Einstein-Yang Mills equations for gauge transformations of second order

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Abstract

Lately a big attention has been paid to the gauge transformations and their applications. The gauge theory of second order was studied by Gh. Munteanu in [16], [17], [18]. In [7] and [8] some generalizations are given. Fundamental results in gauge theory can be found in [1], [2] etc. The transformations of type (1.1) were studied in [9], [10], [17], [18]. Some other types of transformation with more variables and their applications were studied in [3], [9], [13], [14], [15], [16].

Here in the tangent space $T F$ such an adapted basis is constructed, that the horizontal and the two vertical distributions with respect to the given metric structure are mutually orthogonal. The torsion free generalized connection is determined and its coefficients are obtained under condition that the metric structure is parallel or recurrent. The Einstein-Yang Mills equations are also given.

Mathematics Subject Classification: 53B40, 53C60, 53C80, 81T13, 53C07.

Key words: Generalized connection, metric connection, gauge connection.

1. Adapted basis in $TF$

Let $F$ be an $n + m + l$ dimensional $C^\infty$ manifold. Some point $u \in F$ has coordinates $(x^i, y^a, z^p)$ and the allowable coordinate transformations are given by the equations

\[
\begin{align*}
  x^i' &= x^i(x), & i, j, h, k = 1, \ldots, n, \\
  y^a' &= y^a(x, y), & a, b, c, d, e = n + 1, \ldots, n + m, \\
  z^p' &= z^p(x, z), & p, q, r, s, t = n + m + 1, \ldots, n + m + l,
\end{align*}
\]

where

\[
\begin{align*}
  \text{rank } \left[ \frac{\partial x^i'}{\partial x^i} \right] &= n, & \text{rank } \left[ \frac{\partial y^a'}{\partial y^a} \right] &= m, & \text{rank } \left[ \frac{\partial z^p'}{\partial z^p} \right] &= l.
\end{align*}
\]

Proposition 1.1. The coordinate transformations of type (1.1) form a pseudo group.
The Einstein-Yang Mills equations

If the functions \( \mathcal{N}'_{ij}(x', y') \) and \( \mathcal{M}'_{ij}(x', z') \) satisfy the following law of transformation ([18]):

\[
\begin{align*}
N^b_i(x, y) &= N'^{c}_i(x', y') \frac{\partial x'}{\partial x^i} \frac{\partial y^b}{\partial y^c} + \frac{\partial y^b}{\partial x^i} \frac{\partial y^b}{\partial y'^c}, \\
M^p_i(x, z) &= M'^{q}_{i}(x', z') \frac{\partial x'}{\partial x^i} \frac{\partial z^p}{\partial x^q} + \frac{\partial z^p}{\partial x^i} \frac{\partial z^p}{\partial x'^q},
\end{align*}
\]

then the adapted basis of \( TF \) is \( B(\mathcal{N}, \mathcal{M}) = \{ \delta \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial z^p} \} \), where

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^b_i(x, y) \frac{\partial}{\partial y^b} - M^p_i(x, z) \frac{\partial}{\partial z^p}.
\]

Let us denote by \( T_HF, T_{V_1}F, T_{V_2}F \) the subspaces of \( TF \) spanned by \( \{ \delta \frac{\partial}{\partial x^i} \}, \{ \frac{\partial}{\partial y^a} \}, \{ \frac{\partial}{\partial z^p} \} \) respectively, then

\[
TF = T_HF \oplus T_{V_1}F \oplus T_{V_2}F.
\]

**Theorem 1.1.** The horizontal distribution \( T_HF \) is integrable if

\[
\begin{align*}
N^q_{i,j} &= \left( \frac{\partial N^c_i}{\partial x^j} - N^b_j \frac{\partial N^c_i}{\partial y^b} \right) - (i, j) = 0, \\
M^q_{i,j} &= \left( \frac{\partial M^q_i}{\partial x^j} - M^p_j \frac{\partial M^q_i}{\partial z^p} \right) - (i, j) = 0,
\end{align*}
\]

and \((i, j)\) is the expression in the previous bracket, in which \( i \) and \( j \) change their places. \( T_{V_1} \) and \( T_{V_2} \) are integrable distributions.

**Proof.** By direct calculation using the abbreviations \( \delta_i = \frac{\delta}{\delta x^i}, \delta_t = \frac{\partial}{\partial x^i}, \delta_a = \frac{\partial}{\partial y^a}, \)

\[
\delta_p = \frac{\partial}{\partial z^p},
\]

we get

\[
[\delta_i, \delta_j] = N^c_{i,j} \delta_c + M^q_{i,j} \delta_q
\]

and this vector is in \( T_HF \) only if \( N^c_{i,j} = 0 \) and \( M^q_{i,j} = 0 \). On the other side it is obvious, that

\[
[\delta_a, \delta_b] = 0, \quad [\delta_p, \delta_q] = 0.
\]

Putting

\[
\begin{align*}
\delta y^a &= dy^a + N^a_i(x, y) dx^i, \\
\delta z^p &= dz^p + M^p_i(x, z) dx^i,
\end{align*}
\]

the adapted basis \( B^*(\mathcal{N}, \mathcal{M}) = \{ dx^i, \delta y^a, \delta z^p \} \) of \( T^*F \) is formed.

There are so many adapted basis \( B(\mathcal{N}, \mathcal{M}) \) and \( B^*(\mathcal{N}, \mathcal{M}) \) as many solutions have the equations (1.2) and (1.3). In the next section we shall determine such \( \mathcal{N} \) and \( \mathcal{M} \) ((2.9)), that \( T_HF, T_{V_1}F \) and \( T_{V_2}F \) are mutually orthogonal subbundles with respect to the given metric \( G \).

