Linearized geometric dynamics of
Tobin-Benhabib-Miyao economic flow

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Abstract

The aim of this paper is to study the influence of the Euclidean-Lagrangian structure of the state space on the generalized Tobin flow in economics, confirming the existence of optimal economic fluctuations (small oscillations).

Section 1 reviews the generalized Tobin economic flow as formulated by Benhabib and Miyao. Section 2 recalls some tools of single-time geometric dynamics which describe a geodesic motion under a gyroscopic field of forces. Section 3 studies the linearized geometric dynamics produced by the Tobin-Benhabib-Miyao flow and by the Euclidean-Lagrangian structure of the economic state space.

In this way we estimate the time economic evolution for spotlighting the expectations of the agents.

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1 Tobin-Benhabib-Miyao economic flow

The Tobin model [4] regarding the role of money on economic growth has been extended by Benhabib and Miyao [1] to incorporate the role of expectation parameters, and to show that the variation of this parameter produces a Hopf Bifurcation in a three sector economy (see also [5],[6]).

We introduce the state space by the following variables:

• \( k \) = the capital labour ratio;
• \( m \) = the money stock per head;
• \( q \) = the expected rate of inflation.

Then, the model is an ODEs system

\[
\dot{k} = sf(k) - (1 - s)(\theta - q)m - nk, \quad \dot{m} = m(\theta - \bar{p} - n), \quad \dot{q} = \mu(\bar{p} - q),
\]

where
\begin{equation}
\bar{p} = \varepsilon \left( m - L(k, q) \right) + q,
\end{equation}
is the actual rate of inflation. The real functions \( f(k) \) and \( L(k, q) \) are differentiable, and \( s, \theta, n, \mu, \varepsilon \) are parameters \((s = \text{saving ratio}; \theta = \text{rate of money expansion}, n = \text{population growth rate}, \mu = \text{speed of adjustment of expectations}, \varepsilon = \text{speed of adjustment of price level})\). Keeping the parameters \( s, \theta, n, \varepsilon \) like constants, we obtain an ODEs system with one parameter \( \mu \).

An equilibrium point \((k^*(\mu), m^*(\mu), q^*(\mu))\), at which \( \dot{k} = 0 = \dot{m} = \dot{q} \), is the solution of the algebraic system
\begin{equation}
\theta = q + n, L(k, q) = m, s f(k) - (1 - s)m n - k n = 0.
\end{equation}
(suppose we have isolated solutions). Denoting \( x = (x_1, x_2, x_3) = (k - k^*, m - m^*, q - q^*) \), the linearization about the equilibrium point \((0,0,0)\) is
\begin{equation}
\dot{x} = A(\mu) x,
\end{equation}
where \( A(\mu) \) is the Jacobian matrix of the function
\begin{equation}
\left( sf(k) - (1 - s)(\theta - q)m - nk , m(\theta - \bar{p} - n) , \mu(\bar{p} - q) \right),
\end{equation}
computed at \((k = k^*, m = m^*, q = q^*)\), i.e.,
\begin{equation}
\begin{pmatrix}
s f' - n & -(1 - s)n & (1 - s)m \\
\varepsilon m L_1 & -2 \varepsilon m & m(\varepsilon L_2 - 1) \\
-\mu \varepsilon L_1 & \mu \varepsilon & -\mu \varepsilon L_2
\end{pmatrix}
\end{equation}
\( (k^*, m^*, q^*) \),
The characteristic equation is
\begin{equation}
\det \left( A(\mu) - \lambda(\mu) I \right) = -\lambda^3 + c_1 \lambda^2 - c_2 \lambda + c_3 = 0,
\end{equation}
where \( c_1 = \text{tr} A(\mu) \), \( c_2 = \text{sum of principal minors of order two} \), \( c_3 = \det A(\mu) \). If \((-1)^i c_i > 0 \ (i = 1, 2, 3) \) and \( c_1 c_2 < c_3 \), then we have solutions of type \( \lambda_1(\mu) < 0, \lambda_{2,3}(\mu) = \alpha(\mu) \pm i \beta(\mu) \), \( \alpha(\mu) < 0 \), and consequently the equilibrium point is asymptotically stable. In the hypothesis \( c_1 c_2 = c_3 \), we can find \( \mu_0 \) such that \( \lambda_1(\mu_0) < 0 \), and \( \alpha(\mu_0) = 0 \), \( \frac{d\alpha}{d\mu}(\mu_0) \neq 0 \). By the Hopf Bifurcation Theorem, there exist periodic solutions
\begin{equation}
\left( k(t, \mu), m(t, \mu), q(t, \mu) \right), \ t \in \mathbb{R}
\end{equation}
around the equilibrium point \((k^*, m^*, q^*)\).

Medio [2],[3] generalized the previous model and studied the birth of limit cycles given by Hopf Bifurcation, in the framework of \( \lambda - \) matrices and gyroscopic models.

## 2 Geometric dynamics produced by a flow and a Riemannian metric

Our theory [7]-[10] can be applied equally to any kind of flow on a manifold endowed with a geometric structure capable to produce "square of the length" (density of energy) and "derivatives". This geometric structure transforms a flow into a geodesic motion in a gyroscopic field of forces. As an example, we can use a Euclidean-Lagrangian
structure associated to the metric $\delta_{ij}$ or a Riemannian-Lagrangian structure associated to the metric $g_{ij}$.

