A certain complete space-like hypersurface in Lorentz manifolds

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Abstract

In this paper, we find an upper bound of the squared norm of the second fundamental tensor of a complete space-like hypersurface in a Lorentz space form $M_{m+1}^n(c)$ satisfying some curvature conditions. Then it gives naturally an extension of some theorems of Cheng and Nakagawa ([3]), Ishihara ([7]), Li ([8]) and Nishikawa ([9]).

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1 Introduction

In connection with the negative settlement of the Berstein problem by Calabi ([2]), Cheng-Yau ([4]) and Chouque-Bruhat et al.([6]) proved the following famous theorem independently.

Theorem A Let $M$ be a complete space-like Lorentz space form $M_{m+1}^n(c)$, $c > 0$. If $M$ is maximal, then it is totally geodesic.

On the other hand, complete space-like hypersurface with constant mean curvature in a Lorentz space form $M_{m+1}^n(c)$ are investigated by many differential geometers in various view points; for example Akutagawa ([1]), Cheng and Nakagawa ([3]), Li ([8]), Nishikawa([9]) and Ramanathan ([11]). In this paper, we’ll give an upper bound of the squared norm of the second fundamental form, of complete space-like hypersurface with constant mean curvature in a Lorentz space form $M_{m}^n(c)$. Namely, the following assertion is our main theorem.

Main Theorem Let $M'$ be an $(n + 1)$-dimensional Lorentz manifold which satisfies the condition (§) and $M$ be a complete space-like hypersurface with constant mean curvature. If $M$ is not maximal and if it satisfies
A certain complete space-like hypersurface

\[ 2nc_2 + c_1 > 0, \]

then there exist a positive constant \( a_1 \) depending on \( c_1, c_2, c_3, h \) and \( n \) such that \( h_2 \leq a_1 \), where the condition \((\ast)\) means \((5.1), (5.2) \) and \((5.3)\), \( h_2 \) denotes the square norm of the second fundamental form.

As an application of this result, we able to make a generalization of some theorem which are investigated by Cheng and Nakagawa ([3]), Ishihara ([7]), Li ([8]), and Nishikawa ([9]) in new different view point.

2 Definitions

Let \( M' = (M', g') \) be a Lorentz manifold with a Lorentz metric \( g' \) of signature \((-\cdot, +\cdot, +\cdot, \cdots, +)\). \( M' \) has uniquely defined torsion-free affine connection \( \nabla' \) compatible with the metric \( g' \). \( M' \) is called locally symmetric if the curvature tensor \( R' \) of \( M' \) is parallel, that is, \( \nabla' R' = 0 \). Let \( M \) be a hypersurface immersed in \( M' \). \( M \) is said to be space-like if the Lorentz metric \( g' \) of \( M' \) induces a Riemannian metric \( g \) on \( M \). For a space-like hypersurface \( M \) there is naturally defined the second fundamental form (the extrinsic curvature) \( \alpha \) of \( M \). \( M \) is called maximal space-like if the mean(extrinsic) curvature \( H = \text{Tr} \alpha \), the trace of \( \alpha \), of \( M \) vanishes identically. \( M \) is maximal space-like if and only if it is extreme under the variations, with compact support through space-like hypersurfaces, for the induced volume. \( M \) is said to be totally geodesic (a moment of time symmetry) if the second fundamental form \( \alpha \) vanishes identically.