Some \( d \)-tensor gauge \( T \) on \( F \) in the bases \( B \) and \( B^* \) is expressed in the form:

\[
T = T_{\ldots, i, \ldots, a, \ldots} \delta_{\delta x^i} \otimes dx^i \cdots \frac{\partial}{\partial y^a} \otimes \delta y^b \cdots \frac{\partial}{\partial z^p} \otimes \delta z^p \cdots
\]
The components of \( d \)-tensor gauge \( T \), with respect to the coordinate transformations (1.1) are transformed in the following way:

\[
T_{\alpha'\beta'\gamma'\cdots} = T_{\alpha\beta\gamma\cdots} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^{\beta'}}{\partial x^\beta} \cdots \frac{\partial x^{\gamma'}}{\partial x^\gamma} \cdots \frac{\partial z^{\alpha'}}{\partial z^\alpha} \cdots \frac{\partial z^{\beta'}}{\partial z^\beta} \cdots \frac{\partial z^{\gamma'}}{\partial z^\gamma} \cdots
\]

**Theorem 1.2.** The adapted bases \( B \) and \( B^* \) are dual to each other.

### 2. Orthogonality of the subspaces of \( TF \)

The metric tensor \( G \) in \( F \) is a symmetric, positive definite tensor of type \((0, 2)\). In the natural basis of \( T^*F \), \( B^* = \{dx^i, dy^a, dz^p\} \), \( G \) has the form:

\begin{equation}
G = g_{ij} dx^i \otimes dx^j + g_{ib} dx^i \otimes dy^b + g_{aq} dx^a \otimes dz^q +
\end{equation}

\begin{equation}
\bar{g}_{ij} dy^i \otimes dx^j + \bar{g}_{ab} dy^a \otimes dy^b + \bar{g}_{aq} dy^a \otimes dz^q +
\end{equation}

\begin{equation}
\bar{g}_{pq} dz^p \otimes dx^q + \bar{g}_{pb} dz^p \otimes dy^b + \bar{g}_{pq} dz^p \otimes dz^q.
\end{equation}

In the adapted basis \( B^* = \{dx^i, \delta y^a, \delta y^b\} \) of \( T^*F \) the metric tensor \( G \) has the following components:

\begin{equation}
G = g_{ij} dx^i \otimes dx^j + g_{ib} dx^i \otimes \delta y^b + g_{aq} dx^a \otimes dz^q +
\end{equation}

\begin{equation}
\bar{g}_{ij} \delta y^i \otimes dx^j + \bar{g}_{ab} \delta y^a \otimes \delta y^b + \bar{g}_{aq} \delta y^a \otimes dz^q +
\end{equation}

\begin{equation}
\bar{g}_{pq} \delta z^p \otimes dx^q + \bar{g}_{pb} \delta z^p \otimes \delta y^b + \bar{g}_{pq} \delta z^p \otimes \delta z^q.
\end{equation}

**Proposition 2.1.** The components of the metric tensor \( G \) expressed in the bases \( B^* \) and \( B^* \) are connected by formulae:

\begin{equation}
g_{ij} = g_{ij} - \bar{g}_{ij} N^c_i - \bar{g}_{ic} N^c_j - \bar{g}_{rj} M^r_i - \bar{g}_{ir} M^r_j +
\end{equation}

\begin{equation}
\bar{g}_{ab} N^a_i N^b_j + \bar{g}_{aq} N^a_i M^b_j + \bar{g}_{pb} M^a_i N^b_j + \bar{g}_{pq} M^a_i M^b_j,
\end{equation}

\begin{equation}
g_{ib} = \bar{g}_{ib} - \bar{g}_{ab} N^a_i - \bar{g}_{pb} M^a_i
\end{equation}

\begin{equation}
g_{aj} = \bar{g}_{aj} - \bar{g}_{ab} N^a_j - \bar{g}_{aq} M^a_j
\end{equation}

\begin{equation}
g_{iq} = \bar{g}_{iq} - \bar{g}_{aq} N^a_i - \bar{g}_{pq} M^a_i
\end{equation}

\begin{equation}
g_{pj} = \bar{g}_{pj} - \bar{g}_{pb} N^b_j - \bar{g}_{pq} M^b_j
\end{equation}

\begin{equation}
0 = \bar{g}_{ib} - g_{ab} N^a_i, \quad 0 = g_{aj} - \bar{g}_{ab} N^a_j,
\end{equation}

\begin{equation}
0 = \bar{g}_{iq} - g_{pq} M^a_i, \quad 0 = \bar{g}_{pj} - g_{pq} M^b_j.
\end{equation}

The proof follows from (1.8), (1.9), (2.1) and (2.2).

**Proposition 2.2.** If \( T_{\mu i} F, T_{\nu j} F \) and \( T_{\nu i} F \) are mutually orthogonal spaces with respect to the metric tensor \( G \), then (2.3) has the form:

\begin{equation}
g_{ij} = \bar{g}_{ij} - \bar{g}_{ic} N^c_i - \bar{g}_{ic} N^c_j - \bar{g}_{rj} N^r_i - \bar{g}_{ir} N^r_j +
\end{equation}

\begin{equation}
g_{pq} M^a_i M^a_j + g_{ab} N^a_i N^a_j,
\end{equation}

\begin{equation}
0 = \bar{g}_{ib} - g_{ab} N^a_i, \quad 0 = g_{aj} - \bar{g}_{ab} N^a_j,
\end{equation}

\begin{equation}
0 = \bar{g}_{iq} - g_{pq} M^a_i, \quad 0 = \bar{g}_{pj} - g_{pq} M^b_j.
\end{equation}
Proof. From the orthogonality of \( T_H F, T_{V_1} F \) and \( T_{V_2} F \) with respect to the metric tensor \( G \), follows that in (2.2) we have:

\[
(2.8) \quad g_{ib} = 0, \ g_{iq} = 0, \ g_{aj} = 0, \ g_{aq} = 0, \ g_{pj} = 0, \ g_{pb} = 0.
\]

Substituting (2.4) and (2.8) into (2.3) we obtain (2.5), (2.6) and (2.7).

**Theorem 2.1.** If \( T_H F, T_{V_1} F \) and \( T_{V_2} F \) are mutually orthogonal with respect to the metric tensor \( G \) given by (2.1), then:

\[
(2.9) \quad N_i^c = \tilde{g}_{ib} g^{bc}, \ M_j^r = \tilde{g}_{pj} g^{pr},
\]

where \((g^{bc})\) and \((g^{pr})\) are the inverse matrices of \((g_{ab})\) and \((g_{rs})\) respectively. If \( T_{V_1} \) is orthogonal to \( T_{V_2} \) with respect to \( G \) and (2.9) are satisfied, then \( T_H \) is orthogonal to \( T_{V_1} \) and to \( T_{V_2} \).

