QR-Hypersurfaces
of Quaternionic Kähler Manifolds

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003),

Abstract

We prove that the basic manifold of a submersion from a QR-hypersurface of a quaternionic Kähler manifold to an almost quaternionic Hermitian manifold is quaternionic Kähler. Then we prove some results involving the sectional curvatures.

Key words: Kähler manifolds, Hermitian manifolds, quaternionic manifolds, CR-manifolds.

Introduction

Real hypersurfaces of quaternionic space forms have been studied by many authors ([1], [2], [3], [4], [5], [11], [12]) under conditions concerning their shape operator. It is known that real hypersurface of quaternionic Kähler manifolds are not CR-hypersurface in general ([2]).

The study of CR-submanifolds of a quaternionic Kähler manifolds has been carried out in the paper [1]. S. Kobayashi considered the similarity between the total space of a Riemannian submersion and a CR-submanifold of a Kähler manifold in terms of distributions ([9]). In this paper we study Riemannian submersions from QR-hypersurface of a quaternionic Kähler manifold over an almost quaternionic Hermitian manifold (second section). In the last section we study some curvature properties induced on the basic manifold by the submersion.

1 Hypersurfaces of quaternionic Kähler manifolds

We say that a 4(m + 1) – dimensional manifold $\tilde{M}$ with a metric $\tilde{g}$ is a quaternionic Kähler manifold ($m \geq 1$) if there exists a 3-dimensional vector bundle $V$ of tensors of type $(1, 1)$ on $\tilde{M}$ satisfying the following conditions:
(a) In any coordinate neighborhood \( \tilde{U} \) on \( \tilde{M} \) there is a local basis of almost Hermitian structures \( \{ J_a, \tilde{g} \} \), such that \( J_a^2 = -\text{Id}, \ a \in \{1, 2, 3\} \) and \( J_a \circ J_b = -J_b \circ J_a = J_c \) for any cyclic permutation \( (a, b, c) \) of \( (1, 2, 3) \).

(b) For any local section \( \varphi \) of \( V \) and any tangent vector \( X \) to \( \tilde{M} \), \( \tilde{\nabla}_X \varphi \) is also a local section in \( V \), where \( \tilde{\nabla} \) denotes the Levi-Civita connection of \( \tilde{g} \).

Condition (b) is equivalent to the following:

(b') There exist local 1-forms \( \omega_{ab}, \ a, b \in \{1, 2, 3\} \) on \( \tilde{U} \) such that \( \omega_{ab} + \omega_{ba} = 0 \), and

\[
(1) \quad \tilde{\nabla}_x J_a = \omega_{ab}(x) J_b + \omega_{ac}(x) J_c
\]

for any cyclic permutation \( (a, b, c) \) of \( (1, 2, 3) \).

Given two local bases \( \{ J_a \} \) and \( \{ J'_a \} \) of \( \tilde{V} \) defined on coordinate neighborhoods \( \tilde{U} \) and \( \tilde{U}' \) such that \( \tilde{U} \cap \tilde{U}' \neq \emptyset \), we have on \( \tilde{U} \cap \tilde{U}' \):

\[
(2) \quad J'_a = \sum_{b=1}^{3} C_{ab} J_b
\]

where \( [C_{ab}] \) is an element of the special orthogonal group \( SO(3) \) (see [8]).

Let \( M \) be an orientable hypersurface of \( \tilde{M} \) and \( \xi \) a unit normal field defined on \( M \). On \( \tilde{U} \) \( \xi_a = -J_a(\xi), \ a \in \{1, 2, 3\} \) defines a tangent vector field to \( M \). Similarly, we define \( \xi'_a \) on \( \tilde{U}' \) and on \( \tilde{U} \cap \tilde{U}' \neq \emptyset \) we have:

\[
(3) \quad \xi'_a = \sum_{b=1}^{3} C_{ab} \xi_b, \ b \in \{1, 2, 3\}
\]

so that one obtains a distribution \( \mathcal{V} \) on \( M \) which is locally represented by \( \{ \xi_a \} \), \( 1 \leq a \leq 3 \), on \( \tilde{U} \). Let \( \mathcal{H} \) be the orthogonal complementary distribution to \( \mathcal{V} \) with respect to the Riemannian metric \( \tilde{g} \) induced by \( \tilde{g} \) on \( M \).

