Homology and homotopy groups of the complement of certain family of fibers in a critical point problem

Cornel Pintea

Abstract

In this paper we improve the results of [5] by showing that the collection of fibers over a closed countable subset of the base space of a differentiable fibration is not $CS^\infty$-critical, under some different topological conditions on the involved spaces. The example we always have in mind is that of Brieskorn manifolds as total spaces of certain principal fiber bundles.

Mathematics Subject Classification: 55R05, 55Q05, 55N10.
Key words: closed countable sets, homology and homotopy groups of fiber spaces, critical points.

1 Basic results.

We start this section by proving that the complement of a closed countable subset of a given $n$-dimensional manifold has large $n-1$ rational homology group. The manifold $M$ will be with empty boundary all along the paper.

Let $M$ be an $n$-dimensional differentiable manifold and $A \subseteq M$ be a closed countable subset of $M$. Recall that $A$ has uncountably many isolated points, all the other of its points being accumulation points. Assume that $A = I \cup A'$ where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of $A$ and $A'$ is its derived set, namely its set of accumulation points. If $H_{n-1}(M) \simeq 0$, then for each $k \geq 1$ there exists, according to [5, Proposition 2.1], a surjective group homomorphism

$$\delta_k : H_{n-1}(M \setminus A) \to \mathbb{Z}^{k-1} \oplus H_{n-1}(M \setminus A_k),$$

where $A_k = \{a_{k+1}, a_{k+2}, \ldots\} \cup A'$. Moreover, if $M$ is either not compact or compact but not orientable, then $H_{n-1}(M \setminus A) \simeq \mathbb{Z}^k \oplus H_{n-1}(M \setminus A_k)$, for each $k \geq 1$, that is $H_{n-1}(M \setminus A)$ has free abelian subgroups of arbitrarily large rank.

Proposition 1.1 Let $M$ be an $n$-dimensional differential manifold $n \geq 2$ and $A = I \cup A'$ be a closed countable subset of $M$, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of $A$ and $A'$ is its derived set. If $H_{n-1}(M) \simeq 0$, then the group homomorphism

$\delta_k : H_{n-1}(M \setminus A) \to \mathbb{Z}^{k-1} \oplus H_{n-1}(M \setminus A_k),$

where $A_k = \{a_{k+1}, a_{k+2}, \ldots\} \cup A'$. Moreover, if $M$ is either not compact or compact but not orientable, then $H_{n-1}(M \setminus A) \simeq \mathbb{Z}^k \oplus H_{n-1}(M \setminus A_k)$, for each $k \geq 1$, that is $H_{n-1}(M \setminus A)$ has free abelian subgroups of arbitrarily large rank.

is surjective, it actually being a Q-linear mapping.

**Corollary 1.2** Let $M$ be an $n$-dimensional differentiable manifold with $n \geq 2$ and $A = I \cup A'$ be a closed countable subset of $M$, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of $A$ and $A'$ is its derived set. If $V$ is a Q-vector space and $\varphi : V \to H_{n-1}(M \setminus A) \otimes_\mathbb{Z} \mathbb{Q}$ is a surjective Q-linear mapping, then $V$ is not finite dimensional. In particular $H_{n-1}(M \setminus A) \otimes_\mathbb{Z} \mathbb{Q}$ is not finite dimensional.

**Proof.** Assume that $V$ is finite dimensional and $\varphi : V \to H_{n-1}(M \setminus A) \otimes_\mathbb{Z} \mathbb{Q}$ is a surjective Q-linear mapping. It follows that $(\delta_k \otimes id_\mathbb{Q}) \circ \varphi : V \to Q^{k-1} \oplus (H_{n-1}(M \setminus A) \otimes_\mathbb{Z} \mathbb{Q})$ is also a surjective Q-linear mapping for each $k \geq 1$. Consider $q = \dim_\mathbb{Q} V$ and observe that $(\delta_{q+2} \otimes id_\mathbb{Q}) \circ \varphi$ acts surjectively from $V$ to $Q^{q+1} \oplus (H_{n-1}(M \setminus A) \otimes_\mathbb{Z} \mathbb{Q})$. Therefore we have successively

$$q = \dim_\mathbb{Q} V = \dim_\mathbb{Q} \ker [(\delta_{q+2} \otimes id_\mathbb{Q}) \circ \varphi] + \dim_\mathbb{Q} \text{Im} [(\delta_{q+2} \otimes id_\mathbb{Q}) \circ \varphi] \geq$$