Proof. The first assertion follows from (2.6) and (2.7). The existence of inverse matrices follows from the fact, that \( G \) is positive definite. From the symmetry of the metric tensor and (2.9) follow (2.6) and (2.7). From the orthogonality of \( T_{V_1} \) and \( T_{V_2} \) we have \( g_{aq} = \tilde{g}_{aq} = 0, \ g_{pb} = \tilde{g}_{pb} = 0 \). From these relations, (2.6) and (2.9), using (2.3), we obtain \( g_{ib} = 0, \ g_{aj} = 0, \ g_{aq} = 0, \ g_{pj} = 0 \).

**Proposition 2.3.** The nonlinear connections \( N_i^c \) and \( M_j^r \) determined by (2.9) satisfy the transformation laws (1.2) and (1.3).

Proof. Using the relations:

\[
\begin{align*}
dx^i' &= \frac{\partial x^i'}{\partial x^i} dx^i + \frac{\partial y^a'}{\partial y^a} dy^a, \\
&= \frac{\partial x^i'}{\partial x^i} dx^i + \frac{\partial y^a'}{\partial y^a} dy^a, \\
dz^p' &= \frac{\partial z^p'}{\partial z^p} dz^p + \frac{\partial z^p'}{\partial z^p} dz^p
\end{align*}
\]

and the expression similar to (2.1) for the metric tensor \( G \) in \( \bar{B}' = \{dx^i', dy^a', dz^p'\} \) we get

\[
(2.10) \quad \tilde{g}_{ib} = \tilde{g}_{iv} \frac{\partial x^i'}{\partial x^i} \frac{\partial y^b'}{\partial y^b} + \tilde{g}_{av} \frac{\partial x^i'}{\partial x^i} \frac{\partial y^a'}{\partial y^a} + \tilde{g}_{vp} \frac{\partial z^p'}{\partial z^p} \frac{\partial y^b'}{\partial y^b}
\]

\[
(2.11) \quad \tilde{g}_{ab} = \tilde{g}_{av} \frac{\partial x^i'}{\partial x^i} \frac{\partial y^a'}{\partial y^a} \frac{\partial y^b'}{\partial y^b}.
\]

From

\[
g_{ib} = g_{iv} \frac{\partial x^i'}{\partial x^i} \frac{\partial y^b'}{\partial y^b} \quad \text{and} \quad \tilde{g}_{ib} = \tilde{g}_{ib} - \tilde{g}_{ab} N_i^a
\]

follows

\[
(2.12) \quad \tilde{g}_{ib} - \tilde{g}_{ab} N_i^a = (\tilde{g}_{iv} - \tilde{g}_{av} N_i^a) \frac{\partial x^i'}{\partial x^i} \frac{\partial y^b'}{\partial y^b}.
\]

Substituting (2.10) and (2.11) into (2.12) and using the fact that \( T_{V_1} \) is orthogonal to \( T_{V_2} \), i.e. \( g_{pb} = g_{pv} = 0 = \tilde{g}_{pv} \) we get

\[
(2.13) \quad g_{iv} \frac{\partial x^i'}{\partial x^i} \frac{\partial y^v'}{\partial y^v} - g_{av} \frac{\partial x^i'}{\partial x^i} \frac{\partial y^a'}{\partial y^a} N_i^a = -g_{av} N_i^a \frac{\partial x^i'}{\partial x^i} \frac{\partial y^b'}{\partial y^b}.
\]
If we multiply (2.13) with \( \frac{\partial g^{\alpha \beta}}{\partial y^p} \hat{g}^{i_1 \ldots i_q} \frac{\partial g^{i_1 \ldots i_q}}{\partial y^\rho} \) we obtain (1.2). In the similar way we can obtain (1.3).

**Proposition 2.4.** If the adapted basis \( B = \{ \delta_i, \partial_a, \partial_b \} \) is formed by \( N_j^i \) and \( M_j^i \) determined by (2.9), then the horizontal distribution \( T_H F \) is integrable iff:

\[
N_j^i = \left[ \partial_j \hat{g}_{ia} g^{ac} \right] - \hat{g}_{ja} \partial_i (\hat{g}_{ia} g^{ac}) \} - [i,j] = 0,
\]

\[
M_j^i = \left[ \partial_j (\hat{g}_{ip} g^{pq}) - \hat{g}_{jp} \partial_i (\hat{g}_{ip} g^{pq}) \right] - [i,j] = 0.
\]

The proof follows from Theorem 1.1 and (2.9).

The connection between adapted basis and metric structure for the generalized Finsler space was studied in [5], [6].

### 3. Gauge covariant derivatives of the second order

We shall suppose that on \( F \) the metric tensor \( G \) is given by (2.1).

If we form the adapted basis \( B^* \) using the nonlinear connection coefficients \( N_j^i \) and \( M_j^i \) determined by (2.9), as functions of the metric tensor \( G \) and suppose that \( T_{V_1} \) is orthogonal to \( T_{V_2} \), then according to Theorem 2.1 it follows that \( T_{H'F}, T_{V_1} F, T_{V_2} F \) are mutually orthogonal subbundles with respect to \( G \) and in this basis the metric tensor instead of (2.2) has the form:

\[
G = g_{ij} dx^i \otimes dx^j + g_{ab} \delta y^a \otimes \delta y^b + g_{pq} \delta z^p \otimes \delta z^q.
\]

From now on we shall always choose such adapted bases \( B \) and \( B^* \) in which \( N_j^i \) and \( M_j^i \) are determined by (2.9).

**Definition 3.1.** Let \( \nabla : TF \times TF \rightarrow TF \) (\( \times \) is the Descartes' product) be a usual linear connection, such that \( \nabla : (X, Y) \rightarrow \nabla_X Y \in TF, \forall X, Y \in TF \). The operator \( \nabla \) is called generalized gauge connection of the second order.

A generalized gauge connection \( \nabla \) of the second order locally is expressed by

\[
\nabla_{\alpha} \partial_{\beta} = F_{\beta}^{\gamma} \partial_{\gamma},
\]

where \( \alpha, \beta, \gamma, \ldots = 1, \ldots, n+m+l \) and \( \partial_{\alpha} \) are elements of the basis \( B \).