3 Preliminaries

Let \( M \) be a space-like hypersurface in a Lorentz \((n+1)\)-manifold \( M' = (M', g') \). We choose a local field of Lorentz orthonormal frames \( e_0, \cdots, e_n \) are tangent to \( M' \) such that, restricted to \( M \), the vectors \( e_1, \cdots, e_n \) are tangent to \( M \). Here and in the sequel the following convention on the range of indices used throughout this paper, unless otherwise stated:

\[ i, j, k, \cdots = 1, 2, \cdots, n \quad \alpha, \beta, \cdots = 0, 1, 2 \cdots n \]

Let \( \omega_\alpha \) be its dual frame field so that the Lorentz metric \( g' \) can be written as \( g' = -\omega_0^2 + \sum_i \omega_i^2 \). then the connection forms \( \omega_{\alpha\beta} \) of \( M' \) are characterized by the equations

\[
\begin{align*}
  d\omega_i & = -\sum_k \omega_{ik} \wedge \omega_k + \omega_{i0} \wedge \omega_0, \\
  d\omega_0 & = -\sum_k \omega_{0k} \wedge \omega_k + \omega_{0\alpha} \wedge \omega_\alpha = 0.
\end{align*}
\]

The curvature forms \( \Omega'_{\alpha\beta} \) of \( M' \) are given by
\[ \Omega'_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} - \omega_{i0} \wedge \omega_{0j}, \]
\[ \Omega'_{0i} = d\omega_{0i} + \sum_k \omega_{0k} \wedge \omega_{ki}, \]

and we have

\[ \Omega'_{\alpha\beta} = -\frac{1}{2} \sum_{\gamma\delta} R'_{\alpha\beta\gamma\delta} \omega_{\gamma} \wedge \omega_{\delta}, \]

where \( R'_{\alpha\beta\gamma\delta} \) are components of the curvature tensor \( R' \) of \( M' \). We restrict these forms to \( M \).

Then

\[ \omega_0 = 0, \]

and the induced Riemannian metric \( g \) of \( M \) is written as \( g = \sum_i \omega_i^2 \). From formulas (3.1) \(~\) (3.4), we obtain the structure equations of \( M \)

\[ d\omega_i = -\sum_k \omega_{ik} \wedge \omega_k, \quad \omega_{ij} + \omega_{ji} = 0, \]
\[ d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \omega_{i0} \wedge \omega_{0j} + \Omega'_{ij}, \]
\[ \Omega_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \]

where \( \Omega_{ij} \) and \( R_{ijkl} \) denote the curvature forms and the components of the curvature tensor \( R \) of \( M \), respectively. We can also write

\[ \omega_{i0} = \sum_j h_{ij} \omega_j, \]

where \( h_{ij} \) are components of the second fundamental form \( \alpha = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j \) of \( M \). Using (3.6) in (3.5) then gives the Gauss formula

\[ R_{ijkl} = R_{ijkl}' - (h_{ik} h_{jl} - h_{il} h_{jk}). \]

Let \( h_{ijk} \) denote the covariant derivative of \( h_{ij} \) so that

\[ \sum_k h_{ijk} \omega_k = dh_{ij} - \sum_k h_{kj} \omega_{ki} - \sum_k h_{ik} \omega_{kj}. \]

Then, by exterior differentiating (3.6), we obtain the Codazzi equation

\[ h_{ijk} - h_{ikj} = R'_{0ijk}. \]

From the exterior derivative of (3.8), we define the second covariant derivative of \( h_{ij} \) by
A certain complete space-like hypersurface

\[ \sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ijk}\omega_l - \sum_l h_{ijl}\omega_k - \sum_l h_{ijl}\omega_i. \]

Then we obtain the Ricci formula

\[ h_{ijkl} - h_{ijlk} = \sum_m h_{mjk} R_{mikl} + \sum_m h_{mim} R_{mjkl}. \]

(3.10)

The components of the Ricci tensor \( S \) and the scalar curvature \( r \) of \( M \) are given by

\[ S_{ij} = \sum_k R'_{kikj} - hh_{ij} + h_{ij}^2, \]
\[ r = \sum_{j,k} R_{jkjk} - h^2 + h_2, \]

where \( h = \sum_i h_{ii}, h_{ij}^2 = \sum_r r_{ij} h_{rj} \) and \( h_2 = \sum_j h_{jj}^2. \)

