) if

\[G_S \cup \left( \bigcup_{i=1}^n G_i \right) \supset \sigma \cup S\quad \text{and} \quad \tilde{G}_i \cap S = \emptyset\]

\((i = 1, 2, \ldots, n), ([8])\).

**Definition 2.** Let \(T \in B(X)\) and \(S \subset \sigma(T)\) be a compact set. \(T\) is called \(S\)-decomposable (see also [2]) if for any finite, open \(S\)-covering \(\{G_S \cup \{G_i\}_i^*\}\) of \(\sigma(T)\), there exists a system \(\{Y_S \cup \{Y_i\}_i^*\}\) of spectral maximal spaces of \(T\) such that

(i) \(\sigma(T|Y_S) \subset G_S, \quad \sigma(T|Y_i) \subset G_i\) \((1 \leq i \leq n)\),

(ii) \(X = Y_S + \sum_{i=1}^n Y_i\).

\(T\) is strongly \(S\)-decomposable operator if (ii) is replaced by

\[(ii') Z = Z \cap Y_S + \sum_{i=1}^n Z \cap Y_i\]

If \(S = \emptyset\) we have decomposable (strongly) operators where \(Z\) is any spectral maximal space of \(T\).

**Lemma 3.** Let \(T \in B(X)\) be a strongly \(S\)-decomposable operator and \(Y\) a spectral maximal space of \(T\) with \(\sigma(T|Y) \supset S\). If \(Z\) is spectral maximal space of \(T\) \((T\) being the operator induced by \(T\) in \(X = X/Y)\), then \(Z = \varphi^{-1}(\tilde{Z})\) is a spectral maximal space of \(T\), where \(\varphi: X \to \tilde{X}\) is the canonical map.

**Proof.** We have \(S_T = \emptyset\) (see 1.1.9, [2]). If \(Z \supset Y\) and \(Z\) is an invariant to \(T\) linear (closed) subspace of \(X, Y\) is also a spectral maximal space of \(T\) (see [2], 1.2) hence \(S \subset \sigma(T|Y) \supset \sigma(T|Z)\), that is \(X_T|\sigma(T|Z)\) \(\supset Y\) is a spectral maximal space of \(T\). By Lemma 2.6.6 of [2], there follows

\[\sigma(T|X_T(\sigma(T|Z))) = \sigma(T|X_T(\sigma(T|Z))) \setminus \sigma(Y|Y)\]

But \(\sigma(T|X_T(\sigma(T|Z)))\) and \(\sigma(T|Y) = \sigma(T|Z) \cup \sigma(T|Y)\), since \(Y\) is a spectral maximal space of \(T|Z\); consequently we have

\[\sigma(T|X_T(\sigma(T|Z))) = (\sigma(T|Y) \cup \sigma(T|Y)) \setminus \sigma(T|Y) \subset (T|Z)\]

From the equalities \(\tilde{T}|X_T(\sigma(T|Z)) = \tilde{T}|X_T(\sigma(T|Z))\) and \(\tilde{T}|Z = \tilde{T}|Z\) we obtain \(\varphi(X_T(\sigma(T|Z)) \subset \varphi(Z)\), hence \(X_T(\sigma(T|Z)) \subset \tilde{Z}\); then \(Z = X_T(\sigma(T|Z))\), and hence \(Z\) is a spectral maximal space of \(T\).
Theorem 4. Let $T \in B(X)$ be a strongly $S$-decomposable operator and $Y$ a spectral maximal space of $T$ with $\sigma(T|Y) \supset S$. Then $\hat{T}$ is a strongly $S_1$-decomposable operator, where $S_1 = S \cap \sigma(\hat{T})$, and $\hat{T}$ is the operator induced by $T$ in $X/Y$.

