The Geometrical Interpretation of Temporal Cone Norm in Almost Minkowski Manifold

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003),

Abstract

For almost Minkowski manifolds we prove that the norm determined by a unitary vector field which belongs to the timelike cone is the sum of two fundamental forms induced by the Lorentzian metrics on two submanifolds.

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1 Introduction

A Lorentz manifold is a pair \((M, g)\) where \(M\) is an \(n+1\) dimensional smooth paracompact manifold and \(g\) is a global smooth two-times covariant symmetric tensor field which is nondegenerate and has \(n-1\) signature.

A time-normalized space-time \([M, g, Z]\) is a Lorentz manifold \((M, g)\) for which a global unitary (i.e., \(g(Z, Z) = -1\)) tangent vector field \(Z\) of timelike vectors is fixed; this will be denoted by \([M, g, Z]\).

Definition 1 An almost Minkowski manifold is a time-normalized space-time \([M, g, Z]\) provided that the distribution

\[ \Delta \colon x \in M \to \Delta_x \overset{\text{def}}{=} \{ Y \in T_xM \mid g(Y, Z) = 0 \} \]

is totally integrable.

Proposition 2 The necessary and sufficient condition that a time-normalized space-time manifold \([M, g, Z]\) be an almost Minkowski manifold is the existence of a preferential atlas

\[ A = \{(U_\alpha, \chi_\alpha) \mid \alpha \in \Gamma, \chi_\alpha(x) = (x^i), i = 1, n + 1 \partial_{n+1} = Z|u_\alpha\}, \]

where

\[ \frac{\partial g_{in+1}}{\partial x^j} = \frac{\partial g_{jn+1}}{\partial x^i} \quad \forall i, j \in 1, n + 1. \]
Proof. For \( X, Y \) belonging to \( \Delta \) we have
\[
g(X, Z) = g(Y, Z) = 0,
\]
which infers
\[
g(\nabla_Y X, Z) + g(X, \nabla_Y Z) = 0
\]
\[
g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = 0,
\]
where \( \nabla \) is the Levi-Civita connection of Lorentz manifold \((M, g)\). In the local charts of atlas \( A \) where \( Z = \partial_{n+1} \), \( X = X^i \partial_i \) and \( Y = Y^j \partial_j \) we have
\[
g([X, Y], Z) = g(\nabla_X Y, Z) - g(\nabla_Y X, Z) =
\]
\[
= X^i Y^j \left[ \Gamma^{k}_{jn+1} g_{i+1} + \Gamma^{k}_{jn+1} g_{j+1} \right] =
\]
\[
= X^i Y^j \left( \frac{\partial g_{i+1}}{\partial x^j} - \frac{\partial g_{j+1}}{\partial x^i} \right).
\]
Therefore \([X, Y]\) belongs to \( \Delta \) if and only if \( \frac{\partial g_{i+1}}{\partial x^i} = \frac{\partial g_{j+1}}{\partial x^j} \). \qed

Remark 3 The existence of almost Minkowski manifolds is obvious, since it is possible to choose \( Z \) so that \( g(\partial_i, Z) \partial_i \) be irrotational. If \((M, g)\) is stable causal then there exists a real global function \( f \) with the gradient \( \nabla f \) of timelike type, (see e.g. [1], [6]), and the corresponding 1-form of \( Z = 1/\sqrt{-g(\nabla f, \nabla f)} \nabla f \) closed, and therefore it respects the previous conditions.

Remark 4 If the corank one distribution is not integrable, then any two points can be connected by a curve \( \gamma : [0, 1] \to M \) where \( g(\gamma'(t), Z_{\gamma(t)}) = 0 \), according to the Carathéodory theorem, ([3, p.10]).

Definition 5 We define the ordering relation for the elements of \( T_x M, x \in M \):
\[
X \leq Y \iff Y - X \in K_x,
\]
where \( K_x = \{ X \in T_x M \mid g(X, X) < 0, g(X, Z) < 0 \} \) is the interior of the timelike cone of the tangent vectors.

From ([4]) we have:
- \((T_x M, K_x)\) is a Krein space, \( \forall x \in M \)
- The map \( | \cdot |_Z : T_x M \to \mathbb{R}, |X|_Z \defeq \min \{ \lambda \geq 0 \mid -\lambda Z \leq X \leq \lambda Z \} \) is a topological norm of \( T_x M \), named the \( Z \)-norm of the almost Minkowski manifold \([M, g, Z] \).
- An easy calculation implies
\[ |X|_z = |g(X,Z)| + \sqrt{g(X,Z)^2 + g(X,X)}. \]

**Proposition 6** The \( Z \)-norm is invariant to a conformal change of Lorentzian metric.

**Proof.** Let \( g, \hat{g} \) be two conformal metrics (i.e. \( g = \hat{g} \Omega^2 \)). The \( Z \)-norms expressions for the two almost Minkowski manifolds are, ([5])

\[
|X|^g_Z = \frac{|g(X,Z)|}{g(Z,Z)} + \sqrt{\left(\frac{|g(X,Z)|}{g(Z,Z)}\right)^2 + \frac{g(X,X)}{g(Z,Z)}} = \\
= \frac{\hat{g}(X,Z)}{\hat{g}(Z,Z)} + \sqrt{\left(\frac{\hat{g}(X,Z)}{\hat{g}(Z,Z)}\right)^2 + \frac{\hat{g}(X,X)}{\hat{g}(Z,Z)}} = |X|^\hat{g}_Z
\]

**Remark 7** The \( Z \)-norm can be defined if the existent condition of the global timelike vector field \( Z \) is weakened and replaced with the existence of a line element field which is equivalent to the existence of Lorentzian metrics ([2]).