It is called \( d \)-gauge connection of second order if \( \nabla_X Y \) is in \( T_{H'F}, T_{V_1} F, \) or \( T_{V_2} F \) if \( Y \) is in \( T_{H'F}, T_{V_1} F, \) or \( T_{V_2} F \) respectively, \( \forall X \in TF \). It has been studied by many authors, mostly romanian geometers ([2], [16], [17], [18]). The generalized connection in \( K \)-Hamilton spaces and in dual vector bundles were studied in [3] and [4].

**Theorem 3.1.** If the vector fields \( X, Y \) expressed in \( B \) have the form

\[
X = X^{\alpha} \partial_{\alpha} = X^{i} \partial_{i} + \sum_{a} X^{a} \partial_{a} + \sum_{p} X^{p} \partial_{p} ,
\]

\[
Y = Y^{\beta} \partial_{\beta} = Y^{j} \partial_{j} + \sum_{b} Y^{b} \partial_{b} + \sum_{q} Y^{q} \partial_{q} ,
\]

then

\[
\nabla_Y X = X^{\alpha}_{\beta} \partial_{\alpha},
\]

where

\[
X^{\alpha}_{\beta} = \partial_{\beta} X^{\alpha} + F_{\alpha \beta}^{\gamma} X^{\gamma} = \partial_{\beta} X^{\alpha} + F_{\alpha \beta}^{i} X^{i} + F_{\alpha \beta}^{a} X^{a} + F_{\alpha \beta}^{p} X^{p} .
\]
Theorem 3.2. The covariant derivatives are transformed as tensors if all connection coefficients are transformed as tensors except

\[ F^k_{j,i} = F^k'_{j',i'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} + \frac{\partial^2 x^{k'}}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial x^{k'}} \]

(b) \[ F^c_{b,i} = F^c'_{b',i'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial y^c}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial x^i \partial y^{c'}} \frac{\partial y^c}{\partial y^{c'}} - \mathcal{N}^b_i \frac{\partial^2 y^{c'}}{\partial y^b \partial y^{c'}} \frac{\partial y^c}{\partial y^{c'}} \]

c) \[ F^r_{q,i} = F^r'_{q',i'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial z^q}{\partial z^{q'}} + \frac{\partial^2 z^{r'}}{\partial x^i \partial z^q} \frac{\partial z^r}{\partial z^{r'}} - \mathcal{M}_{q,i} \frac{\partial^2 z^{r'}}{\partial z^q \partial z^{r'}} \frac{\partial z^r}{\partial z^{r'}} \]

(d) \[ F^c_{b,a} = F^c'_{b',a'} \frac{\partial y^{b'}}{\partial y^b} \frac{\partial y^c}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial y^b \partial y^{c'}} \frac{\partial y^c}{\partial y^{c'}} \]

(e) \[ F^r_{q,p} = F^r'_{q',p'} \frac{\partial z^q}{\partial z^{q'}} \frac{\partial z^p}{\partial z^{p'}} \frac{\partial z^r}{\partial z^{r'}} + \frac{\partial^2 z^{r'}}{\partial z^q \partial z^{r'}} \frac{\partial z^r}{\partial z^{r'}} \]

The torsion tensor \( T(X, Y) \) is defined in the usual way by:

\[ T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \]

Theorem 3.3. The torsion tensor for the generalized gauge connection of the second order has the form:

\[ T(X, Y) = T^k \delta_k + T^c \partial_c + T^r \partial_r, \]

where

\[ T^\alpha = T^\alpha_j Y^j X^i + T^\alpha_b Y^b X^j + T^\alpha_q Y^q X^i + \]

\[ T^\alpha_b' Y^b X^j + T^\alpha_a' Y^a X^b + T^\alpha_q' Y^q X^i + \]

\[ T^\alpha_p' Y^p X^i + T^\alpha_p' Y^p X^b + T^\alpha_q' Y^q X^i, \]

where \( \alpha = k \) or \( \alpha = c \) or \( \alpha = r \). The components of the torsion tensor are expressed as the difference of the corresponding connection coefficients for instance

\[ T^k_{j,i} = F^k_{j,i} - F^k_{i,j} \]

\[ T^k_{j,b} = F^k_{j,b} - F^k_{b,j}, \ldots \]

except the following

\[ T^c_{j,i} = F^c_{j,i} - F^c_{i,j} - \mathcal{N}^c_{i,j} \]

\[ T^c_{j,b} = F^c_{j,b} - F^c_{b,j} + (\partial_b \mathcal{N}^c) \]

\[ T^c_{b,i} = F^c_{b,i} - F^c_{i,b} - (\partial_b \mathcal{N}^c) \]

\[ T^c_{j,r} = F^c_{j,r} - F^c_{r,i} - \mathcal{M}^c_{i,j} \]

\[ T^c_{j,q} = F^c_{j,q} - F^c_{q,j} + \partial_q \mathcal{M}^r_{i,j} \]

\[ T^c_{p,i} = F^c_{p,i} - F^c_{i,p} - \partial_p \mathcal{M}^r_i \]
Theorem 3.4. The generalized gauge connection of the second order is torsion free if all connection coefficients are symmetric in the lower indices except the following:

\begin{align*}
(a) \quad & F_{j}^{c} = F_{b}^{c} - \partial_{b}N_{j}^{c}, \quad F_{j}^{r} = F_{q}^{r} - \partial_{q}M_{j}^{r} \\
(b) \quad & F_{j}^{c} = F_{i}^{c} + N_{i}^{c} \\
(c) \quad & F_{j}^{r} = F_{i}^{r} + M_{i}^{r}
\end{align*}

As \( N_{i}^{c} \) and \( M_{i}^{r} \) are tensors, so all \( F \)'s appeared in (3.8)(b) and (3.8)(c) are also tensors. When the horizontal distribution is integrable, then \( N_{i}^{c} = 0 \) and \( M_{i}^{r} = 0 \) (Theorem 1.1) and then \( F_{j}^{c} = F_{i}^{c} \) and \( F_{j}^{r} = F_{i}^{r} \).