We see that for each \( x \in M \), \( \mathcal{H}_x = J_a - \text{invariant} \), but \( \mathcal{V}_x \) is not an anti-invariant subspace of \( T_x M \) with respect \( J_a, a = \{1, 2, 3\} \). It is easy see that \( J_a(\mathcal{V}_x) = T_x M^\perp \), \( x \in M \), where \( T_x M^\perp \) is the normal space at \( x \) to the hypersurface \( M \) in \( \tilde{M} \). In general, when the previous conditions are satisfied, we say that \( M \) is a QR-hypersurface of \( \tilde{M} \) (see [3]). Now, let \( B \) be the second fundamental form of \( M \) in \( \tilde{M} \). Then, for any \( E, F \in \Gamma(TM) \) we have the Gauss formula

\[
(4) \quad \tilde{\nabla}_E F = \nabla_E F + B(E, F),
\]

where \( \tilde{\nabla} \) and \( \nabla \) are the Levi-Civita connections on \( \tilde{M} \) and \( M \), respectively.

If \( L \) denotes the fundamental tensor of Weingarten with respect to \( \xi \), we have the Weingarten formula

\[
(5) \quad \tilde{\nabla}_E \xi = -L(E),
\]

and for any \( E, F \in \Gamma(TM) \) the following formula

\[
(6) \quad g(L(E), F) = g(B(E, F), \xi)
\]
The integrability of the distributions $\mathcal{V}$ and $\mathcal{H}$ on $M$ has been studied by A. Bejancu ([2]). We recall that the vertical distribution $\mathcal{V}$ is integrable if and only if
\begin{equation}
B(U, X) = 0
\end{equation}
for any $U \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$. If (7) is satisfied, we say that $M$ is a mixed geodesic QR-hypersurface of $\tilde{M}$.

2 Riemannian submersions of QR-hypersurfaces

Let $M$ be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $\tilde{M}$. We denote by $(M', g', J'_a)$, $a \in \{1, 2, 3\}$, an almost quaternionic Hermitian manifold (i.e. satisfying the condition (a)). We say that a Riemannian submersion $\pi: M \to M'$ is a QR-submersion if the following conditions are satisfied:

i) $\mathcal{V}$ is the kernel of $\pi_*$;

ii) for each $x \in M$, $\pi_x : H_x \to T_{\pi(x)}M'$ is an isometry with respect to each complex structure of $H_x$ and $T_{\pi(x)}M'$, where $T_{\pi(x)}M'$ denotes the tangent space to $M'$ at $\pi(x)$.

As in the paper [10], the letters $U, V, W, W'$ will always denote vertical vector fields and $X, Y, Z, Z'$ horizontal vector fields. A horizontal vector field $X$ on $M$ is said to be basic if it is $\pi$-related to a vector field $X'$ on $M'$.

We denote by $T$ and $A$ O'Neill’s fundamental tensors (see [13], [11]).

Lemma 2.1 Let $X$ and $Y$ be basic vector fields on $M$. Then the following conditions hold:

a) The horizontal component $h[X, Y]$ of $[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;

b) $h(\nabla_X Y)$ is basic vector field corresponding to $\nabla'_X Y'$ where $\nabla$ and $\nabla'$ are the the Levi-Civita connections on $M$ and $M'$, respectively;

c) $[X, U] \in \Gamma(\mathcal{V})$, for any vertical field $U \in \Gamma(\mathcal{V})$;

where $h$ denotes the horizontal component of a vector $E$ on $M$.

