d—Connections Compatible with Homogeneous Metric on the Cotangent Bundle

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003),

Abstract

In this paper is studied the cotangent bundle $\tilde{T}^*M = T^*M\setminus \{0\}$ with a 0-homogeneous lift $G$. The connection compatible with the homogeneous metric is determined.

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1 Introduction

Let $(T^*M, \pi^*, M)$ be the cotangent bundle, where $M$ is a $C^\infty$-differentiable, real $n$-dimensional manifold. If $(U, \varphi)$ is a local chart on $M$ and $(x^i)$ are the coordinates of a point $p \in M$, then a point $u \in \pi^*\{U\}$, $\pi^*(u) = p$ has the coordinates $(x^i, p^r)$, $(i, r = 1, n)$. The natural basis of the module $\mathcal{X}(T^*M)$ is given by $(\partial_i = \frac{\partial}{\partial x^i}, \partial^r = \frac{\partial}{\partial p^r})$. Given a nonlinear connection $N$ on $T^*M$ ([1]) there exist a single system of functions $N_{ia}(x, p)$ such that $\delta_k = \partial_k + N_{ka}(x, p)\partial^a$, $(a = 1, n)$ and $(\delta_k, \partial^a)$ is a local basis of $\mathcal{X}(T^*M)$, which is called the adapted basis to $N$. We have the dual basis $(dx^i, dp_a = dp_a - N_{ka}(x, p)dx^k)$. For $X \in \mathcal{X}(T^*M)$ is obtained a unique decomposition $X = hX + vX$, $hX \in H$, $vX \in V$; $(V$ is the vertical distribution) and for $\omega \in \mathcal{X}^*(T^*M)$ we have $\omega = h\omega + v\omega$, where $(h\omega)(X) = \omega(hX)$, $(v\omega)(X) = \omega(vX)$. In the adapted basis $(\delta_k, \partial^a)$ we have $X = X^k\delta_k + X^a\partial^a$ and $\omega = \omega^i dx^i + \omega^a dp_a$. The homogeneous lift of the Riemannian and Finslerian metrics on the tangent bundle have been studied by Acad. Radu Miron ([3], [4]), while the properties of homogeneous structures on cotangent bundle were studied by P. Stavre and the author ([5], [6], [7]). More specific, details on the homogeneous lift of a Cartan metric on cotangent bundle and on integrability conditions of homogeneous almost complex structures are given in [6], the properties of the homogeneous lift of a Riemann metric on cotangent bundle are studied in [7], and the homogeneous almost product structure case is developed in [5].
2 Existence of metrical $d$-connections

Let $(M, g_{ij}(x))$ be a Riemannian space and $(T^*M, \pi^*, M)$ its cotangent bundle. We introduce $g^{rs}(x)$ with $g_{ik}(x)g^{ks}(x) = \delta^r_s$.

We consider
\[
\hat{N}_{kr}(x, p) \overset{\text{def}}{=} p_s \gamma^s_{rk}(x),
\]
where $\gamma^s_{rk}(x)$ are the Christoffel symbols of $g$. Evidently $\{\hat{N}_{kr}(x, p)\}$ are the coefficients of a nonlinear connection on $\tilde{T}^*M = T^*M \setminus \{0\}$ which is 1- homogeneous on the fibres. Using $\hat{N}_{kr}$ we consider
\[
\delta^k_r = \partial^k_r + \hat{c}^N_{kr}(x, p) \partial^r \quad \text{and} \quad \delta p^k_r = dp^k_r - \hat{c}^N_{ik}(x, p) dx^i.
\]

We have
\[
\ast G = h \ast G + v \ast G, \quad \ast G = g_{ij}(x) dx^i \otimes dx^j + g^{rs}(x) \delta p^r \otimes \delta p^s.
\]

If we define the homothety $h_t: (x, p) \rightarrow (x, tp)$, $\forall t \in \mathbb{R}$, then
\[
\left(\ast G \circ h^t\right)(x, p) = g_{ij}(x) dx^i \otimes dx^j + t^2 g^{rs}(x) \delta p^r \otimes \delta p^s \neq \ast G(x, p).
\]

**Proposition 1** $\ast G$ is globally defined Riemannian metric on $\tilde{T}^*M$ and is not homogeneous on the fibres of $T^*M$.