Proof. Let $\{G_i\} \cup \{G_i\}_{i=1}^n$ be an open $S_1$-covering of $\sigma(\hat{T})$ and $G_S = G_{S_1} \cup \rho(\hat{T})$; we can suppose that $G_i \cap D = \emptyset$ ($i = 1, 2, \ldots, n$). Then $\{G_i\} \cup \{G_i\}_{i=1}^n$ is a $S$-covering of $\sigma(T)$. Let $\{Y_i\} \cup \{Y_i\}_{i=1}^n$ be the corresponding system of spectral maximal spaces of $T$, such that

$$\sigma(T|Y_i) \subset G_{S_i}, \quad \sigma(T|Y_i) \subset G_i, \quad (i = 1, 2, \ldots, n)$$

and

$$X = Y_S + \sum_{i=1}^n Y_i.$$ 

We shall set $S_\alpha = x \sigma(T|Y_S) \cup \sigma(T|Y), \quad S_\alpha = \sigma(T|Y_S) \cup \sigma(T|Y), \quad (i = 1, 2, \ldots, n)$; $Z_S = X_T(\sigma_S), (i = 1, 2, \ldots, n)$, and $Z_i = X_T(\sigma_i), (i = 1, 2, \ldots, n)$ are spectral maximal spaces of $T$ (we have $\sigma_S \supset S, \quad \sigma_i \supset S$, see Theorem 2.1.3, [2]) and $Y \subset Z_S, Y \sigma Y_i$. Consequently $Z_S, Z_i$ are spectral maximal spaces of $T$ ([4], 3.2) and by Lemma 2.6.6 from [2], we obtain

$$\sigma(T|Z_S) = \sigma(T|Y_S) \supset \sigma(T|Y),$$

and analogously

$$\sigma(T|Z_i) = \sigma(T|Y_i) \supset \sigma(T|Y_i), \quad (i = 1, 2, \ldots, n).$$

If $\hat{T}$ is an arbitrary spectral maximal space of $\hat{T}$, then $Z = \varphi^{-1}(\hat{T})$ is a spectral maximal space of $T$ (where $\varphi$ is the canonical map; see the preceding lemma). Hence

$$\hat{T} = \varphi(\hat{T}|Z) = \varphi(Y_S \cap Z) + \varphi(Y_1 \cap Z) + \ldots + \varphi(Y_n \cap Z) = Z.$$

But from the inclusions $\hat{Y}_S \subset \hat{Z}_S \subset \hat{Y}_i \subset \hat{Z}_i$, $\varphi(Y_S \cap Z) \subset \hat{Y}_S \cap \hat{Z}$, $\varphi(Y_1 \cap Z) \subset \hat{Y}_1 \cap \hat{Z}$, $\varphi(Y_n \cap Z) \subset \hat{Y}_n \cap \hat{Z}$, and hence $\hat{T}$ is strongly $S_1$-decomposable.

Corollary 5. Let $T \in B(X)$ be a strongly $S$-decomposable operator and $Y$ a spectral maximal space of $T$ such that $\sigma(T) \cap S = \emptyset$; then $\hat{T}$ is a strongly decomposable operator.

Proof. It follows by the preceding Theorem, since $S_1 = \emptyset$.

Proposition 6. Let $T_\alpha \in B_\alpha(X)$ two strongly $S_\alpha$-decomposable operators ($\alpha = 1, 2$); then $T = T_1 \oplus T_2$ is a strongly $S$-decomposable operator, where $S = S_1 \cup S_2$.

Proof. By Proposition 2.6.2 and [Theorem 2.2.3, [2]] it follows that it will suffice to show that $T$ satisfies strongly the condition $\beta_S$ (see Definition, [2]). Let $Y$ be a spectral maximal space of $T$ and $G = \{G_{S_\alpha}\} \cup \{G_i\}_{i=1}^n$ an open $S'$-covering of $\sigma(T|Y)$, where $S' = S \cap \sigma(T|Y)$. Then, in accordance with Proposition 2.1.7 from [2], we have $Y = Y_1 \oplus Y_2$, where $Y_\alpha$ is a spectral maximal space of $T_\alpha (\alpha = 1, 2)$. If $y \in Y$, then $y = y_1 \oplus y_2$, with $y_\alpha \in Y_\alpha, (\alpha = 1, 2)$; since $T_\alpha (\alpha = 1, 2)$ are strongly $S$-decomposable it follows that $T_\alpha|Y_\alpha$ verifies the condition $\beta_S$, where $S'_\alpha = S_\alpha \cap \sigma(T_\alpha|Y_\alpha), (\alpha = 1, 2)$. Consequently

$$y_\alpha = y_\alpha^3 + y_1^\alpha + \ldots + y_n^\alpha \quad (\alpha = 1, 2).$$
and
\[ \gamma_T(y_{2i}^\alpha) = \gamma_{T_{\alpha}|Y_i}(y_{2i}^\alpha) \subset G_{S'}, \quad (\alpha = 1, 2), \]
\[ \gamma_T(y_i^\alpha) = \gamma_{T_{\alpha}|Y_i}(y_i^\alpha) \subset G_i, \quad (\alpha = 1, 2; \ i = 1, 2, \ldots, n). \]