Using (1.2), (1.3) and (b), (c) from Theorem 3.2 it can be proved that \( F_{j}^{c} \) and \( F_{j}^{r} \) given by (3.8)(a) are transformed as tensors, as was stated in Theorem 3.2. By the proof the relation

\[
(\partial_{c}(\partial_{b}y^{c}))((\partial_{b}y^{c})) + (\partial_{b}(\partial_{a}y^{b}))((\partial_{c}y^{a})((\partial_{b}y^{b})) = \partial_{c}[(\partial_{b}y^{c})((\partial_{b}y^{b})] = \partial_{c}\delta_{b}^{c} = 0
\]

(similar for \( z \)) is used.

The proofs of Theorems 3.2, 3.3 and 3.4 are given in [8] and the curvature theory of \( \nabla \) is given in [7].

4. Recurrent gauge connection of second order

As before we shall use such adapted basis \( B^{*} \) in which the nonlinear connections are given by (2.9) and the metric tensor \( G \) has the form (3.1).

Definition 4.1. The generalized gauge connection \( \nabla \) of second order is recurrent (metric) if

\[
\begin{align*}
g_{\alpha\beta|\gamma} &= w_{\gamma}g_{\alpha\beta} \quad (g_{\alpha\beta|\gamma} = 0), \\
g_{\alpha\beta|\gamma} &= \partial_{\gamma}g_{\alpha\beta} - F_{\alpha}^{\kappa}g_{\kappa\beta} - F_{\beta}^{\kappa}g_{\alpha\kappa}.
\end{align*}
\]

Theorem 4.1. The connection coefficients of the recurrent gauge connection of the second order are determined by

\[
\begin{align*}
2F_{\alpha}^{\gamma}_{\beta} &= g^{\kappa\gamma}(\gamma_{\alpha\kappa\beta} - w_{\alpha\kappa\beta} + \tilde{r}_{\alpha\kappa\beta}),
\end{align*}
\]

where

\[
\begin{align*}
\gamma_{\alpha\kappa\beta} &= \partial_{\beta}g_{\alpha\kappa} + \partial_{\kappa}g_{\alpha\beta} - \partial_{\alpha}g_{\kappa\beta} \\
w_{\alpha\gamma\beta} &= w_{\alpha}g_{\gamma\beta} + w_{\beta}g_{\alpha\gamma} - w_{\gamma}g_{\alpha\beta},
\end{align*}
\]

\[
\begin{align*}
g_{\alpha\beta|\gamma} &= \partial_{\gamma}g_{\alpha\beta} - F_{\alpha}^{\kappa}g_{\kappa\beta} - F_{\beta}^{\kappa}g_{\alpha\kappa}.
\end{align*}
\]
The Einstein-Yang Mills equations

\[ \tilde{T}_{\alpha\gamma\beta} = \tilde{T}_{\alpha}^\rho g_{\rho\beta} + \tilde{T}_{\beta}^\rho g_{\rho\alpha} + \tilde{T}_{\alpha\beta}^\gamma g_{\rho\gamma}, \]

(4.6)

\[ \tilde{T}_{\alpha}^\rho = F_{\alpha}^\rho - F_{\gamma}^\gamma. \]

(4.7)

From (3.1) and (4.1) follows

\[ g_{\alpha j|\gamma} = 0, \quad g_{\rho p|\gamma} = 0, \quad g_{\alpha q|\gamma} = 0, \]

(4.8)

where

\[ \gamma = i, \quad \text{or} \quad \gamma = b, \quad \text{or} \quad \gamma = s. \]

**Theorem 4.2.** The connection coefficients of the metric gauge connection of the second order are given by

\[ 2F_{\alpha}^\gamma = g^{\kappa\gamma}(\gamma_{\alpha\kappa\beta} + \tilde{T}_{\alpha\gamma\beta}). \]

(4.9)

**Theorem 4.3.** The connection coefficients of the recurrent torsion free gauge connections of the second order are given by (4.3) in which \( \tilde{T}_{\alpha\gamma\beta} = 0 \) except when in (4.6)

\[ \tilde{T}_{j}^c_i = -\partial_b N^c_i = -\tilde{T}_{j}^c_b, \quad \tilde{T}_{j}^r_q = -\partial_q M^r_j = -\tilde{T}_{j}^r_i. \]

(4.10)

appear.

The proof follows from Theorem 3.4.

If the horizontal distribution \( T_H F \) is integrable, then according to the Theorem 1.1 for the torsion free connection (in (4.10)) we have \( \tilde{T}_{j}^c_i = 0 \) and \( \tilde{T}_{j}^r_i = 0 \).

From (4.3) we can obtain all \( 3^3 \) types of connection coefficients. In the following we shall give some explicit expressions for (4.3). In all calculations it is important that such an adapted basis \( B^* \) is used in which the metric tensor \( G \) is determined by (3.1).

**Case 1.** For \( (\alpha, \beta, \gamma) = (i, j, a) \) we get

\[ 2F_{i}^{a}^{j} = g^{ab}(\gamma_{ibj} - \omega_{ibj} + \tilde{T}_{ibj}), \]

(4.11)

where

\[ \gamma_{ibj} = -\partial_b g_{ij}, \quad \omega_{ibj} = -\omega_b g_{ij}, \]

\[ \tilde{T}_{ibj} = \tilde{T}_{i}^{h} b.g_{bj} + \tilde{T}_{j}^{h} b.g_{bh} + \tilde{T}_{i}^{c} g_{bc}. \]

Using (4.7) and (4.10) for the torsion free connection the above equation has the form

\[ \tilde{T}_{ibj} = N^c_i g_{bc}, \]

because in this case

\[ \tilde{T}_{i}^{h} b = F_{i}^{h} b - F_{b}^{h} i = T_{i}^{h} b = 0. \]