$$\geq \dim_\mathbb{Q} \text{Im} [(\delta_{q+2} \otimes id_\mathbb{Q}) \circ \varphi] = \dim_\mathbb{Q} [Q^{q+1} \oplus (H_{n-1}(M \setminus A) \otimes_\mathbb{Z} \mathbb{Q})] \geq$$

$$\geq \dim_\mathbb{Q} Q^{q+1} = q + 1.$$

**Proposition 1.3** ([5]) If $p : E \to M$ is a fibration whose base space $M$ is an $n$-dimensional differentiable manifold and $A$ is a closed countable subset of $M$, then the pair $(E, E \setminus p^{-1}(A))$ is $(n-1)$-connected, that is $\pi_q(E, E \setminus p^{-1}(A)) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$. In particular we get that $\pi_q(E, E \setminus p^{-1}(A)) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$ and that the natural Hurewitz group homomorphism $\chi_n : \pi_n(E, E \setminus p^{-1}(A)) \to H_n(E, E \setminus p^{-1}(A))$ is surjective. On the other hand the inclusion $E_{\setminus \pi^{-1}(A)} : E \setminus p^{-1}(A) \hookrightarrow E$ is $(n-1)$-connected, that is the induced group homomorphisms $\pi_q(E \setminus p^{-1}(A)) : \pi_q(E) \to \pi_q(E)$ is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$. Hence the morphism $\chi_n$ is an isomorphism if $E$ is simply connected and $n \geq 3$.

**Remark 1.4** If $M$ is an $n$-dimensional differentiable manifold and $A$ is a closed countable subset of $M$, then, by considering in Proposition 1.3 the particular fibration $id_M : M \to M$, the pair $(M, M \setminus A)$ is $(n-1)$-connected, that is $\pi_q(M, M \setminus A) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$. In particular we get that $H_q(M, M \setminus A) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$ and the natural group homomorphism $\chi : \pi_n(M, M \setminus A) \to H_n(M, M \setminus A)$ is surjective. On the other hand the inclusion $i_{M \setminus A} : M \setminus A \hookrightarrow M$ is $(n-1)$-connected, that is the induced group homomorphism $\pi_q(i_{M \setminus A}) : \pi_q(M \setminus A) \to \pi_q(M)$ is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$. Hence the morphism $\chi$ is an isomorphism if $M$ is simply connected and $n \geq 3$.

**Corollary 1.5** Let $p : E \to M$ be a fibration whose base space $M$ is an $n$-dimensional differentiable manifold and $A$ be a closed countable subset of $M$. If the total space $E$ is simply connected, $H_n(E) \otimes_\mathbb{Z} \mathbb{Q} \simeq 0 \simeq \pi_{n-1}(E) \otimes_\mathbb{Z} \mathbb{Q}$, then the group homomorphism $h_{n-1} \otimes id_\mathbb{Q} : \pi_{n-1}(E \setminus p^{-1}(A)) \otimes_\mathbb{Q} \mathbb{Q} \to H_{n-1}(E \setminus p^{-1}(A)) \otimes_\mathbb{Q} \mathbb{Q}$, which is actually $\mathbb{Q}$-linear, is injective, where $h_{n-1}$ is the natural group homomorphism.
Proof. Indeed, the following ladder with exact rows and commutative rectangles:

\[
\begin{array}{c}
\pi_n(E) \rightarrow \pi_n(E, E^{p^{-1}}(A)) \rightarrow \pi_{n-1}(E^{p^{-1}}(A)) \rightarrow \pi_{n-1}(E) \rightarrow \pi_{n-1}(E, E^{p^{-1}}(A)) \\
\downarrow h_n^E \quad \chi_n \quad \downarrow h_{n-1} \quad h_{n-1}^E \quad \chi_{n-1} \\
H_n(E) \rightarrow H_n(E, E^{p^{-1}}(A)) \rightarrow H_{n-1}(E^{p^{-1}}(A)) \rightarrow H_{n-1}(E) \rightarrow H_{n-1}(E, E^{p^{-1}}(A))
\end{array}
\]

leads us, simply by tensorizing with \(Q\) on the right, to the new one

\[
\begin{array}{c}
\pi_n(E) \otimes Q \rightarrow \pi_n(E, E^{p^{-1}}(A)) \otimes Q \rightarrow \pi_{n-1}(E^{p^{-1}}(A)) \otimes Q \rightarrow \pi_{n-1}(E) \otimes Q \rightarrow \pi_{n-1}(E, E^{p^{-1}}(A)) \otimes Q \\
\downarrow h_n^E \otimes id_Q \quad \chi_n \otimes id_Q \quad \downarrow h_{n-1} \otimes id_Q \quad h_{n-1}^E \otimes id_Q \quad \chi_{n-1} \otimes id_Q \\
H_n(E) \otimes Q \rightarrow H_n(E, E^{p^{-1}}(A)) \otimes Q \rightarrow H_{n-1}(E^{p^{-1}}(A)) \otimes Q \rightarrow H_{n-1}(E) \otimes Q \rightarrow H_{n-1}(E, E^{p^{-1}}(A)) \otimes Q
\end{array}
\]