Let us now denote the covariant derivative of \( R'_{\alpha\beta\gamma\delta} \), as a curvature tensor of \( M' \), by \( R'_{\alpha\beta\gamma\delta} \). Then restricting on \( M \), \( R'_{0ijkl} \) is given by

\[ R'_{0ijkl} = R'_{00ik} h_{jl} - R'_{00jk} h_{il} - \sum_m R'_{mijk} h_{ml}, \]

(3.11)

where \( R'_{0ijkl} \) denotes the covariant derivative of \( R'_{0ijk} \) as a tensor on \( M \) so that

\[ \sum_l R'_{0ijkl}\omega_l = dR'_{0ijkl} - \sum_l R'_{0ljk}\omega_l - \sum_l R'_{0ijkl}\omega_l - \sum_l R'_{0ijkl}\omega_l. \]

For the sake of brevity, a tensor \( h_{ij}^{2m} \) and a function \( h_{2m} \) on \( M \), for any integer \( m(\geq 2) \), are introduced as follows:

\[ h_{ij}^{2m} = \sum_{i_1,\ldots,i_{m-1}} h_{ii_1i_2} h_{i_1i_2} \cdots h_{i_{m-1}ji}, \]
\[ h_{2m} = \sum_i h_{ii}^{2m}. \]

First of all, let us introduce a fundamental property for the generalized maximal principle due to Omori ([10]) and Yau([13]).

**Theorem 3.1 ([10], [13])** Let \( M \) be a complete Riemannian manifold whose Ricci curvature is bounded from below on \( M \). Let \( F \) be a \( C^2 \)-function bounded from below on \( M \), then for any \( \epsilon > 0 \), there exists a point \( p \) such that

\[ |\nabla F(p)| < \epsilon, \quad \Delta F(p) > -\epsilon \quad \text{and} \quad \inf F + \epsilon > F(p). \]

We also know the following result ([5]).
Theorem 3.2 ([5]) Let $M$ be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let $F$ be a polynomial of the variable $f$ with constant coefficients such that

$$F(f) = c_0 f^n + c_1 f^{n-1} + \cdots + c_k f^{n-k} + c_{k+1},$$

where $n > 1$, $1 \geq n - k \geq 0$ and $c_0 > c_{k+1}$. If a $C^2$-positive function $f$ satisfies

$$\Delta f \geq F(f),$$

then we have

$$F(f_1) \leq 0,$$

where $f_1$ denotes the supremum of the given function $f$.

4 The Laplacian operator

Let $M$ be a space-like hypersurface of an $(n+1)$-dimensional Lorentz manifold $M'$. Then the Laplacian $\Delta h_{ij}$ of the components $h_{ij}$ of $\alpha$ is defined by

$$\Delta h_{ij} = \sum_k h_{ijkk}.$$

From (3.9) we have

$$\Delta h_{ij} = \sum_k h_{ijk} + \sum_k R_{0ijkk},$$

and from (3.10) it follows that

$$h_{kijk} = h_{kikj} + \sum_m h_{mi} R_{mkjk} + \sum_m h_{km} R_{mijk}.$$  

By using (3.9), replace $h_{kikj}$ in (4.2) by $h_{kikj} + R_{0ikj}$ and substitute the right hand side of (4.2) into $h_{kijk}$ in (4.1). Then we get

$$\Delta h_{ij} = \sum_k (h_{kkij} + R_{0kij} + R_{0ijkk})$$

$$+ \sum_k (\sum_m h_{mi} R_{mkjk} + \sum_m h_{km} R_{mijk}).$$

From (3.7), (3.11) and (4.3) we have
A certain complete space-like hypersurface

\( \Delta h_{ij} = \sum_k h_{kkij} + \sum_k R'_0 kij; + \sum_k R'_0 ijk; \)

\( + \sum_k (h_{kk} R'_0 ij0 + h_{ij} R'_0 k0k) \)

\( + \sum_{m,k} (h_{mj} R'_0 mkik + 2 h_{mk} R'_0 mijk + h_{mi} R'_0 mkjk) \)