Let us start with an arbitrary flow

$$\dot{x}_i = X_i(x), \ i = 1, 2, 3$$

on the Euclidean space $\mathbb{R}^3$. Modifying the prolongation by derivation in a suitable way, we obtain a gyroscopic prolongation

$$\frac{d^2x_i}{dt^2} = \delta^{jk} \left( \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) \frac{dx_k}{dt} + \frac{\partial f}{\partial x_i},$$

where

$$f = 0.5 \delta^{ij} X_i X_j$$

is the density of economic energy. This new prolongation determines a geometric dynamics, i.e., a geodesic motion in a gyroscopic field of forces. Another way to realize a geometric dynamics is to consider the least squares Lagrangian

$$L(x, \dot{x}) = 0.5 \delta^{ij} (\dot{x}_i - X_i(x)) (\dot{x}_j - X_j(x))$$

and to write the Euler-Lagrange equations which are just (9). Automatically, the geometric dynamics conserves the Hamiltonian

$$H(x, \dot{x}) = 0.5 \delta^{ij} (\dot{x}_i - X_i(x)) (\dot{x}_j + X_j(x)).$$

The linearization of the first order ODEs system (8) around an equilibrium point $(0, 0, 0)$ is of the form

$$\dot{x} = Ax.$$

Then the linearization of the second order ODEs system (9), around the same equilibrium point, is

$$\frac{d^2x}{dt^2} = (A - A^T) \frac{dx}{dt} + A^T A x.$$ 

For this second order system, the equilibrium point is

$$x(t) = 0, \frac{dx}{dt}(t) = 0.$$

Of course, the second order ODEs system (9) can be linearized also around a nonzero critical point of the density of energy $f$, but this study will be made in a further paper.
3 Linearized geometric dynamics around Tobin-Benhabib-Miyao flow

The Tobin-Benhabib-Miyao flow and the Euclidean-Lagrangian structure of the state space determine a geometric dynamics. We shall analyse its linearization around the equilibrium point which is described by a second order ODEs system of type (14). The solution of this system is of the form $x = u e^{\lambda t}$, $t \in \mathbb{R}$, where $u$ is a nonzero solution of the linear system

$$(-A^T A - \lambda(A - A^T) + \lambda^2 I)u = 0.$$  

In other words, $\lambda$ is a latent value of the $\lambda$-matrix in parenthesis, i.e., solution of the equation

$$\det\left(-A^T A - \lambda(A - A^T) + \lambda^2 I\right) = 0,$$

and $u$ is a latent vector (nonzero solution of the equation 15). On the other hand, the following proposition is true.

**Theorem.** If $\lambda$ is a proper value of a real matrix $A$, then $\lambda$ and $-\lambda$ are latent values of the previous $\lambda$-matrix. Consequently the latent values satisfy $\sum \lambda = 0$.

**Proof.** We use the decomposition

$$0 = \det\left(-A^T A - \lambda(A - A^T) + \lambda^2 I\right) = \det(\lambda I - A)\det(\lambda I + A^T).$$

In order to obtain informations about the nature of the latent values $\lambda$, associated to the latent vector $u$, we build an equation of degree two satisfied by $\lambda$.

Let $\lambda$ be a real latent value and $u$ be the corresponding real latent vector. Pre-multiplying by $u^T$ we find

$$\lambda^2 = \frac{u^T A^T A u}{u^T u}.$$  

Let $\lambda$ be a complex latent value and $u$ be the associated complex latent vector. Pre-multiplying by $\bar{u}^T$ (conjugate transpose of $u$) gives

$$m \lambda^2 + i g \lambda + n = 0,$$

where

$$m = \bar{u}^T u > 0, \ i g = -\bar{u}^T (A - A^T) u, \ n = -\bar{u}^T A^T A u < 0.$$  

The discriminant of the equation (19) is

$$\Delta(\mu) = -g^2 - 4 mn.$$  

**Theorem.** 1) The ODEs system (14) has saddle point properties around the equilibrium point iff $\Delta(\mu) > 0$.

2) If $\Delta(\mu) < 0$, no saddle point properties exist.

Let $\mu_0$ be such that $\Delta(\mu_0) = 0$ and $\Delta(\mu) > 0$ or $< 0$ if $\mu < \mu_0$ respectively $\mu > \mu_0$ with $\frac{d\Delta}{d\mu}(\mu_0) < 0$. These are just the conditions in the "Hopf Bifurcation Theorem"
for second order systems”. Consequently, when $\mu$ passes $\mu_0$, the system will undergo a bifurcation and lose its saddle point properties. Assuming simple latent values around $\mu_0$, we find:

**Corollary.** 1) if $g(\mu) = 0$ for $\mu \in N_c(\mu_0)$, then certain latent value lying on the real axis crosses imaginary axis from left to right, causing "total instability";

2) if $g(\mu) \neq 0$, then the loss of stability is of the "flutter type”, i.e., a pair of complex conjugate latent values crosses the imaginary axis from the left, causing Hopf Bifurcation and giving birth to closed orbits around the equilibrium point.

**Remark.** This limit cycle is optimal because it fulfils all optimality requirements, including the transversality condition

\[
\lim_{t \to \infty} k(t) q(t) e^{-\mu t} = 0.
\]

Thus an economy satisfying all standard neo-classical competitive conditions such as perfect foresight, zero profit, market clearing, can exhibit permanent small oscillations in prices and capital stocks. Therefore we recovered the concept of optimal economic fluctuations.

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