with exact rows and commutative rectangles as well. The exactness of the rows follows by using the universal-coefficient theorem [7, Theorem 5.2.14]. In the above diagram we have omitted, for simplicity, the notation \(\otimes_Z\). Because \(H_n(E) \otimes_Z Q \simeq 0 \simeq \pi_{n-1}(E) \otimes_Z Q\), it follows that \(h_n^E \otimes id_Q\) is surjective and \(h_{n-1}^E \otimes id_Q\) is injective. On the other hand \(\chi_n \otimes id_Q\) is, according to corollary 1.3, an isomorphism such that, by combining all these facts with the fifth Lemma in [4, pp. 46], one can conclude that \(h_{n-1} \otimes id_Q\) is indeed injective. \(\Box\)

Remark 1.6 Let \(M\) be an \(n\)-dimensional differentiable manifold and \(A\) be a closed countable subset of \(M\). If the natural group homomorphism \(h_{n-1}^M : \pi_{n-1}(M) \rightarrow H_{n-1}(M)\) is surjective, then the natural group homomorphism \(h_{n-1}^A : \pi_{n-1}(M \setminus A) \rightarrow H_{n-1}(M \setminus A)\) is also surjective and the \(Q\)-linear mapping \(\chi_n \otimes id_Q : \pi_{n-1}(M \setminus A) \otimes_Z Q \rightarrow H_{n-1}(M \setminus A) \otimes_Z Q\) is surjective as well. In particular, according to corollary 1.2, \(\pi_{n-1}(M \setminus A) \otimes_Z Q\) is not finite dimensional.

2 Application to critical points

In this section we will show that, under some topological conditions on the total space, on the base spaces and on the fiber of a fibration, the collection of fibers over a closed countable subset of the base space is not critical with respect to certain class of special real functions.

Let \(M, N\) be differentiable manifolds and \(f : M \rightarrow N\) be a differentiable mapping. Denote by \(C(f)\) its critical set and by \(R(f)\) its set of regular points, while the set of its critical values \(f(C(f))\) will be denoted by \(B(f)\).

Definition 2.1 We say that a differentiable mapping \(f \in C^\infty(M, N)\) separates the critical values by the regular ones if \(B(f) \cap f(R(f)) = \emptyset\). Denote by \(CS^\infty(M, N)\) the set of all of these mappings. A closed subset \(C\) of \(M\) is said to be \(CS^\infty(M, N)\)- (properly) critical if \(C(f) = C\) for some (proper) mapping \(f \in CS^\infty(M, N)\).

Remark 2.2 Let \(p : \tilde{N} \rightarrow N\) be a covering mapping and \(\tilde{f} : M \rightarrow \tilde{N}\) be a differentiable mapping. If \(\tilde{f} \notin CS^\infty(M, \tilde{N})\), then \(p \circ \tilde{f} \notin CS^\infty(M, N)\). Therefore if \(g \in CS^\infty(M, N)\) and \(\tilde{g} \in C^\infty(M, \tilde{N})\) is a lifting of \(g\), then \(\tilde{g} \in CS^\infty(M, \tilde{N})\).

Indeed, since \(\tilde{f} \notin CS^\infty(M, \tilde{N})\), it follows that there exist \(x_0 \in C(f)\), \(x_1 \in R(f)\) such that \(\tilde{f}(x_0) = \tilde{f}(x_1)\). Because \(p\) is a local diffeomorphism, it implies that \(x_0 \in\)
Proposition 2.3 (i) \( f \in \text{CS}^\infty(M, N) \) iff \( C(f) = f^{-1}(B(f)) \).

(ii) If \( M \) is a connected differentiable manifold and \( f \in \text{CS}^\infty(M, \mathbb{R}) \) is such that \( R(f) = M \setminus C(f) \) is also connected, then \( f(R(f)) = (m_i, M_j) \), where \( m_i = \inf x \in M f(x), M_j = \sup x \in M f(x) \) and \( B(f) \subseteq \{ m_i, M_j \} \cap \mathbb{R} \). Moreover, if \( M \) is compact, then \( m_i, M_j \in \mathbb{R} \) and \( B(f) = \{ m_i, M_j \} \).

