Euler - Savary’s Formula on Minkowski Geometry

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003),

Abstract
We consider a base curve, a rolling curve and a roulette on Minkowski plane and give the relation between the curvatures of these three curves. This formula is a generalization of the Euler - Savary’s formula of Euclidean plane.

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1 Introduction

On the Euclidean plane $E^2$, we consider two curves $c_B$ and $c_R$. Let $P$ be a point relative to $c_R$. When $c_R$ rolls without splitting along $c_B$, the locus of the point $P$ makes a curve, say $c_L$. On this set of curves, $c_B$, $c_R$, $c_L$ are called the base curve, rolling curve and roulette, respectively. For example, if $c_B$ is a straight line, $c_R$ is a quadratic curve and $P$ is a focus of $c_R$, then $c_L$ is the Delaunay curve that are used to study surfaces of revolution with the constant mean curvature.

Since this "rolling situation" makes up three curves, it is natural to ask questions: what is the relation between the curvatures of these curves, when given two curves, can we find the third one? Many geometers studied these questions and generalized the situation [3]. Today the relation of the curvatures is called as the Euler - Savary’s formula.

However, the "rolling situation" on the Minkowski geometry is not studied yet. Only the Delaunay curve is considered to study surfaces of revolution with the constant mean curvature [1]. The purpose of this paper is to give answers to the above-mentioned general questions on the Minkowski geometry. After the preliminaries of section 2, in section 3, we consider the associated curve that is the key concept to study the roulette, for, the roulette is one of associated curves of the base curve. Section 4 is devoted to give the Euler - Savary’s formula on the Minkowski plane. In the final section, we determine the third curve from other two.
2 Preliminaries

Let $L^2$ be the Minkowski plane with metric $g = (+, -)$. A vector $X$ of $L^2$ is said to be spacelike if $g(X, X) > 0$ or $X = 0$, timelike if $g(X, X) < 0$ and null if $g(X, X) = 0$ and $X \neq 0$.

A curve $c$ is a smooth mapping $c : I \rightarrow L^2$ from an open interval $I$ into $L^2$. Let $t$ be a parameter of $c$. By $c(t) = (x(t), y(t))$, we denote the orthogonal coordinate representation of $c(t)$. The vector field $\frac{dc}{dt} = (\frac{dx}{dt}, \frac{dy}{dt}) =: X$ is called the tangent vector field of the curve $c(t)$. If the tangent vector field $X$ of $c(t)$ is a spacelike, timelike, or null, then the curve $c(t)$ is called spacelike, timelike, or null, respectively.

In the rest of this paper, we mostly consider non-null curves. When the tangent vector field $X$ is non-null, we can have the arc length parameter $s$ and have the Frenet formula

\[ \frac{dX}{ds} = KY, \quad \frac{dY}{ds} = kX, \]

where $k$ is the curvature of $c(s)$ (cf. [2]). The vector field $Y$ is called the normal vector field of the curve $c(s)$. Remark that we have the same representation of the Frenet formula regardless of whether the curve is spacelike or timelike.

If $\phi(s)$ is the slope angle of the curve, then we have $\frac{d\phi}{ds} = k$.

3 Associated curve

In this section, we give general formulas of the associated curve. Let $c(s)$ be a non-null curve with the arc length parameter $s$, and $\{X, Y\}$ the Frenet frame of $c(s)$.

If we put

\[ c_A = c(s) + u_1(s)X + u_2(s)Y, \]

then $c_A(s)$ generally makes a curve. This curve is called the associated curve of $c(s)$.

Remark that $\{u_1(s), u_2(s)\}$ is a relative coordinate of $c_A(s)$ with respect to $\{c(s), X, Y\}$.

If we put

\[ \frac{dc_A}{ds} = \frac{\delta u_1}{ds}X + \frac{\delta u_2}{ds}Y, \]

then, since

\[ \frac{dc_A}{ds} = \frac{dc}{ds} + \frac{du_1}{ds}X + \frac{du_2}{ds}Y + u_1 \frac{dX}{ds} + u_2 \frac{dY}{ds} = \left(1 + \frac{du_1}{ds} + ku_2\right)X + \left(ku_1 + \frac{du_2}{ds}\right)Y, \]

by virtue of (2.1), we have

\[ \frac{\delta u_1}{ds} = \frac{du_1}{ds} + ku_2 + 1, \]
\[ \frac{\delta u_2}{ds} = \frac{du_2}{ds} + ku_1. \]

Let $s_A$ be the arc length parameter of $c_A$. Then, from
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\[
\frac{dc_A}{ds} = \frac{dc_A}{ds} \frac{ds_A}{ds} = v_1 X + v_2 Y,
\]
\[
v_1 := \frac{du_1}{ds} + ku_2 + 1, \quad v_2 := \frac{du_2}{ds} + ku_1,
\]
the Frenet frame \(\{Z, W\}\) of \(c_A\) has following equations;
\[
\begin{align*}
\frac{dZ}{ds_A} &= k_A W, \\
\frac{dW}{ds_A} &= k_A Z,
\end{align*}
\]
where \(k_A\) is the curvature of \(c_A\).