We consider the function
\[
H(x, p) = g^{rs}(x) p_r p_s.
\]

Obviously $H$ is 2-homogeneous on the fibres of cotangent bundle $\tilde{T}^*M$.

If $\hat{G}$ is defined by
\[
\hat{G} = g_{ij}(x) dx^i \otimes dx^j + h^{rs}(x, p) \delta p^r \otimes \delta p^s,
\]

where $a > 0$ is a constant, then we get:

**Proposition 2** The following properties hold:

1. The pair $(\tilde{T}^*M, \hat{G})$ is a Riemannian space depending only on the metric $g$.
2. $\hat{G}$ is 0-homogeneous on the fibres of $\tilde{T}^*M$.
3. The distribution $N$ and $V$ are orthogonal with respect to $\hat{G}$

\[
\hat{G} (hX, vY) = 0, \quad \forall X, Y \in \mathcal{X}(T^*M).
\]

where
\[
h^{rs}(x, p) = \frac{a^2}{H} g^{rs}(x).
\]

From [1] we have:

**Definition 1** A linear connection $D$ on $T^*M$ is called metrical $d$–linear connection with respect to $\hat{G}$ if $D \hat{G} = 0$ and $D$ preserves by parallelism the horizontal distribution $N$. 

We will prove the existence of metrical $d-$linear connections. In the adapted frame we have:

$$
D_{\delta_k} \delta_j = F^i_{jk} \delta_i + \tilde{F}^{(r)}_{j(r)k} \partial^r, \quad D_{\delta_k} \partial^r = - \tilde{F}^{(r)}_{k} \delta_i - F^v_{(j)k} \partial^j,
$$

(8)

$$
D_{\delta_k} \bar{\delta}_j = C^{i(k)}_j \delta_i + \tilde{C}^{(k)}_{j(r)} \partial^r, \quad D_{\delta_k} \bar{\partial}^r = - C^{(r)(k)}_j \delta_i - C^{(r)(k)}_{(j)} \partial^j,
$$

where $F^i_{jk}, \tilde{F}^{(r)}_{j(r)k}, F^v_{(j)k}, C^{i(k)}_j, \tilde{C}^{(k)}_{j(r)}, C^{(r)(k)}_j, C^{(r)(k)}_{(j)}$ are the coefficients of $D$.

**Theorem 1** There exists a metrical $d-$linear connection $D$ on $\tilde{T}^\ast M$ with respect to $\tilde{G}$, which depends only on the metric tensor $g$; its components are

$$
\begin{align*}
\tilde{F}^{(r)}_{j(r)k} &= F^{(r)}_{k} = C^{(k)}_{j(r)} = C^{(r)(k)}_j = C^{(r)(k)}_{(j)} = 0, \\
\tilde{F}^{(r)}_{j(r)k} &= \gamma^i_{jk}(x), \\

\end{align*}
$$

(9)

$$
\tilde{C}^{(r)(k)}_{(j)} = \frac{1}{H}[(\delta^k_j p^r + \delta^r_j p^k - g^r_k p^r)],
$$

where $g^{rm} p_m = p^r$.

**Proof.** In the general case of a vector bundle we have a canonical metrical connection given by [2],

$$
\begin{align*}
F^i_{jk} &= \frac{1}{2} g^{is} (\delta_j g_{si} + \delta_k g_{js} - \delta_s g_{jk}), \\
F^{(r)}_{j(r)k} &= \partial^r N_{jk} + \frac{1}{2} h^{rs} h_{js}[k], \\

\end{align*}
$$

(9)

$$
C^{i(k)}_j = \frac{1}{2} g_{js} g^{ks} \partial^s g^r, \\
C^{(r)(k)}_{(j)} = - \frac{1}{2} h_{js} (\partial^r h^k + \delta^k h^{rs} - \partial^s h^{rk}),
$$

where $\cdot|\cdot$ and $\cdot||\cdot$ are the $h-$, and $v-$ covariant derivative with respect to the Berwald connection $(B^r_{jk} = \partial^r N_{jk}, 0)$.

