On some special vector fields

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Abstract

We introduce the notion of $F$-distinguished vector fields in a deformation algebra, where $F$ is a $(1,1)$-tensor field. The aim of this paper is to study these special vector fields and, using their properties, to characterize spherical hypersurfaces, when $F$ is the shape operator. The last section is devoted to the relation between the geometrical properties of Weyl manifolds and the algebraic properties of Weyl algebras.

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1 $F$-distinguished vector fields

Let $M$ be a connected paracompact, smooth manifold of dimension $n \geq 2$. Let $TM$ be the tangent bundle of $M$ and $T^r_s(M)$ be the $C^\infty(M)$-module of tensor fields of type $(r,s)$ on $M$. We denote $T^1_0(M)$ (respectively $T^0_1(M)$) by $\mathcal{X}(M)$ (respectively $\Lambda^1(M)$).

Let $A$ be a $(1,2)$-tensor field on $M$. The $C^\infty(M)$-module $\mathcal{X}(M)$ becomes a $C^\infty(M)$-algebra if we consider the multiplication rule given by $X \circ Y = A(X,Y)$, $\forall X, Y \in \mathcal{X}(M)$. This algebra is denoted by $\mathcal{U}(M, A)$ and it is called the algebra associated to $A$. If $\nabla$ and $\nabla'$ are two linear connections on $M$, then $\mathcal{U}(M, \nabla - \nabla')$ is called the deformation algebra defined by the pair $(\nabla, \nabla')$ [9].

Let $(M,g)$ be a Riemannian manifold and $F$ be a $(1,1)$-tensor field on $M$.

Definition 1.1 $X \in \mathcal{X}(M)$ is called a $(\nabla,F)$-Killing vector field if

\begin{equation}
(1.1) \quad g(\nabla_Z F(X), Y) + g(Z, \nabla_Y F(X)) = 0, \forall Y, Z \in \mathcal{X}(M)
\end{equation}

holds.

One should remark that this is equivalent to the condition that $F(X)$ is a $\nabla$-Killing vector field.
Definition 1.2 Let $A$ be a $(1,2)$-tensor field on $M$. $X$ is called a $F$-distinguished vector field in the algebra $\mathcal{U}(M, A)$ if one has
\[
g(A(Z, F(X)), Y) + g(Z, A(Y, F(X))) = 0, \forall Y, Z \in \mathcal{X}(M).
\]

In the particular case when $F$ is the identity tensor field of type $(1,1)$ one gets the known notion of distinguished vector fields on $M$ [10].

Let $\tilde{\nabla}$ be the Levi-Civita connection, associated to $g$ and $\nabla, \nabla$ be linear connections on $M$, given by
\[
\nabla = \tilde{\nabla} - \frac{1}{2}A, \quad \nabla = \tilde{\nabla} + \frac{1}{2}A.
\]

Proposition 1.1 Let $X \in \mathcal{U}(M, A)$. The following assertions are equivalent:

i) $X$ is a $(\nabla, F)$-Killing vector field and a $F$-distinguished vector field in the algebra $\mathcal{U}(M, A)$;

ii) $X$ is a $(\nabla, F)$-Killing vector field and a $F$-distinguished vector field in the algebra $\mathcal{U}(M, A)$;

iii) $X$ is a $(\nabla, F)$ and $(\nabla, F)$-Killing vector field.

Proof. i)\(\Leftrightarrow\)ii) Let $X$ be $F$-distinguished vector field in the algebra $\mathcal{U}(M, A)$. Hence $g(A(Z, F(X)), Y) + g(Z, A(Y, F(X))) = 0, \forall Y, Z \in \mathcal{X}(M)$. Since $A = \nabla - \tilde{\nabla}$, then $g(\nabla_Z F(X), Y) + g(Z, \nabla_Y F(X)) = 0$ \(\Leftrightarrow\) $g(\tilde{\nabla}_Z F(X), Y) + g(Z, \tilde{\nabla}_Y F(X)) = 0$.

ii) \(\Leftrightarrow\)i) It is a consequence of (1.1) and (1.2).