## 2 The \( Z \)-norm dependence on the first fundamental form of the hypersurface normal to \( Z \)

Consider the preferential atlas \( A \) of Proposition 2 (this exists, cf. [4]). We note by \( S \) the integral submanifold of distribution \( \Delta \) with \( p \in S \). Obviously \( S \) is a hypersurface imbedded in \( M \) with inclusion map \( \theta : S \to M \). Let \( n \in T^*_q M, q \in S \) be the 1-form \( n(X) = g(X,Z), \forall X \in T_q M \). This implies \( n(\theta_*X) = 0, \forall X \in T_q M \) and if we denote \( H_q = \theta_*(T_q S) \), this is a hyperplane in \( T_q M \). If \( Z \) is be tangent to \( \theta(S) \), then there exist \( X \in T_q S \setminus \{0\} \) such that \( \theta_*(X) = Z \) and \(-1 = g(Z,Z) = g(\theta_*(X),Z) = 0\), which is impossible. Therefore \( Z \) is not in the tangent space of \( \theta(S) \). If \( \{E_1, \ldots, E_n\} \) is a basis in \( T_q S \), then \( \{Z, \theta_*(E_1), \ldots, \theta_*(E_n)\} \) is linearly independent and hence is a basis for \( T_q M \). The components of \( g \) with respect to this basis are

\[
(g_{ab}) = \begin{pmatrix}
g(Z,Z) & 0 & \cdots & 0 \\
0 & \theta_*(E_1), \theta_*(E_1) & \cdots & \theta_*(E_1), \theta_*(E_n) \\
\vdots & \vdots & \ddots & \vdots \\
0 & \theta_*(E_n), \theta_*(E_1) & \cdots & \theta_*(E_n), \theta_*(E_n)
\end{pmatrix} = \\
= \begin{pmatrix}
-1 & 0 \\
0 & \theta_*(E_1), \theta_*(E_1)
\end{pmatrix}
\]

Because \( g \) has one negative eigenvalue, then \( \theta^* g \) is positively definite. Let’s consider \( \theta^* : T^*_q S \to T^*_q S, \) and \( H^*_q = \{\omega \in T^*_q M \mid \omega(Z) = 0\} \).

From \( \theta^*|_{H^*_q} : H^*_q \to T^*_q S, \) being obviously a bijection, we denote its inverse by \( \bar{\theta}_*: T^*_q S \to H^*_q \). Therefore there exist two bijections \( \theta_* \) and \( \bar{\theta}_* \) between \( T_q S \) and \( H_q \) and respectively between \( T^*_q S \) and \( H^*_q \). This map can be extended in a usual way to a map \( \theta \) of arbitrary tensors on \( S \) to \( \theta(S) \) in \( M \). Since \( n \) is normal to hypersurface \( \theta(S), \) for a given tensor \( T \in T^*_{\theta(q)} S \) we obtain that \( \theta(T) \) has zero transvections with \( n \) in all indices.
\[
(\tilde{\theta}T)_{\beta_1...\beta_r}^{i_1...i_r} n_m = (\tilde{\theta}T)_{\beta_1...\beta_r}^{i_1...i_r} g^{mp} n_p = 0.
\]

Denote by \(h\) the metric on \(\theta(S)\), defined by \(h = \tilde{\theta}(\theta^*g)\). In the preferential atlas \(A\) the components of \(h\) are

\[
h_{ab} = g_{ab} + n_a n_b = g_{ab} + g_{a n+1} g_{b n+1}, \forall a, b \in 1, n+1.
\]

**Proposition 8** The \((1,1)\) tensor associate to \(h\) having the components \(h^b_a\) is a projection operator, \(h^b_a = \delta^b_a + g_{a n+1} \delta^b_{n+1}\) and

\[\begin{align*}
\text{a)} & \quad \text{The projection of } X \in T_Q M \text{ onto the subspace } H_Q \text{ is } \\
& \quad h^b_a X^a \partial_b = X + g(X, Z) Z.
\end{align*}\]

\[\begin{align*}
\text{b)} & \quad \text{The projection of } \omega \in T_Q M \text{ onto the subspace } H^*_Q \text{ is } \\
& \quad h^b_a \omega \partial^a = \omega + \omega(Z) n.
\end{align*}\]