We define a skew-symmetric tensor field $C$ by
\begin{equation}
\hat{\nabla}_x Y = h\hat{\nabla}_x Y + C(X, Y)
\end{equation}
for all $X, Y \in \Gamma(\mathcal{H})$.

The second fundamental form $B$ of $M$ in $\tilde{M}$ is:
\[ B(E, F) = \tilde{\nabla}_E F - \nabla_E F \]

for all \( E, F \in \Gamma(TM) \).

**Theorem 2.2** Let \( M \) be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold \( \tilde{M} \). If \( \pi : M \to M' \) is a QR-submersion of \( M \) on an almost quaternionic Hermitian manifold, then \( M' \) is a quaternionic Kähler manifold.

**Proof.** By using Gauss formula and (1) we obtain

\[ h(\nabla_X J_a Y - J_a(h\nabla_X Y) = \omega_{ab}(X) J_b Y + \omega_{ac}(X) J_c Y, \]

for any local basic vector fields \( X, Y \) on \( M \) and for any cyclic permutation \((a, b, c)\) of \((1, 2, 3)\). Then we can define 1-forms \( \omega'_{ab} \) on \( M' \) by

\[ \omega'_{ab}(X') \circ \pi = \omega_{ab}(X), \quad a, b, c \in \{1, 2, 3\}, \]

for any local vector field \( X' \) on \( M' \) and \( X \) a real basic vector field on \( M \) such that \( \pi_* X = X' \).

On the other hand, by the definition of a QR-submersion we have

\[ \pi_* \circ J_a = J'_a \circ \pi_*. \]

Using Lemma 2.1, from (3)-(5) we obtain

\[ h(\nabla'_X J'_a Y') = \omega'_{ab}(X') J'_b Y' + \omega'_{ac}(X') J'_c Y', \]

where \( \nabla' \) is the Levi-Civita connection on \( M' \) and \( X', Y' \) any local vector fields on \( M' \). We conclude that \((M', J'_a, g')\) is a quaternionic Kähler manifold. \(\square\)

### 3 Totally umbilical QR-hypersurfaces

In the sequel we shall denote by \( \langle \cdot, \cdot \rangle \) the scalar product induced on the tangent spaces of \( M \) and \( \tilde{M} \) by the Riemannian metric \( g \). We recall that a hypersurface \( M \) of \( \tilde{M} \) is totally umbilical if the first and the second fundamental forms are proportional, that is

\[ B(E, F) = \langle E, F \rangle H \]

for any \( E, F \in \Gamma(TM) \), where \( H \) is the mean curvature vector of \( M \), defined by the formula,

\[ H = \frac{1}{4m + 3} \text{Trace} B. \]

We have the Gauss equation:

\[ \tilde{R}(E, E', F, F') = R(E, E'F, F') - \langle B(E, F), B(E', F') \rangle + \langle B(E', F'), B(F, E') \rangle. \]

Taking account of the formula (1), the Gauss equation for a totally umbilical hypersurface \( M \) in \( \tilde{M} \) becomes:
\( \tilde{R}(E, E', F, F') = R(E, E', F, F') - \langle E, F \rangle \langle E', F' \rangle + \langle F, F' \rangle \|H\|^2, \)

where \( \|H\|^2 = \langle H, H \rangle. \)

We see that, if \( M \) is a totally umbilical QR-hypersurface of \( \tilde{M} \), then it is a mixed geodesic QR-hypersurface, i.e. \( B(V, X) = 0 \) for any \( V \in \Gamma(V) \) and \( X \in \Gamma(H) \). Consequently, the vertical distribution \( V \) is integrable.

Moreover, it is easy to check that each leaf of \( V \) is totally geodesic in \( M \) (see, for example [3], p. 121). Then we conclude that the first fundamental tensor \( T \) of the Riemannian submersion \( \pi : M \to M' \) vanishes, because \( T_U V \) is the second fundamental form of each fibre for any \( U, V \in \Gamma(V) \) (see [7], [13]).