Theorem 2.4 Let \( F \hookrightarrow E \overset{p}{\rightarrow} M^n \) be a differential fibration with compact total space and commutative fundamental group of the fiber \( F \). If \( A \) is a closed countable subset of \( M \), \( n \geq 3 \), \( E \) is simply connected and \( H_n(E), H_{n-1}(M), \pi_{n-1}(E) \otimes \mathbb{Q} \) are trivial and \( \pi_{n-2}(F) \) is finitely generated, then \( p^{-1}(A) \) is neither \( \text{CS}^\infty(E, \mathbb{R}) \)-critical nor \( \text{CS}^\infty(E, S^1) \)-critical.

Proof. Assume that there exists a mapping \( f \in \text{CS}^\infty(E, \mathbb{R}) \) such that \( C(f) = p^{-1}(A) \). This means that \( B(f) = \{ m_i, M_j \} \) and that its restriction

\[
E \setminus C(f) \rightarrow \text{Im} f \setminus B(f) = (m_f, M_f), \quad p \mapsto f(p)
\]

is a proper submersion, that is, via Ehresmann’s theorem, a locally trivial fibration whose compact fiber we are denoting by \( F \). Its base space \( (m_f, M_f) \) being contractible, it follows that the inclusion \( i_x : F \hookrightarrow E \setminus C(f) \) is a weak homotopy equivalence, namely the induced group homomorphisms \( \pi_q(i_x) : \pi_q(F) \rightarrow \pi_q(E \setminus C(f)) \) are all isomorphisms. Consequently, using the Whitehead theorem [3, pp. 167] or [7, pp. 399], it follows that the induced group homomorphisms \( H_q(i_x) : H_q(F) \rightarrow H_q(E \setminus C(f)) = H_q(E \setminus p^{-1}(A)) \) are also isomorphisms. Consequently \( H_q(i_x) \otimes id_Q : H_q(F) \otimes Q \rightarrow H_q(E \setminus C(f)) \otimes Q = H_q(E \setminus p^{-1}(A)) \otimes Q \) is a \( Q \)-linear isomorphism. Therefore \( H_q(E \setminus p^{-1}(A)) \otimes Q \) is finite dimensional for all \( q \) since \( H_q(F) \) is finite generated as homology group of a compact manifold. The hypothesis \( H_{n-1}(M) \cong 0 \) ensures us that \( H_{n-1}(M) \otimes id_Q \) is obviously surjective and implicitly that \( \pi_{n-1}(M \setminus A) \otimes Q \) is not finite dimensional while the hypothesis \( H_n(E) \cong 0 \) and the triviality of \( \pi_{n-1}(E) \otimes Q \) ensure us that the \( Q \)-linear mapping \( h_{n-1} \otimes id_Q : \pi_{n-1}(E \setminus p^{-1}(A)) \otimes Q \rightarrow H_{n-1}(E \setminus p^{-1}(A)) \otimes Q \) is injective. Consequently \( \pi_{n-1}(E \setminus p^{-1}(A)) \otimes Q \) is a finite dimensional \( Q \)-vector space as a subspace of the finite dimensional \( Q \)-vector space \( H_{n-1}(E \setminus p^{-1}(A)) \otimes Q \).

The exact homotopy sequence of the fibration \( F \hookrightarrow E \setminus p^{-1}(A) \overset{p}{\rightarrow} M \setminus A \)

\[
\cdots \rightarrow \pi_{n-1}(E \setminus p^{-1}(A)) \rightarrow \pi_{n-1}(M \setminus A) \overset{\partial_{n-1}}{\rightarrow} \pi_{n-2}(F) \rightarrow \cdots
\]

produces the exact sequence

\[
\cdots \rightarrow \pi_{n-1}(E \setminus p^{-1}(A)) \otimes Q \rightarrow \pi_{n-1}(M \setminus A) \otimes Q \rightarrow \pi_{n-2}(F) \otimes Q \rightarrow \cdots
\]

simply by tensoring with \( Q \). Its exactness follows by using the universal-coefficient theorem. But since \( \pi_{n-1}(E \setminus p^{-1}(A)) \otimes Q \) and \( \pi_{n-2}(F) \otimes Q \) are finite dimensional they force \( \pi_{n-1}(E \setminus p^{-1}(A)) \otimes Q \) to be finite dimensional too, not being the case as
follows from remark 1.6. In order to prove the $C^{∞}(M, S^1)$-non-criticality of $p^{-1}(A)$ we assume that there exists a mapping $f \in C^{∞}(M, S^1)$ such that $C(f) = p^{-1}(A)$.