Let \(\theta\) (resp. \(\omega\)) be the slope angle of \(c\) (resp. \(c_A\)). Then
\[
(3.4) \quad k_A = \frac{d\omega}{ds_A} = \frac{d\omega}{ds_A} \frac{ds}{ds_A} = \left(k + \frac{d\phi}{ds}\right) \frac{1}{\sqrt{v_1^2 - v_2^2}},
\]
where \(\phi = \omega - \theta\).

If \(c_A\) is space-like, then we can put
\[
\begin{align*}
cosh \phi &= \frac{v_1}{\sqrt{v_1^2 - v_2^2}}, \\
sinh \phi &= \frac{v_2}{\sqrt{v_1^2 - v_2^2}}.
\end{align*}
\]

Since
\[
\frac{d\phi}{ds} = \frac{d}{ds} \left(\cosh^{-1} \frac{v_1}{\sqrt{v_1^2 - v_2^2}} \right),
\]
(3.4) reduces to
\[
k_A = \left(k + \frac{v_1 v'_2 - v'_1 v_2}{v_1^2 - v_2^2}\right) \frac{1}{\sqrt{v_1^2 - v_2^2}},
\]
where dash represents the derivative with respect to \(s\).

If \(c_A\) is time-like, since \(\sinh \phi = \frac{v_2}{\sqrt{v_1^2 - v_2^2}}\), we have
\[
k_A = \left(k + \frac{v_1 v'_2 - v'_1 v_2}{v_2^2 - v_1^2}\right) \frac{1}{\sqrt{v_1^2 - v_2^2}}.
\]

4 Euler - Savary’s formula

In this section, we consider the roulette and give the Euler - Savary’s formula.

Let \(c_B\) (resp. \(c_R\)) be the base (resp. rolling) curve and \(k_B\) (resp. \(k_R\)) the curvature of \(c_B\) (resp. \(c_R\)). Let \(P\) be a point relative to \(c_R\). By \(c_L\), we denote the roulette of the locus of \(P\).

We can consider that \(c_L\) is an associated curve of \(c_B\), then the relative coordinate \(\{x, y\}\) of \(c_L\) with respect to \(c_B\) satisfies
\[
\frac{\delta x}{ds_B} = \frac{dx}{ds_B} + k_B y + 1,
\]
\[
\frac{\delta y}{ds_B} = \frac{dy}{ds_B} + k_B x,
\]
by virtue of (3.2).

Since \( c_R \) rolls without splitting along \( c_B \), at each point of contact, we can consider \( \{x, y\} \) is a relative coordinate of \( c_L \) with respect to \( c_R \) for a suitable parameter \( s_R \). In this case, the associated curve is reduced to a point \( P \). Hence it follows that
\[
\frac{\delta x}{ds_R} = \frac{dx}{ds_R} + k_{RY} + 1 = 0,
\]
\[
\frac{\delta x}{ds_R} = \frac{dx}{ds_R} + k_{RY} = 0.
\]

Substituting these equations into (4.1), we have
\[
\frac{\delta x}{ds_B} = (k_B - k_R)y, \quad \frac{\delta y}{ds_B} = (k_B - k_R)x,
\]
so
\[
\frac{\delta x}{\delta y} = \frac{x}{y}.
\]

**Proposition 4.1** Let \( c_R \) rolls without splitting along \( c_B \) from the starting time \( t = 0 \). Then at each time \( t = t_0 \) of this motion, the normal at the point \( c_L(t_0) \) passes through the point of contact \( c_B(t_0) = c_R(t_0) \).

Suppose that \( c_L \) is spacelike. Then, from (4.3),
\[
0 < \left( \frac{\delta x}{ds_B} \right)^2 - \left( \frac{\delta y}{ds_B} \right)^2 = (k_B - k_R)^2(y^2 - x^2).
\]

Hence we can put
\[
x = \sinh \phi, \quad y = \cosh \phi.
\]

Differentiating these equations, we have
\[
\frac{dx}{ds_R} = \frac{dr}{ds_R} \sinh \phi + r \cosh \phi \frac{d\phi}{ds_R} = -1 - k_R r \cosh \phi,
\]
\[
\frac{dy}{ds_R} = \frac{dr}{ds_R} \cosh \phi + r \sinh \phi \frac{d\phi}{ds_R} = -k_R r \sinh \phi,
\]
by virtue of (4.2). From these equations, it follows that
\[
r \frac{d\phi}{ds_R} = - \cosh \phi - k_r r.
\]

Therefore, substituting this equation into (3.4), we have
\[
rk_L = \pm 1 - \frac{\cosh \phi}{r|k_B - k_R|}.
\]
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If $c_L$ is timelike, by similar calculation, we have

$$rk_L = \pm 1 + \frac{\sinh \phi}{r|k_B - k_R|}.$$ 

We can easily see that the case $c_L$ is null makes a contradiction.