\( - \sum_{m,k} (h_{mi} h_{mj} h_{kk} + h_{km} h_{mj} h_{ik} - h_{km} h_{mk} h_{ij} - h_{mi} h_{mk} h_{kj}). \)

5 Some curvature conditions

Let \( M' \) be an \( (n + 1) \)-dimensional Lorentz manifold and let \( M \) be a space-like hypersurface of \( M' \). For a point \( x \) in \( M \) let \( \{ e_0, e_1, \cdots, e_n \} \) be a local field of orthonormal frames of \( M' \) around of \( x \) in such a way that, restricted to \( M \), the vectors \( e_1, \cdots, e_n \) are tangent to \( M \) and the other is normal to \( M \). Accordingly, \( e_1, \cdots, e_n \) are space-like vectors and \( e_0 \) is a time-like one. For linearly independent vectors \( u \) and \( v \) in the tangent space \( T_x M' \), by which the non-degenerate plane section is spanned, we denote by \( K'(u, v) \) the sectional curvature of the plane section in \( M' \) and by \( R' \) or \( \text{Ric}'(u, u) \) the Riemannian curvature tensor on \( M \) or the Ricci curvature in the direction of \( u \) in \( M' \), respectively. Let us denote by \( \nabla' \) the Riemannian connection on \( M' \). We assume that the ambient space \( M' \) satisfies the following three conditions: For some constants \( c_1, c_2 \) and \( c_3 \)

\[
K'(u, v) = \frac{c_1}{n}, \quad (5.1)
\]

\[
K'(u, v) \geq c_2, \quad (5.2)
\]

\[
|\nabla' R'| \leq \frac{c_3}{n}, \quad (5.3)
\]

When \( M' \) satisfies the above conditions (5.1), (5.2) and (5.3), it is said simply for \( M' \) to satisfy the (*) condition.

Remark 5.1 It can be easily seen that 1=0, then the ambient space \( M' \) is locally symmetric.

Remark 5.2 If \( M \) is maximal, then the condition (5.1) can be replaced by

\[
\text{Ric}'(v, v) \geq c_1, \quad (5.4)
\]

for any time-like vector \( v \).
If $M'$ satisfies the conditions (5.4), (5.2) and (5.3), it is said simply for $M'$ to satisfy the condition $(\ast')$.

**Remark 5.3** If $M'$ is a Lorentz space form $M_1^{n+1}(c)$ of index 1 and of constant curvature $c$, then it satisfies the condition $(\ast')$, where $-\frac{c_1}{n} = c_2 = c$.

Now we assume that the ambient space $M'$ satisfies the condition $(\ast)$ and the mean curvature of the hypersurface $M$ is constant. Then the Laplacian of the squared norm $h_2$ of the second fundamental form $\alpha$ of $M$ is given by

$$\triangle h_2 = 2|\nabla \alpha|^2 + 2 \sum_{i,j} (R_{0ik;j} + R_{0ij;k}) + 2 \sum_k (h_{kk}R'_{0ij0} + h_{ij}R'_{0k0k}) + \sum_{k,m} (2h_{km}R_{mijk} + h_{mj}R_{mkjk} - h h_{ij}^2) h_{ij},$$

where $\nabla \alpha$ is the covariant derivative of the second fundamental form $\alpha$ and $|\nabla \alpha|$ is the norm of $\nabla \alpha$ which is defined by $\sum_{i,j,k} h_{ij}h_{ijk}$. Hence, by (4.4) and the assumption $\sum_k h_{kk}=0$, we have

$$\triangle h_2 = 2|\nabla \alpha|^2 + 2 \sum_{i,j} (R_{0ik;j} + R_{0ij;k}) + 2 \sum_k (h_{kk}R'_{0ij0} + h_{ij}R'_{0k0k}) + \sum_{k,m} (2h_{km}R_{mijk} + h_{mj}R_{mkjk} + h_{mi}R'_{mkjk}) - 2(h h_{ij}^2) h_{ij}.$$