**Proof.**

\[
h^b_a = h_{ac} g^{cb} = (g_{ac} + g_{an+1} g_{cn+1}) g^{cb} = \delta^b_a + g_{a n+1} \delta^b_{n+1}
\]

\[
h^b_a h^c_b = (\delta^b_a + g_{a n+1} \delta^b_{n+1}) (\delta^c_b + g_{b n+1} \delta^c_{n+1}) = h^a_c
\]

\[X + g(X, Z) Z = X^a \partial_a + X^b g_{b n+1} \partial_{n+1} = X^a (\delta^a_a + g_{a n+1} \delta^a_{n+1}) \partial_b = h^b_a X^a \partial_b
\]

\[\omega + \omega(Z) n = \omega_a dx^a + \omega_{n+1} g_{n+1} dx^a = \omega_a (\delta^a_a + g_{a n+1} \delta^a_{n+1}) dx^a = h^a_a \omega_a dx^a
\]

**Remark 9** Analogously we can project the tensor \(T \in T()_q S\) to

\[
H(T) \in H_q \otimes ... \otimes H_q \otimes H^*_q \otimes ... \otimes H^*_q \overset{\text{r times}}{\overset{\text{s factors}}{\overset{\text{r factors}}{\overset{\text{s factors}}{\overset{\text{def}}{H^*_q (q)}}}}}
\]

\[
H(T) = T + (-1)^{r+s+1} T(Z^*, ..., Z^*, Z, ..., Z) Z \otimes ... \otimes Z \otimes Z^* \otimes ... \otimes Z^*
\]

One can than verify the relation

\[H(H(T)) = H(T), \forall T \in T()_q S.
\]

**Proposition 10** \((S, \theta^* g)\) is a totally geodesic submanifold

**Proof.** In the above notation, the coordinates of the second fundamental form of \(S\) are \((1, p. 46)\)

\[
\chi_{ab} = h^c_a h^d_b n_c ; d = h^c_a h^d_b g_{cn+1} ; d = 0.
\]

**Remark 11** We will denote the covariant differentiation with respect to the Levi-Civita connection of \((S, \theta^* g)\) by double stroke.

Then for any tensor \(T \in T()_q S\) we have:

\[
H(T)_{\beta_1...\beta_r}^{i_1...i_r} = T_{\beta_1...\beta_r}^{i_1...i_r} ; h^c_a h^d_b h^e_c h^f_d = 0
\]

where \(T\) is an extension of \(H(T)\) to a neighborhood of \(\theta(S)\). This formula is correct because the double stroke of the induced metric is zero and the torsion vanishes,

\[
h_{ab||c} = (g_{ef} + g_{en+1} g_{fn+1}) ; h^c_a h^f_b h^i_c = 0
\]

\[
f_{||ab} = h^c_a h^d_b f_{cd} = h^c_a h^d_b f_{dc} = f_{||ba}.
\]
For $p \in M$, we denote by $s$ the 1-dimensional submanifold which passes through $p$ and $T_q s = \langle Z_q \rangle$, $\forall q \in s$. The local imbedding map $i : s \hookrightarrow M$ is the inclusion which determines the applications $i_{*,q} : T_q s \to T_q M$ and $i_q^* : T_q^* M \to T_q^* s$. Denote $N_q = i_{*,q}(T_q s)$, $N_q^* = \{ \omega \in T_q^* M \mid \omega (X) = g(X, Z), \lambda \in \mathbb{R} \}$; it is obvious that $i_{*,q} : T_q s \to N_q$ and $i^*_q \mid N_q^* : N_q^* \to T_q^* s$ are bijections.

For $l = \left[ i^*_q \mid N_q^* \right]^{-1} \circ (i^*_q g)$, we have $l(X, Y) = -XY$, where

$$X = \Xi Z_q, \ Y = \Xi Z_q, \ X, Y \in T_q s,$$

$L$ must be a two symmetric 2-form negatively definite on $i(s)$ and for $\forall q \in \theta(S) \cap i(s)$

$$T_q M = H_q \oplus N_q, \ T_q^* M = H_q^* \oplus N_q^*.$$

**Proposition 12** The $Z$-norm on the almost Minkowski manifold $[M, g, Z]$ is the sum of the Riemannian norms associated to the projections onto the submanifolds $(\theta(h), h)$ and $(i(s), -l)$.

**Proof.** In $(\theta(S), h)$ the Riemannian norm is

$$|X|_h = \sqrt{h(X, X)} = \sqrt{g(X, X) + g(X, Z)^2}, \ \forall X \in H_q.$$

In $(i(s), -l)$ the Riemannian norm is

$$|X|_l = \sqrt{-l(X, X)} = |X| = |g(X, Z)|, \ \forall X \in N_q.$$

If $X \in T_q M = H_q \oplus N_q$, by Proposition 8 a) we have $X = X_1 + X_2$, where $X_1 = X + g(X, Z)Z$ and $X_2 = -g(X, Z)Z$;

$$h(X_1, X_1) = g(X_1, X_1) + g(X_1, Z)^2 = g(X, X) + g(X, Z)^2$$

$$l(X_2 X_2) = -g(X, Z)^2$$

Then

$$|X_1|_h + |X_2|_l = \sqrt{-l(X_2, X_2)} + \sqrt{h(X_1, X_1)} = |g(X, Z)| + \sqrt{g(X, X) + g(X, Z)^2} \overset{!}{=} |X|_Z.$$

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