This yields
\[ y = y_1^1 \oplus y_2^2 = (y_{21}^1 + y_1^1 + \ldots + y_n^1) + (y_{22}^2 + y_1^2 + \ldots + y_n^2) = \]
\[ (y_{21}^1 \oplus y_{22}^2) + (y_1^1 \oplus y_1^2) + \ldots + (y_n^1 \oplus y_n^2) = y_S' + y_1 + \ldots + y_n \]

and
\[ \gamma_T(y_i^\alpha) = \gamma_{T_{\alpha}|Y}(y_i^\alpha) = \gamma_{T_{\alpha}|Y_1}(y_i^1) \cup \gamma_{T_{\alpha}|Y_2}(y_i^2) \subset G_i, \quad (1 \leq i \leq n) \]

hence \( T \) satisfies strongly the condition \( \beta_S \).

Definition 7. A \( S \)-decomposable operator \( T \in B(X) \) is said to be *almost strongly \( S \)-decomposable* if for any spectral maximal space \( Y \) of \( T \) such that \( \sigma(T|Y) \cap S = \emptyset \) or \( \sigma(T|Y) \supset S \), we have that restriction \( T|Y \) is a decomposable respectively \( S \)-decomposable operator.

Remark 8. The need to state the definition is justified by the following: being given a \( S \)-decomposable (strongly \( S \)-decomposable) operator, we know about the existence of the spectral maximal spaces \( Y \) of \( T \), that have the property that \( \sigma(T|Y) \cap S = \emptyset \) or \( \sigma(T|Y) \supset S \); these are the spaces which result from the relations \( Y \oplus X_T(S) = X_T(\sigma(T|Y) \cup S) \) or \( Y = X_T(\sigma(T|Y)) \). However, we know nothing about the existence of the spectral maximal spaces \( Y \) of \( T \) that have the property that \( \sigma(T|Y) \cap S = S' \) is a separated part of \( S \) (open and closed in \( S \)). Obviously strongly \( S \)-decomposable operators are almost strongly \( S \)-decomposable. It seems that strong \( S \)-decomposability (unlike the strong decomposability) has no such favourable demeanour as the one of the \( S \)-decomposability (considering the properties from Definition 2.2.1 and Proposition 2.2.17, [2]).

Proposition 9. Let \( T = T_1 \oplus T_2 \in B(X_1 \oplus X_2) \) be a strongly \( S \)-decomposable operator; then \( T_{\alpha}, \ (\alpha = 1, 2) \) are almost strongly \( S_{\alpha} \)-decomposable, where \( S_{\alpha} = S \cap \sigma(T_{\alpha}), \ (\alpha = 1, 2) \).

Proof. It will suffice to prove that if \( F \subset \sigma(T_1) \) and \( F \cap S_1 = \emptyset \) or \( F \supset S_1 \), then we also have \( F \cap S = \emptyset \) or, respectively, \( (F \cup S) \cap \sigma(T_1) \supset S_1 \). If \( F \cap S_1 = \emptyset \), we also have
\[ F \cap S = (F \cap S) \cap \sigma(T_1) = F \cap (S \cap \sigma(T_1)) = F \cap \sigma(T_1) = \emptyset, \]
hence when \( \sigma(T_1|Y) \cap S_1 = \emptyset \) we also have \( \sigma(T_1|Y) \cap S = \emptyset \) (where \( Y \) is a spectral maximal space of \( T_1 \)). But it also follows that
\[ X_{T_1 \oplus T_2}(\sigma(T_1|Y_1) \cup S) = X_{T_1}(\sigma(T_1|Y_1) \cup S) \oplus X_{T_2}(\sigma(T_1|Y_1) \cup S) = \]
\[ = [Y_1 + X_{T_1}(S)] \oplus [Y_2 + X_{T_2}(S)] = X_{T_1 \oplus T_2}(S) + Y \]
and we can easily verify that \( Y = Y_1 \oplus Y_2, \ T \in T|Y_1 \oplus Y_2 \) being decomposable, by Proposition 2.2.6 from [2], there follows that \( T_1|Y_1 \) is decomposable. Let now \( Y_1 \) be a maximal space of \( T_1 \) such that \( \sigma(T_1|Y_1) \supset S_1 \). Then we have
\[ X_{T_1 \oplus T_2}(\sigma(T_1|Y_1) \cup S) = X_{T_1}(\sigma(T_1|Y_1) \cup S) \oplus X_{T_2}(\sigma(T_1|Y_1) \cup S) = \]
\begin{align*}
&= X_{T_1}((\sigma(T_1|Y_1) \cup S)) \cap \sigma(T_1)) \oplus X_{T_2}(\sigma(T_1|Y_1) \cup S) = Y_1 \oplus X_{T_2}(\sigma(T_1|Y_1) \cup S)
\end{align*}
whence it results \( T_1|Y_1 \) is \( S_1 \)-decomposable. Analogously, we verify that \( T_2 \) is almost strongly \( S_2 \)-decomposable.