Remark 1.1 Let $A^i_{jk}, g_{ij}$ and $X^i$ be the local components of $A$, $g$ and $X$, respectively, in a local system of coordinates. The formula (1.2) becomes
\[
(1.3) \quad (A^p_{js} g_{pk} + A^p_{ks} g_{jp}) F^{s}_i X^i = 0.
\]
The integral curves of $F$-distinguished vector fields, called $F$-distinguished curves, verify the following differential system of equations
\[
(1.4) \quad (A^p_{js} g_{pk} + A^p_{ks} g_{jp}) F^{s}_i \frac{dx^i}{dt} = 0.
\]

Remark 1.2 Let $(M, g)$ be a Riemannian manifold, $\tilde{\nabla}$ be the Levi-Civita connection associated to $g$ and $\pi \in \Lambda^1(M)$. Let $\nabla$ be the Lyra connection associated to $\pi$, hence
\[
(1.5) \quad \nabla_X Y = \tilde{\nabla}_X Y + \pi(Y) X - g(X, Y) P, \forall X, Y \in \mathcal{X}(M),
\]
where $P$ is the dual vector field associated to $\pi$ i.e. $g(P, Z) = \pi(Z), \forall Z \in \mathcal{X}(M)$.

Then $A = \nabla - \tilde{\nabla}$ verifies
\[
(1.6) \quad A^i_{jk} = \delta^i_k \pi_j - g_{jk} \pi^i,
\]
where $\pi^i = g^{ik} \pi_k$. So, from (1.6) we notice that (1.3) is satisfied. Hence all the elements of the Lyra algebra $\mathcal{U}(M, A)$ are $F$-distinguished vector fields.
2 On spherical hypersurfaces

Let $M^n$ be a hypersurface in the Euclidean space $\mathbb{E}^{n+1}$. Let us denote by $g, b$ and $h$ the first, the second and the third fundamental forms on $M$, respectively. We suppose that $b$ is nondegenerated. Let $\nabla, \nabla'$ and $\nabla''$ be the Levi-Civita connections associated to $g, b$ and $h$, respectively. Let us denote by

$$A = \frac{1}{\nabla} - \frac{2}{\nabla'}, \quad A' = \frac{2}{\nabla} - \frac{3}{\nabla'}, \quad A'' = \frac{1}{\nabla} - \frac{3}{\nabla''}$$

We note that

$$b(A(X,Y), Z) = b(A'(X,Y), Z) = 2b(A''(X,Y), Z) = -\frac{1}{2} (\nabla_g b)(Y,Z).$$

We suppose that the $(1,1)$-tensor field $F$ is the shape operator of the hypersurface $M$. Then $F^s_i = b^{s\eta}g_{\eta i}$.

**Remark 2.1** The deformation algebras $U(M, A), U(M, A')$ and $U(M, A'')$ have the same $F$-distinguished vector fields.

Indeed, this is a consequence of (1.3) and (2.1).

**Remark 2.2** Let $M^2$ be a surface in the Euclidean space $\mathbb{E}^3$, given by

$$x = (a + b \cos x^1) \cos x^2,$$
$$y = (a + b \cos x^1) \sin x^2,$$
$$z = b \sin x^1,$$

where $a > b > 0, a$ and $b$ are constants, $x^2 \in \mathbb{R}$ and $x^1 \in \mathbb{R} \setminus \{2k + 1\frac{\pi}{2}\}$. $k \in \mathbb{Z}$. One has the following nonvanishing components of $A, A'$ and $A''$:

$$A^{12}_{22} = \frac{2a \sin x^1}{b}, \quad A^{21}_{21} = A^{22}_{12} = \frac{2a \sin x^1}{(a + b \cos x^1) \cos x^1},$$

$$A^{12}_{22} = -\frac{a \sin x^1}{b}, \quad A^{21}_{21} = A^{22}_{12} = -\frac{a \sin x^1}{(a + b \cos x^1) \cos x^1},$$

$$A^{12}_{22} = \frac{a \sin x^1}{b}, \quad A^{21}_{21} = A^{22}_{12} = \frac{a \sin x^1}{(a + b \cos x^1) \cos x^1}.$$
Moreover, (2.2) implies

\[(2.3) \quad (g_{r} b^{sr} \nabla_{r} b_{ik} + g_{sk} b^{sr} \nabla_{r} b_{ji}) b^{iq} g_{ql} = 0. \]

Then (2.3) lead to \( \nabla_{r} b_{jk} = 0 \). Hence one gets i) ([8]).

ii) \( \Rightarrow i) \) is obvious.