Thus we get

$$(5.5) \quad \triangle h_2 = 2|\nabla \alpha|^2 + 2 \sum_{i,j} h_{ij}(R'_{0ik;j} + R'_{0ij;k}) + 2 \sum_k (h_{kk}R'_{0ij0} + h_{ij}R'_{0k0k}) + 4 \sum_{i,j,k,m} h_{ij}h_{km}R'_{mijk} + \sum h_{mj}^2 R_{mkjk} - 2(h h_{ij}^2),$$

where we have denoted by $h_{ij}^3 = \sum h_{ir}h_{rj}$ and $h_3 = \sum h_{ii}^3$. Since the matrix $H=(h_{ij})$ can be diagonalized, the component of $h_{ij}$ of $H$ can be expressed by

$$(5.6) \quad h_{ij} = \lambda_i \delta_{ij},$$

where $\lambda_i$ is the principle curvature on $M$. By definition, we see

$$\lambda_i^2 \leq h_2 = \sum_i \lambda_i^2,$$

and hence we have
A certain complete space-like hypersurface

(5.7) \[-\sqrt{h_2} \leq \lambda_i \leq \sqrt{h_2},\]

(5.8) \[-h_2 \leq \lambda_i \lambda_j \leq h_2.\]

Now, we estimate (5.5) from above. First, we treat with the second term of (5.5).

It is seen that we have

\[-2 \sum_{i,j,k} (R'_{0ik;j} + R'_{0ijk;k}) h_{ij} = -2 \sum_{j,k} \lambda_j (R'_{0jk;j} + R'_{0j;j;k}) \leq 2 \sum_{j,k} |\lambda_j| (|R'_{0jk;j}| + |R'_{0j;j;k}|).\]

So by (5.3) and (5.7) we have

(5.9) the second term of (5.5) \(\geq -4c_3 \sqrt{h_2}\).

Next, we consider the third term of (5.4). It is estimated as follows:

\[2 \sum_{i,j} h_{ij} R'_{0ij0} + h_2 \sum_{k} R'_{0k0k} = 2 \sum_{k} (h \lambda_k R'_{0k0k} + h_2 \sum_{k} R'_{0k0k}) = 2 \sum_{k} (h_2 - h \lambda_k) R'_{0k0k} = 2 \sum_{k} (h_2 - h \lambda_k) \frac{c_1}{n},\]

where we have used (5.1). Hence we have

(5.10) the third term of (5.5) \(= \frac{2c_1 (nh_2 - h^2)}{n}\).

It is evident that if the ambient space \(M'\) is a Lorentz space form \(M^{n+1}_c\) of constant curvature \(c\) and if the hypersurface \(M\) is maximal, then it also holds under (5.4), namely if \(M'\) satisfies the condition \((*)\), then the third term of (5.5) \(\geq 2c_1 h_2\). Last we estimate the fourth term of (5.5). We have by (5.2)

\[4 \sum_{i,j,k,m} h_{ij} h_{km} R'_{0ijk} + \sum_{j,k,m} h_{ij} h_{km} R'_{0ijk} = 4 \sum_{j,k} (\lambda_j \lambda_k R'_{0jkj} + \lambda_j^2 R'_{0jk}) \]

\[= 4 \sum_{j,k} (\lambda_j^2 - \lambda_j \lambda_k) R'_{0jkj} = 2 \sum_{j,k} (\lambda_j - \lambda_k)^2 R'_{0jkj} \geq 2c_2 \sum_{j,k} (\lambda_j - \lambda_k)^2.\]

Accordingly, we obtain

(5.11) the fourth term of (5.5) \(\geq 4c_2 (nh_2 - h^2)\),

where we have used the formula
\[
\sum_{j,k} (\lambda_j - \lambda_k)^2 = 2n \sum_j \lambda_j^2 - 2 \sum \lambda_j \lambda_k = 2n \sum_j \lambda_j^2 - 2(\sum \lambda_j)^2
\]

and the definitions of \( h_2 = \sum_j \lambda_j^2 \) and \( h^2 = (\sum \lambda_j)^2 \). Thus, substituting (5.9), (5.10) and (5.11) into (5.5), we can prove the following.