If the horizontal distribution \( T_H F \) is integrable, then \( \tilde{T}_{ibj} = 0(N^c_i = 0) \). In this case for the torsion free metric connection (4.11) reduces to the form
Case 2. For \((\alpha, \beta, \gamma) = (i, j, r)\) we get
\[
2 F^r_{ij} = g^{rp} (\gamma_{ipj} - \omega_{ipj} + \hat{T}^h_{ipj}), \tag{4.12}
\]
where
\[
\gamma_{ipj} = -\partial_p g_{ij}, \quad \omega_{ipj} = -\omega_p g_{ij},
\hat{T}^h_{ipj} = \hat{T}^h_{ipj} g_{hi} + \hat{T}^h_{ipj} g_{ip}.
\]
For the torsion free connection we have
\[
\hat{T}^h_{ipj} = 0, \quad \hat{T}^h_{ipj} = 0, \quad \hat{T}^{q}_{ij} = \mathcal{M}^{q}_{ij}
\]
and if beside the above conditions the horizontal distribution \(T_H F\) is integrable
\((\mathcal{M}^{q}_{ij} = 0)\), then \(\hat{T}^{q}_{ij} = 0\). In this case (4.12) reduces to
\[
2 F^r_{ij} = g^{rp} (-\partial_p g_{ij} - \omega_p g_{ij}).
\]
For the metric connection in the above equation \(\omega_p = 0\).

Case 3. For \((\alpha, \beta, \gamma) = (j, b, c)\) we get
\[
2 F^c_{jb} = g^{cd} (\gamma_{jdb} - \omega_{jdb} + \hat{T}^d_{jdb}), \tag{4.13}
\]
where
\[
\gamma_{jdb} = \partial_j g_{db}, \quad \omega_{jdb} = \omega_j g_{db},
\hat{T}^d_{jdb} = \hat{T}^d_{jdb} g_{db} + \hat{T}^d_{jdb} g_{jd} + \hat{T}^d_{jdb} g_{cd}.
\]
For the torsion free connection we have
\[
\hat{T}^d_{b d} = 0, \quad \hat{T}^d_{j db} = -\partial_d N^c_{j d}
\]
and
\[
\hat{T}^d_{j db} = - (\partial_d N^c_{j d}) g_{db} - (\partial_d N^c_{j d}) g_{cd}.
\]
For the torsion free metric connection (4.13) has the form:
\[
2 F^c_{jb} = g^{cd} (\partial_j g_{db} - g_{ab} \partial_d N^a_{j d} - g_{ad} \partial_b N^a_{j d}).
\]

The other coefficients of the generalized recurrent second order gauge connection \(\nabla\) can be obtained in the similar manner.

5. Einstein-Yang Mills equations of the second order

Let \(L(x, y, z)\) be a Lagrangian defined on the compact set \(\Omega \subset \mathbb{R}^{n+m+l}\). As it is a scalar field we have
The Einstein-Yang Mills equations

\[ (5.1) \quad L(x, y, z) = L(x', y', z'). \]

We shall suppose, that the adapted basis \( B = \{ \delta_i, \partial_a, \partial_p \} \) of \( T(F) \) and its dual basis \( B^* = \{ dx^i, \delta y^a, \delta z^p \} \) are chosen in such a way, that \( T_{H} F, T_{V_1} F, T_{V_2} F \) are mutually orthogonal with respect to \( G \), i.e. when (2.9) is satisfied. The metric tensor in this basis has the form

\[ (5.2) \quad G = g_{ij} dx^i \otimes dx^j + g_{ab} \delta y^a \otimes \delta y^b + g_{pq} \delta z^p \otimes \delta z^p. \]

We have

\[ (5.3) \quad g = \det G(x, y, z) = |g_{ij}| \cdot |g_{ab}| \cdot |g_{pq}|. \]

As

\[ g_{ij} = g_{ij}' \frac{\partial x^i}{\partial x^i'} \frac{\partial x^j}{\partial x^j'}, \]

\[ g_{ab} = g_{ab}' \frac{\partial y^a}{\partial y^a'} \frac{\partial y^b}{\partial y^b'}, \]

\[ g_{pq} = g_{pq}' \frac{\partial z^p}{\partial z^p'} \frac{\partial z^q}{\partial z^q'}, \]

and the determinant of the Jacobian matrix is

\[ (5.4) \quad |J| = \left| \frac{D(x', y', z')}{D(x, y, z)} \right| = \left| \frac{\partial x^i}{\partial x^i} \right| \left| \frac{\partial y^a}{\partial y^a} \right| \left| \frac{\partial z^p}{\partial z^p} \right| \]

from (5.3) we obtain

\[ (5.5) \quad |G(x, y, z)| = |g_{ij}'||g_{ab}'||g_{pq}'||J|^2 = |G(x', y', z')||J|^2. \]

Let us define the Lagrangian density by

\[ (5.6) \quad \mathcal{L}(x, y, z) = L(x, y, z) \sqrt{G(x, y, z)} = \sqrt{g} L(x, y, z). \]

The substitution of (5.1) and (5.5) into (5.6) results

\[ (5.7) \quad \mathcal{L}(x, y, z) = L(x', y', z') \sqrt{|G(x', y', z')||J|} = \mathcal{L}(x', y', z') |J|. \]

The elementary volume element \( d\omega \) in \( \Omega \) is

\[ d\omega(x, y, z) = dx^1 \wedge \ldots \wedge dx^n \wedge dy^{n+1} \wedge \ldots \wedge dy^{n+m} \wedge dz^{n+m+1} \wedge \ldots \wedge dz^{n+m+l}. \]

It is known, that

\[ (5.8) \quad d\omega(x', y', z') = |J| d\omega(x, y, z). \]

**Proposition 5.1.** The integral of action

\[ (5.9) \quad I = \int_{\Omega} \mathcal{L}(x, y, z) d\omega \]
does not depend on coordinate system if and only if $L(x, y, z)$ satisfies the relation

$$L(x, y, z) = |J|L(x', y', z').$$  

(5.10)

Proof. $I$ is invariant if

$$L(x, y, z) d\omega(x, y, z) = L(x', y', z') d\omega(x', y', z').$$  

(5.11)

The substitution of (5.8) into (5.11) results (5.10). The proof in the opposite direction is obvious.