**Theorem 4.1** On the Minkowski plane $L^2$, suppose that a curve $c_R$ rolls without splitting along a curve $c_B$. Let $c_L$ be a locus of a point $P$ that is relative to $c_R$. Let $Q$ be a point on $c_L$ and $R$ a point of contact of $c_B$ and $c_R$ corresponds to $Q$ relative to the rolling relation. By $(r, \phi)$, we denote a polar coordinate of $Q$ with respect to the origin $R$ and the base line $c_B'$. Then curvatures $k_B, k_R$ and $k_L$ of $c_B, c_R$ and $c_L$, respectively, satisfies

$$rk_L = \pm 1 - \frac{\cosh \phi}{r|k_B - k_R|} \quad \text{(when } c_L \text{ is spacelike)},$$

$$rk_L = \pm 1 + \frac{\sinh \phi}{r|k_B - k_R|} \quad \text{(when } c_L \text{ is timelike)}.$$ 

## 5 Determining the curve

Since the roulette is a locus of a point, it is determined by the base curve and the rolling curve. In this section, we consider the converse problem. First suppose that a base curve $c_B$ and a roulette $c_L$ is given.

Let $(x(s_B), y(s_B))$ be the orthogonal coordinates of the base curve $c_B$ with the arc length parameter $s_B$. For a point $Q$ of $c_B$, draw the normal to the roulette $c_L$. Let $R$ be the foot of this normal with the orthogonal coordinate $(f(s_B), g(s_B))$. Then the length of $QR$ is

$$QR = \sqrt{|(f(s_B) - x(s_B))^2 - (g(s_B) - y(s_B))^2|}. \quad (5.1)$$

If we consider (5.1) on the rolling curve $c_R$, this equation represents the length of the point $P$ relative to $c_R$ and a point of $c_R$. Hence the orthogonal coordinate $(u(s_B), v(s_B))$ of $c_R$ is given by the equations

$$u(s_B)^2 - v(s_B)^2 = (f(s_B) - x(s_B))^2 - (g(s_B) - y(s_B))^2,$$

$$\left(\frac{du}{ds_B}\right)^2 - \left(\frac{dv}{ds_B}\right)^2 = \pm 1,$$

the sign of $\pm 1$ depends on spacelike or timelike of $c_R$.

Next suppose that a rolling curve $c_R$ and a roulette $c_L$ is given.

Let $(x(s_L), y(s_L))$ be the orthogonal coordinate of $c_L$ with arc length parameter $s_L$. Suppose that the polar coordinate $r(s_R)$ of $c_R$ is given by the arc length parameter $s_R$ of $c_R$.

Since the normal of $c_L$ is $\left(\frac{dy}{ds_L}, \frac{dx}{ds_L}\right)$, a point $(u, v)$ of the base curve $c_B$ is given by
\[ u = x(s_L) \pm r(s_R) \frac{dy}{ds_L}, \]
\[ v = y(s_L) \pm r(s_R) \frac{dx}{ds_L}. \]

(5.2)

Then, from
\[
\frac{du}{ds_R} = \frac{dx}{ds_L} \frac{ds_L}{ds_R} \pm \frac{dr}{ds_R} \frac{dy}{ds_L} \pm \frac{d}{s_L} \left( \frac{dy}{ds_L} \right) \frac{ds_L}{ds_R},
\]
\[
\frac{dv}{ds_R} = \frac{dy}{ds_L} \frac{ds_L}{ds_R} \pm \frac{dr}{ds_R} \frac{dx}{ds_L} \pm \frac{d}{s_L} \left( \frac{dx}{ds_L} \right) \frac{ds_L}{ds_R},
\]

we have
\[
\frac{du}{ds_R} = \frac{dx}{ds_L} \left( 1 \pm r k_L \right) \frac{ds_L}{ds_R} \pm \frac{dr}{ds_R} \frac{dy}{ds_L},
\]
\[
\frac{dv}{ds_R} = \frac{dy}{ds_L} \left( 1 \pm r k_L \right) \frac{ds_L}{ds_R} \pm \frac{dr}{ds_R} \frac{dx}{ds_L},
\]

where \( k_L \) is the curvature of \( c_L \).

Since \( s_R \) is also the arc length of \( c_B \), it follows that
\[
\left( \frac{du}{ds_R} \right)^2 - \left( \frac{dv}{ds_R} \right)^2 = \left( \frac{ds_L}{ds_R} \right)^2 \left( 1 \pm r k_L \right)^2 - \frac{dr^2}{ds_R} = \pm 1,
\]

where the sign of \( \pm 1 \) depends on spacelike or timelike of \( c_B \). From this differential equation, we can solve \( s_L = s_L(s_R) \). Substituting this equation into (5.2), we can have the orthogonal coordinate of \( c_B \).

The solvability of these differential equations is easily checked. For example, we have solutions like that: \( c_B \) is \( x \)-axis, \( c_R \) is quadratic curve and \( c_L \) is "Delaunay curve" (cf. [1]).

**References**


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