**Lemma 5.4** Let \( \mathcal{M}' \) be an \((n + 1)\)-dimensional Lorentz manifold satisfying the condition \((\ast)\) and \( M \) a space-like hypersurface of \( \mathcal{M}' \). If its mean curvature is constant, then we have

\[
\triangle h_2 \geq -4c_3 \sqrt{h_2} + 2(2nc_2 + c_1)(nh_2 - h^2) - 2(hh_3 - h_2^2).
\]

In particular, if \( M \) is maximal, we have

\[
\triangle h_2 \geq -4c_3 \sqrt{h_2} + 2(2nc_2 + c_1)h_2 + 2h_2^2.
\]

Also, if \( \mathcal{M}' = \mathcal{M}_{1+n}(c) \), then we obtain

\[
\triangle h_2 \geq 2c(nh_2 - h^2) - 2(hh_3 - h_2^2).
\]

### 6 Proof of Main Theorem

Let \( \mathcal{M}' \) be an \((n + 1)\)-dimensional Lorentz manifold and let \( M \) be a complete hypersurface of \( \mathcal{M}' \) with constant mean curvature. Assume that the ambient space satisfies the condition \((\ast)\). The condition \((\ast)\) is defined by (5.1), (5.2) and (5.3). Now, by (5.12) in Lemma 5.1 the function \( h_2 \) satisfies

\[
\triangle h_2 \geq -4c_3 \sqrt{h_2} + \frac{2(2nc_2 + c_1)(nh_2 - h^2)}{n} - 2(hh_3 - h_2^2).
\]

Moreover, we obtain

\[
(6.1) \quad -2hh_3 = -2h \sum_i h_i^3 = -2h \sum_j \lambda_j^3 \geq -2h \sum_j \sqrt{h_2}^3 = -2nhh_2 \sqrt{h_2},
\]

from which together with (5.12) it follows that

\[
(6.2) \quad \triangle h_2 \geq -4c_3 \sqrt{h_2} + 2(2nc_2 + c_1)(h_2 - \frac{h^2}{n}) - 2nhh_2 \sqrt{h_2} + 2h_2^2.
\]

Now we define a non-negative function \( f \) by \( f^2 = h_2 \). Then it turns out to be

\[
(6.3) \quad \triangle f^2 \geq 2[f^4 - nhf^3 + (2nc_2 + c_1)f^2 - 2c_3f - \frac{h^2}{n}(2nc_2 + c_1)].
\]

**Proof of the Main Theorem**

Let \( \lambda_1, \ldots, \lambda_n \) be principal curvatures on \( M \). The Ricci tensor \( S_{ij} \) is expressed by
\[ S_{ij} = \sum_k (R'_{kikj} - h_{ij} h_{kk} + h_{ik} h_{jk}). \]

So we have
\[ S_{jj} \geq (n-1)c_2 - h\lambda_j + \lambda_j^2 \geq (n-1)c_2 - \frac{h^2}{4}, \]
which yields the Ricci curvature of \( M \) is bounded from below. For the function \( f \) defined by \( f^2 = h_2 \), by (6.3) we have
\[ \Delta f^2 \geq F(f^2), \]
where the function \( F(x) \) is defined by
\[ F(x) = 2[x^2 - nhx^3 + (2nc_2 + c_1)x - 2c_3 x^2 - \frac{h^2}{n}(2nc_2 + c_1)]. \]

By comparing with (3.12), we get
\[ n = 2, \quad n - k = \frac{1}{2}, \quad c_0 = 2, \quad c_{k+1} = -\frac{2h^2(2nc_2 + c_1)}{n}, \]
where we have used \( 2nc_2 + c_1 > 0 \). Now we are able to apply Theorem 3.2 to the function \( f^2 \). Then we obtain
\[ (6.4) \quad F(f_1^2) \leq 0, \]
where \( f_1^2 \) denotes the supremum of the given function \( f^2 \).