From (5.6) and (5.7) it follows, that one example for $L$, which gives coordinate invariant integral of action is

$$L(x, y, z) = \sqrt{gL(x, y, z)}.$$  

(5.12)

Proposition 5.2. For arbitrary $C^2$ function $L(x, y, z)$ the following relation is valid:

$$dL = (\delta_i L)\delta x^i + (\delta_a L)\delta y^a + (\delta_p L)\delta z^p.$$  

(5.13)

Proof. As $L = L(x, y, z)$ we have:

$$dL = (\partial_i L)dx^i + (\partial_a L)dy^a + (\partial_p L)dz^p.$$  

(5.14)

From (1.4), (1.8) and (1.9) we have

$$\delta_i = \partial_i - N^b_i \partial_b - M^p_i \partial_p, \quad \delta_a = \partial_a, \quad \delta_p = \partial_p.$$

(5.15)

$$\delta x^i = dx^i, \quad \delta y^a = dy^a + N^a_i dx^i, \quad \delta z^p = dz^p + M^p_i dx^i.$$

The substitution of (5.14), (5.15) into (5.13) results (5.12).

We shall suppose that the Lagrangian $L(x, y, z)$ is the function of $\phi^A(x, y, z)$, $\partial_i \phi^A(x, y, z)$, $\partial_a \phi^A(x, y, z)$ and $\partial_p \phi^A(x, y, z)$, where $\phi^A(x, y, z)$ are scalar fields, i.e.

$$\phi^A(x, y, z) = \phi^A(x', y', z') \quad A = 1, 2, \ldots, p.$$  

(5.16)

For the simplification, we shall consider only one function $\phi = \phi(x, y, z)$ and use these abbreviations:

$$\partial_i \phi = \partial_i \phi(x, y, z), \quad \partial_a \phi = \partial_a \phi(x, y, z), \quad \partial_p \phi = \partial_p \phi(x, y, z).$$  

(5.17)

Now we have

$$L(\phi, \partial_i \phi, \partial_a \phi, \partial_p \phi) = \sqrt{g}L(\phi, \partial_i \phi, \partial_a \phi, \partial_p \phi)$$

and the integral of action has the form

$$I(\phi) = \int_\Omega L(\phi, \partial_i \phi, \partial_a \phi, \partial_p \phi)d\omega.$$  

(5.19)
We are looking for such functions $\phi$, for which $I(\phi)$ has maximal or minimal value, i.e. for which $\delta I(\phi) = 0$.

For the simplicity we shall suppose that $\Omega$ is the $n+m+l$ dimensional rectangle, such that

$$\int_{\Omega} d\omega = \int_{a_1}^{b_1} dx^1 \int_{a_2}^{b_2} dx^2 \cdots \int_{a_n}^{b_n} dx^n,$$

and the variation of $\phi$ on the boundary of $\Omega$ is equal to zero, i.e.

$$\delta \phi(b^i, y, z) = \delta \phi(a^i, y, z) = 0, \quad i = 1, 2, \ldots, n$$

$$\delta \phi(x, b^a) = \delta \phi(x, a^a) = 0, \quad a = n + 1, \ldots, n + m$$

$$\delta \phi(x, y, b^p) = \delta \phi(x, y, a^p) = 0, \quad p = n + m + 1, \ldots, n + m + l$$

where for instance

$$\delta \phi(b^1, y, z) = \delta \phi(b^1, x^2, \ldots, x^n, y, z),$$

$$\delta \phi(x, y, b^{n+m+1}) = \delta \phi(x, y, z^{n+m+1}, \ldots, b^{n+m+1}).$$

From the variation principle we have

$$\delta I(\phi) = \int_{\Omega} \delta L(x, y, z)d\omega =$$

$$\int_{\Omega} \left[ \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_i \phi)} \delta (\partial_i \phi) + \frac{\partial L}{\partial (\partial_a \phi)} \delta (\partial_a \phi) + \frac{\partial L}{\partial (\partial_p \phi)} \delta (\partial_p \phi) \right] d\omega.$$

From (5.19) it can be seen that $L$ is function of independent variables $\phi, \partial_i \phi, \partial_a \phi, \partial_p \phi$. To express this fact we shall write in (5.24)

$$\frac{\partial L}{\partial (\partial_i \phi)} = \frac{dL}{d(\partial_i \phi)}, \quad \frac{\partial L}{\partial (\partial_a \phi)} = \frac{dL}{d(\partial_a \phi)}, \quad \frac{\partial L}{\partial (\partial_p \phi)} = \frac{dL}{d(\partial_p \phi)}.$$

From (5.14) it follows

$$L(\phi, \partial_i \phi, \partial_a \phi, \partial_p \phi) = L(\phi, \delta_i \phi, \partial_a \phi, \partial_p \phi),$$

$$\delta_i \phi = \partial_i \phi - N^a_i \partial_a \phi - M^p_i \partial_p \phi,$$

so we have

$$\frac{dL}{d(\partial_i \phi)} = \frac{\partial L}{\partial (\partial_i \phi)} \frac{\partial (\delta_i \phi)}{\partial (\partial_i \phi)} = \frac{\partial L}{\partial (\delta_i \phi)} = A.$$
From (5.24)-(5.31) we obtain

\begin{align}
\frac{d\mathcal{L}}{d(\partial_{a}\phi)} &= \frac{\partial \mathcal{L}}{\partial(\partial_{a}\phi)} \frac{\partial(\delta_{a}\phi)}{\partial(\partial_{a}\phi)} + \frac{\partial \mathcal{L}}{\partial(\partial_{a}\phi)} = \frac{\partial \mathcal{L}}{\partial(\partial_{a}\phi)} (-N_{a}^{i}) + \frac{\partial \mathcal{L}}{\partial(\partial_{a}\phi)} = B. \\
\frac{d\mathcal{L}}{d(\partial_{p}\phi)} &= \frac{\partial \mathcal{L}}{\partial(\partial_{p}\phi)} \frac{\partial(\delta_{p}\phi)}{\partial(\partial_{p}\phi)} + \frac{\partial \mathcal{L}}{\partial(\partial_{p}\phi)} = \frac{\partial \mathcal{L}}{\partial(\partial_{p}\phi)} (-M_{p}^{i}) + \frac{\partial \mathcal{L}}{\partial(\partial_{p}\phi)} = C.
\end{align}

We shall suppose

\begin{equation}
\delta(\partial_{i}\phi) = \partial_{i}(\delta\phi), \delta(\partial_{a}\phi) = \partial_{a}(\delta\phi), \delta(\partial_{p}\phi) = \partial_{p}(\delta\phi).
\end{equation}