**Theorem 10.** Let \( T = T_1 \oplus T_2 \in B(X_1 \oplus X_2) \) be a strongly decomposable operator. Then \( T_1 \) and \( T_2 \) are strongly decomposable.

**Proof.** This follows by Propositions 4 and 7.

**Proposition 11.** Let \( T \in B(X) \) be a strong \( S \)-decomposable operator and \( P \in B(X) \) a projection commuting with \( T \). Then \( T|PX \) is almost strong \( S \)-decomposable, where \( S = \sigma(T)PX \cap S \).

**Proof.** We have \( X = X_1 \oplus X_2, T = T_1 \oplus T_2 \), where \( X_1 = PX, X_2 = (I - P)X \), \( T_1 = T|X_1, T_2 = T|X_2 \) and by Proposition 7 we have that \( T|PX \) is almost strong \( S_1 \)-decomposable.

**Corollary 12.** Let \( T \in B(X) \) be a strongly decomposable operator and \( P \in B(X) \) a projection. Then \( T|PX \) is strongly decomposable.

**Proof.** This follows from the preceding proposition.

**Proposition 13.** Let \( T \in B(X) \) be a strong \( S \)-decomposable operator and let \( \sigma \) be a separated part of \( \sigma(T) \). Then \( T|E(\sigma,T)X \) is strongly \( S_1 \)-decomposable, where \( S_1 = S \cap \sigma \) (for \( E(\sigma,T) \) see Corollary 2.2.8 from [2])

**Proof.** \( X_1 = E(\sigma,T)X \) is a spectral maximal space of \( T \). Let \( Y_1 \) be a spectral maximal space of \( T|X_1 \). Then by [2, Proposition 1.2] this is also a spectral maximal space of \( T \), hence \( T|X_1 \) is \( S_1 \)-decomposable, where \( S_1' = \sigma(T|Y_1) \cap S \). But \( \sigma(T|Y_1) \cap S_1 = \sigma(T|Y_1) \cap (\sigma \cap S) = (\sigma(T|Y_1) \cap \sigma) \cap S = S_1' \), hence \( (T|X_1)|Y_1 \) is \( S_1' \)-decomposable, that is \( T|E(\sigma,T)X \) is strongly \( S_1 \)-decomposable.

**Proposition 14.** Let \( T \in B(X) \) be a strong \( S \)-decomposable operator and let \( f: G \to C \) (\( G \supset \sigma(T) \) open and connected) be an analytic function, injective on \( \sigma(T) \). Then \( f(T) \) is almost strong \( S_1 \)-decomposable.

**Proof.** From the equalities \( X_{f(T)}(F) = X_T(f^{-1}(F)) \) (where \( F \supset S_1 = f(S) \)) and
\[
X_{f(T)}(F \cup S_1) = X_T(f^{-1}(F) \cup S) = Y_F \oplus X_T(S) = Y_F \oplus X_{f(T)}(S_1)
\]
(where \( F \cap S_1 = \emptyset \) and by Proposition 2.2.9 from [2], it follows that the spectral maximal spaces \( Y \) of \( f(T) \) that have the property \( \sigma(f(T)|Y) \supset S_1 \) or \( \sigma(f(T)|Y) \cap S_1 = \emptyset \) are also spectral maximal spaces of \( T \). One further performs the proof as for Proposition 2.2.9 from [2], since a \( S_1 \)-covering of \( \sigma(f(T)) \) is easily transformed through \( f^{-1} \) into a \( S \)-covering of \( \sigma(T) \).

**References**


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