We define the function \( y = y(x) \) of the variable \( x \) by
\[ y = y(x) = x^4 - nhx^3 + (2nc_2 + c_1)x^2 - 2c_3 x^2 - \frac{h^2}{n}(2nc_2 + c_1). \]

By the assumption \( 2nc_2 + c_1 > 0 \) and the fact that the hypersurface is not maximal, the algebraic equation \( y(x) = 0 \) with constant coefficients has positive roots because \( y(0) < 0 \) and it converges to infinity as \( x \) tends to infinity. We denote by \( \sqrt{a_1} (a_1 > 0) \) the minimal root among the positive roots. So it depends only on the constant coefficients, namely, it depends on \( c_1, c_2, c_3, h \) and \( n \), and by definition we see that
\[ y|_{[0, \sqrt{a_1})} < 0. \]

From the above equation together with (6.4) it follows that we have \( 0 \leq f_1 \leq \sqrt{a_1} \). Since the squared norm \( h_2 \) of the second fundamental form is given by \( h_2 = f^2 \), we have
\[ \sup h_2 = f_1^2 \leq a_1. \]

So we get the conclusion. \( \square \)

If the hypersurface \( M \) is maximal, then we have by (6.3)
\[ \Delta f^2 \geq 2\{f^4 + (2nc_2 + c_1)f^2 - 2c_3 f\} = F(f^2), \]
where a non-negative function \( f \) is defined by \( f^2 = h_2 \). By a similar method to the proof of our Main Theorem, we have
\[ (6.5) \quad F(f_1^2) \leq 0, \]
where $f_1$ denotes the supremum of the function $f$. We define a function $y$ of the variable $x$ by

$$y = y(x) = x\{x^3 + (2nc_2 + c_1)x - 2c_3\}.$$  

By the direct calculus, there exists a unique positive root of the equation $y(x) = 0$, say $\sqrt{a_1}$, if $c_3 > 0$.

**Corollary 6.1** Let $M'$ be an $(n+1)$-dimensional Lorentz manifold which satisfies the condition $(\ast)$ and let $M$ be a complete space-like maximal hypersurface. If $M'$ is not locally symmetric, then there exists a positive constant $a_1$ depending on $c_1, c_2, c_3(>0), h$ and $n$ such that $h_2 \leq a_1$.

**Remark 6.2** Corollary 6.1 was proved by Li ([8]) under the additional condition $c_3^2 + \frac{(2nc_2 + c_1)^3}{27} < 0$.

**Remark 6.3** In the case where the ambient space is locally symmetric and it satisfies the condition $(\ast')$, the constant $a_1$ is the positive root of the algebraic equation

$$F(x^2) = x^2\{x^3 + (2nc_2 + c_1)\} = 0,$$

which yields that if $2nc_2 + c_1 \geq 0$, then $F|_{(0, \infty)} > 0$, which means that we have no positive roots. In the case where $2nc_2 + c_1 < 0$ there exists a unique positive root of the equation $y(x) = 0$, say $\sqrt{a_1}$. In the first case, considering (6.5) we have $f_1 = 0$. By definition of $f_1 = \sup f$ for the non-negative function $f$, we see that $f$ vanishes identically on $M$. It yields that $M$ is totally geodesic. So if it satisfies $2nc_2 + c_1 < 0$, then we have $a_1 = -(2nc_2 + c_1)$. This result was derived by Li ([8]). The first assertion of Corollary 6.1 was also proved by Nishikawa([9]). In particular, when $M' = H_{n+1}^{(c)}$, this reduces to Ishihara's theorem ([7]).

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A certain complete space-like hypersurface


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