From (5.24)-(5.31) we obtain

\begin{equation}
\delta \mathcal{L} = \frac{d\mathcal{L}}{d\phi} \delta\phi + \frac{d\mathcal{L}}{d(\partial_{i}\phi)} \delta(\partial_{i}\phi) + \frac{d\mathcal{L}}{d(\partial_{a}\phi)} \delta(\partial_{a}\phi) + \frac{d\mathcal{L}}{d(\partial_{p}\phi)} \delta(\partial_{p}\phi) =
\end{equation}

\begin{align}
&\frac{\partial \mathcal{L}}{\partial\phi} \delta\phi + A\delta(\partial_{i}\phi) + B\delta(\partial_{a}\phi) + C\delta(\partial_{p}\phi) = \\
&\frac{\partial \mathcal{L}}{\partial\phi} \delta\phi + A\partial_{i}(\delta\phi) + B\partial_{a}(\delta\phi) + C\partial_{p}(\delta\phi) = \\
&\frac{\partial \mathcal{L}}{\partial\phi} \delta\phi + \partial_{i}(A\delta\phi) + \partial_{a}(B\delta\phi) + \partial_{p}(C\delta\phi) \\
&= (\partial_{i} A)\delta\phi - \partial_{a}(B\delta\phi) - \partial_{p}(C\delta\phi).
\end{align}

Using (5.20) and (5.21) we have

\begin{equation}
\int_{\Omega} \partial_{i}(A\delta\phi)d\omega = \int_{\Omega} dy^{a} \int_{a^{i}} dz^{p} \int_{b^{i}} \partial_{i}(A\delta\phi)dx^{i} =
\end{equation}

\begin{align}
&\int_{a^{i}} dy^{a} \int_{a^{p}} dz^{p} (A\delta\phi)\bigg|_{a^{i}}^{b^{i}} = 0.
\end{align}

In the similar way we obtain

\begin{equation}
\int_{\Omega} \partial_{a}(B\delta\phi)d\omega = 0 \quad \int_{\Omega} \partial_{p}(C\delta\phi)d\omega = 0.
\end{equation}

From (5.32)-(5.34) it follows

\begin{equation}
\delta I(\phi) = \int_{\Omega} \delta \mathcal{L}d\omega = \int_{\Omega} \left( \frac{\partial \mathcal{L}}{\partial\phi} - \partial_{i} A - \partial_{a} B - \partial_{p} C \right) \delta\phi d\omega.
\end{equation}

The extrem value of integral of action is obtained, when \( \delta\mathcal{L} = 0 \). If we in (5.35) substitute \( A, B \) and \( C \) from (5.28), (5.29) and (5.30) we obtain

\begin{align}
\delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial\phi} - \partial_{i} \left( \frac{\partial \mathcal{L}}{\partial(\delta_{i}\phi)} \right) - \partial_{a} \left( \frac{\partial \mathcal{L}}{\partial(\delta_{a}\phi)} \right) - \partial_{p} \left( \frac{\partial \mathcal{L}}{\partial(\delta_{p}\phi)} \right) \\
&+ N_{a}^{i} \partial_{a} \left( \frac{\partial \mathcal{L}}{\partial(\delta_{a}\phi)} \right) + \partial_{p} N_{a}^{i} \\
&+ M_{p}^{i} \partial_{p} \left( \frac{\partial \mathcal{L}}{\partial(\delta_{p}\phi)} \right) + \partial_{p} M_{p}^{i}.
\end{align}
Theorem 5.1. The Einstein-Yang Mills equation for the gauge transformation (1.1) are given by

\begin{equation}
\delta L = \frac{\partial L}{\partial \phi} - \delta_i \left( \frac{\partial L}{\partial (\delta_i \phi)} \right) - \partial_a \left( \frac{\partial L}{\partial (\partial_a \phi)} \right) - \partial_p \left( \frac{\partial L}{\partial (\partial_p \phi)} \right)
+ \frac{\partial L}{\partial (\delta_i \phi)} (\partial_a N^a_i + \partial_p M^p_i) = 0.
\end{equation}

As from (5.18) $L = \sqrt{g} L$, and $\sqrt{g}$ is not the function of $\phi, \partial_i \phi, \partial_a \phi$ and $\partial_p \phi$ from (5.36) we get

\begin{equation}
\frac{\partial (\sqrt{g} L)}{\partial \phi} - \delta_i \left( \sqrt{g} \frac{\partial L}{\partial (\delta_i \phi)} \right) - \partial_a \left( \sqrt{g} \frac{\partial L}{\partial (\partial_a \phi)} \right) - \partial_p \left( \sqrt{g} \frac{\partial L}{\partial (\partial_p \phi)} \right)
+ \sqrt{g} \frac{\partial L}{\partial (\delta_i \phi)} (\partial_a N^a_i + \partial_p M^p_i) = 0.
\end{equation}

Theorem 5.2. The Einstein-Yang Mills equation for the gauge transformation (1.1) expressed as function of the Lagrangian $L$ and metric function is given by

\begin{equation}
\left[ \frac{\partial L}{\partial \phi} - \delta_i \left( \frac{\partial L}{\partial (\delta_i \phi)} \right) - \partial_a \left( \frac{\partial L}{\partial (\partial_a \phi)} \right) - \partial_p \left( \frac{\partial L}{\partial (\partial_p \phi)} \right) \right]
+ \frac{\partial L}{\partial (\delta_i \phi)} (\partial_a N^a_i + \partial_p M^p_i) - \frac{1}{\sqrt{g}} \left[ \frac{\partial L}{\partial (\delta_i \phi)} \delta_i \sqrt{g} + \frac{\partial L}{\partial (\partial_a \phi)} \partial_a \sqrt{g} + \frac{\partial L}{\partial (\partial_p \phi)} \partial_p \sqrt{g} \right] = 0.
\end{equation}

Proof. The proof follows from (5.37).

Acknowledgement The author expresses her hearthy thanks to Professor Z. Shen, Indiana University for his valuable suggestions. The present research was partly supported by Science Fund of Serbia grant number 1262.

References


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