But $g = g(x)$, so $\delta_j g_{si} = \delta_j g_{si}$ and $\partial^k g^{is} = 0 \Rightarrow \gamma^i_{jk}(x)$ and $C^{(r)(k)}_{(j)} = 0$.

From $h^r(x, p) = \frac{a^2}{H} g^{rs}(x)$ it follows $h_{rs}(x, p) = \frac{H}{a^2} g_{rs}(x)$. But

$$
\partial^r h^k(x, p) = \partial^r \left( \frac{a^2}{H^2} g^{ks} p_m p_r \right) g^{km} p_m = - \frac{2a^2}{H^2} g^{ks} g^{rm} p_m,
$$

$$
C^{(r)(k)}_{(j)} = - \frac{1}{2} h_{js} (\partial^r h^k + \partial^k h^{rs} - \partial^s h^{rk}) =
$$

$$
= - \frac{1}{2} \frac{H}{a^2} g_{js}(x) \left( - \frac{2a^2}{H^2} g^{ks} g^{rm} p_m - \frac{2a^2}{H^2} g^{rs} g^{km} p_m + \frac{2a^2}{H^2} g^{rk} g^{sm} p_m \right)
$$
\[ C^{(r)(k)}_{(j)} = \frac{1}{H} (\delta^r_j p + g^r j p_k - g^r k p_j), \quad \text{and} \quad g^{rm} p_m = p^r. \]

\[ F^{(r)}_{(j)k} = \partial^r \tilde{\eta} \tilde{N}_{jk} + \frac{1}{2} h^{rs} h_{jst} \tilde{N}_{rk} + \frac{1}{2} h^{rs} [\delta_k h_{js} - \partial^m (\tilde{N}_{sk}) h_{jm} - \partial^m (\tilde{N}_{sj}) h_{km}]. \]

Since \( N_{jk} = \gamma^r_{jk} p_r \) and

\[ F^{(r)}_{(j)k} = \gamma^r_{jk} + \frac{1}{2} \frac{a^2}{2 H} g^{rs}(x) [\frac{H}{a^2} \partial_k g_{js} + \frac{1}{a^2} g_{js} \partial_k g^{ml} p_m p_l + \frac{2}{a^2} \gamma^m_{kl} p_m g_{js} g^{lm} p_m - \frac{H}{a^2} \gamma^m_{sk} g_{jm} - \frac{H}{a^2} \gamma^m_{jk} g_{sm}], \]

we obtain

\[ F^{(r)}_{(j)k} = \gamma^r_{jk} + \frac{1}{2} \frac{a^2}{2 H} g^{rs} \partial_k g_{js} + \frac{1}{2} \frac{a^2}{2 H} \partial_k g^{ml} p_m p_l \delta^r_j + \frac{1}{2} \frac{a^2}{2 H} g^{ms} g^{ri} p_m p_i \partial_k g_{ks} \delta^r_j + \frac{1}{4} \frac{a^2}{2 H} g^{rs} \partial_k g_{ij} + \frac{1}{4} \frac{a^2}{2 H} g^{rs} \partial_k g_{ij}. \]

But \( g^{ms} \partial_k (g_{ls}) = -\partial_k (g^{ms}) g_{ls} \), and consequently

\[ F^{(r)}_{(j)k} = \gamma^r_{jk} + \frac{1}{4} g^{rs} (\partial_k g_{js} + \partial_j g_{sk} - \partial_s g_{kj}) + \frac{1}{2} \frac{H}{a^2} \partial_k g^{ml} p_m p_l \delta^r_j - \frac{1}{2} \partial_k g^{ms} p_m p_s \delta^r_j - \frac{1}{4} \frac{H}{a^2} g^{ri} g_{ms} p_m p_i \partial_k g_{ks} \delta^r_j + \frac{1}{4} g^{rs} \partial_k g_{ij} + \frac{1}{4} g^{rs} \partial_k g_{ij}. \]

which ends the proof.

\[ \square \]

**References**


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