Consider a lifting $\tilde{f} : M \rightarrow \mathbb{R}$ with respect to the covering mapping $exp : \mathbb{R} \rightarrow S^1$, recall that $\tilde{f} \in C^{∞}(M, \mathbb{R})$ an observe that $C(\tilde{f}) = C(f) = p^{-1}(A)$, such that we have got a contradiction with the $C^{∞}(M, \mathbb{R})$-non-criticality of $p^{-1}(A)$.

**Example 2.5.** Consider the integers $n \geq 2$ and $d \geq 1$ and the *Brieskorn manifolds* $W_d^{2n-1}$ as the $(2n-1)$-dimensional real algebraic submanifolds of $\mathbb{C}^{n+1}$ defined by the equations
\[
\sum_{i=0}^{d} z_i^2 + \cdots + z_n^2 = 0 \quad \text{and} \quad |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1.
\]

The topology of Brieskorn manifolds is mostly known it being simply connected for $n \geq 3$ and a homotopy sphere for both $n \geq 3, d \geq 1$ odd. On the other hand, if $n = 2m$ is even, then $W_d^{2m-1}$ is a rational homology sphere whose only nontrivial integral homology groups are given by $H_{2m-1}(W_d^{2m-1}) \cong \mathbb{Z}$ and $H_0(W_d^{2m-1}) \cong H_{4m-1}(W_d^{2m-1}) \cong \mathbb{Z}$, [1, Corollary 5.3, pp. 275], [6].

All manifolds $W_d^{2n-1}$ are invariant under the standard linear action of $O(n)$ on the $(z_1, \ldots, z_n)$-coordinates. If $n = 2m$ is even there is a free circle action on $W_d^{2m-1}$ given by the action of the circle group $S^1 = Z(U(m)) \subset O(2m)$ where $Z$ denotes the center. Moreover if $n = 4m$, then $Sp(1)$ realised as subgroup of $O(4m)$ by the scalar multiplication on $\mathbb{R}^{4m} \cong H^m$, acts also freely on $W_d^{8m-1}$. The quotient manifolds $N^{4m-2}_{d} := W_d^{2m-1}/S^1$ and $\tilde{N}^{8m-4}_{d} := W_d^{8m-1}/Sp(1)$ are simply connected and their integral cohomology groups are given by
\[
H^k(N^{4m-2}_{d}) \approx H^k(CP^{2m-1}), \quad H^k(\tilde{N}^{8m-4}_{d}) \approx H^k(HP^{2m-1}), \quad \text{see [6]}.
\]

Therefore if $F \hookrightarrow E \xrightarrow{p} M^n$ is one of the fibrations
\[
S^1 \hookrightarrow W_d^{2m-1} \rightarrow N_{d}^{4m-2}, \quad Sp(1) \hookrightarrow W_d^{8m-1} \rightarrow \tilde{N}_{d}^{8m-4},
\]
then it satisfies the conditions of theorem 2.4. Consequently for a closed countable subset $A$ of the base space, $p^{-1}(A)$ is neither $C^{∞}(E, \mathbb{R})$-critical nor $C^{∞}(E, S^1)$-critical. Indeed we have successively
\[
H_{4m-2}(W_d^{4m-1}) \cong 0 \approx H^1(CP^{2m-1}) \approx H^1(N^{4m-2}_{d}) \approx H_{4m-3}(N^{4m-2}_{d}),
\]
\[
H_{8m-4}(W_d^{8m-1}) \cong 0 \approx H^1(HP^{2m-1}) \approx H^1(\tilde{N}^{8m-4}_{d}) \approx H_{8m-5}(\tilde{N}^{8m-4}_{d}),
\]
the last isomorphisms ($\ast$) and ($\ast\ast$) being ensured by the Poincaré duality. Finally, the triviality of $\pi_{4m-3}(W_d^{4m-1}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and of $\pi_{8m-5}(W_d^{8m-1}) \otimes_{\mathbb{Z}} \mathbb{Q}$ follows from [2, pp. 123], while the triviality of $\pi_{4m-3}(S^1) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\pi_{8m-5}(Sp(1)) \otimes_{\mathbb{Z}} \mathbb{Q} = \pi_{8m-6}(S^3) \otimes_{\mathbb{Z}} \mathbb{Q}$ is obvious since $\pi_{4m-3}(S^1) \cong 0$ and $\pi_{8m-6}(S^3)$ is finite [3, 318].

**References**


Cornel Pintea

"Babeș - Bolyai" University, Faculty of Mathematics
400084, M. Kogălniceanu 1, Cluj-Napoca, Romania

e-mail address: cpintea@math.ubbcluj.ro