3 Weyl manifolds

Let \( g \) be a semi-Riemannian metric on \( M \) and let \( \tilde{g} = \{e^{u}g | u \in C^{\infty}(M)\} \) be the conformal class defined by \( g \).

Let \( W \) be a Weyl structure on the conformal manifold \( (M, \tilde{g}) \) i.e. a mapping \( W : \tilde{g} \mapsto \Lambda^{1}(M) \). Hence \( W(e^{u}g) = W(g) - du, \forall u \in C^{\infty}(M) \). The triple \( (M, \tilde{g}, W) \) is called a Weyl manifold. There exists a unique torsion free connection \( \nabla \), compatible with the Weyl structure \( W \).

\[(3.1) \quad \nabla g + W(g) \otimes g = 0, \]

given by

\begin{equation}
2g(\nabla_{X}Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + \\
+ W(g)(X)g(Y, Z) + W(g)(Y)g(X, Z) - W(g)(Z)g(X, Y) + \\
+ g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X). \tag{3.2}
\end{equation}

\( \nabla \) is called the Weyl conformal connection. Let \( \triangleleft \) be the Levi-Civita connection associated to \( g \) and \( A = \nabla - \triangleleft \). \( U(M, A) \) is called the Weyl algebra. One has

\[(3.3) \quad 2g(A(X, Y), Z) = W(g)(X)g(Y, Z) + W(g)(Y)g(X, Z) - W(g)(Z)g(X, Y). \]

The torsion free connections \( \nabla \) and \( \triangleleft \) are called projectively equivalent if their unperturbed geodesic coincide [5].

The goal of this section is to study the Weyl algebra. Our algebraic approach gives some insights of geometrical nature.

**Theorem 3.1** Let \( (M, \tilde{g}, W) \) be a Weyl manifold. Let \( R, S \) and \( R, S \) be the curvature tensor field and the Ricci tensor field associated to \( \nabla \) and \( \triangleleft \), respectively. Let \( F \) be a \((1, 1)\)-tensor field. We suppose that the mapping \( F_{p} : T_{p}M \mapsto T_{p}M \) is surjective, \( \forall p \in M \). Then the following assertions are equivalent:

i) Every element of the algebra \( U(M, A) \) is a \( F \)-distinguished vector field.

ii) The algebra \( U(M, A) \) is associative.

iii) \( \nabla \) and \( \triangleleft \) are projectively equivalent.
iv) \( R = \circ R \), when \( S \) is nondegenerated.

v) \( S = \circ S \), when \( S \) is nondegenerated and the 1-form \( W(g) \) is exact.

vi) \( \nabla = \circ \nabla \).

**Proof.** i) \( \Rightarrow \) vi). Let \( X \) be a \( F \)-distinguished vector contained in the Weyl algebra \( \mathcal{U}(M, A) \). From (1.2) and

\[
2g(A(Z, F(X)), Y) = W(g)(Z)g(F(X), Y) + W(g)(F(X))g(Y, Z) - W(g)(Y)g(Z, F(X)),
\]

\[
2g(A(F(Y), Y), Z) = W(g)(Y)g(F(X), Z) + W(g)(F(X))g(Y, Z) - W(g)(Z)g(F(X), Y)
\]

one gets

\[
(3.4) \quad W(g)(F(X))g(Z, Y) = 0, \forall X, Y, Z \in \mathcal{X}(M).
\]

Since the mapping \( F_p : T_p M \rightarrow T_p M \) is surjective, \( \forall p \in M \), (3.3) and (3.4) imply \( g(A(X, Y), Z) = 0, \forall X, Y, Z \in \mathcal{X}(M) \). Therefore \( A = 0 \) i.e. vi).

vi) \( \Rightarrow \) i) If \( A = 0 \), then (1.2) is satisfied.

ii) \( \Leftrightarrow \) iii) \( \Leftrightarrow \) iv) \( \Leftrightarrow \) v) \( \Leftrightarrow \) iv) [6].

**References**


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