Let us now recall the following two Gray-O’Neill curvature equations for a Riemannian submersion:

\( R(U, V, U', V') = \tilde{R}(U, V, U', V') + \langle T_U V', T_V U' \rangle + \langle C(Y, X'), C(X, Y') \rangle - \langle C(X, X'), C(Y, Y') \rangle, \)

for all \( U, V, U', V' \in \Gamma(V) \) and \( X, Y, X', Y' \in \Gamma(H) \), where, for any quadruplet of horizontal vector fields \( (X, Y, X', Y') \), \( R^\ast(X, Y, X', Y') = R'(\pi_* X, \pi_* Y, \pi_* X', \pi_* Y') \circ \pi \), with \( R' \) Riemannian curvature on the fibres of \( \mathcal{H} \). Here \( R' \) is the Riemannian curvature of the metric \( g' \) on \( M' \).

**Lemma 3.1** Let \( M \) be a totally umbilical, not totally geodesic, QR-hypersurface of a quaternionic Kähler manifold. Then the tensor field \( C \) which measures the integrability of the horizontal distribution \( H \), is given by the formula

\( C(X, Y) = \|H\| \sum_{a=1}^{3} \langle X, J_a Y \rangle \xi_a. \)

**Proof.** Using (1), (4), (5) and (7), we obtain

\( J_a(LX) - \nabla_X \xi_a = \omega_{ac}(X) \xi_b - \omega_{ab}(X) \xi_c, \)

for any \( X \in \Gamma(H) \). Now, by (6) in (8), we have

\( \langle \nabla_X Y, \xi_a \rangle = \langle B(X, J_a Y), \xi \rangle, \)

for any \( X, Y \in \Gamma(H) \), and \( a \in \{1, 2, 3\} \). Taking into account that the mean curvature vector \( H \) of \( M \) is a global vector field and it is non vanishing on \( M \) (see [3]), we take \( \xi = \frac{H}{\|H\|} \). Then we have

\( C(X, Y) = V \nabla_X Y = \|H\| \sum_{a=1}^{3} \langle X, J_a Y \rangle \xi_a, \)
from which formula (7) follows.

**Theorem 3.1** Let $M$ be a totally umbilical, not totally geodesic, QR-hypersurface of a quaternionic Kähler manifold. Then,

(a) $\tilde{K}(U, V) = K(U, V) - \|H\|^2$, where $\{U, V\}$ is an orthonormal basis of the vertical $2$-plane $\alpha$, $\alpha \subset \mathcal{V}_x$, $x \in M$, and $\tilde{K}, K$ denote the sectional curvatures of $\alpha$ on $\tilde{M}, M$, respectively.

(b) $K(X, Y) = K'(X', Y') - 3\|H\|^2 \sum_{a=1}^{3} (X, J_a Y)^2$, where $X, Y$ is an orthonormal basis of a horizontal $2$-plane $\alpha \subset \mathcal{H}_x$, $K(X, Y)$ denoting the sectional curvature of $\alpha$, and $K'(X', Y')$ denotes the sectional curvature in $M'$ of the $2$-plane spanned by $X' = \pi_* X$ and $Y' = \pi_* Y$.

**Proof.** Property a) is easily obtained from (4) and (5). From (6), as an immediate consequence of the skew-symmetry of $C$, we have

$$R(X, Y, X, Y) = R'(X', Y', X', Y') - 3\|C(X, Y)\|^2.$$  

(11)

Lemma 3.1 and (11) directly give b). \hfill \Box

We recall that a totally umbilical, not totally geodesic, hypersurface $M$ of a Riemannian manifold $\tilde{M}$ is an extrinsic hypersphere if the mean curvature vector field $H$ is parallel with respect to the linear normal connection $\nabla^\perp$ or, equivalently, $\|H\| = c$ is a constant $c \neq 0$ on $M$.

Then we have the following

**Theorem 3.2** Let $M$ be an extrinsic hypersurface of a flat quaternionic Kähler manifold $\tilde{M}$ and $\pi : M \to M'$ a QR-submersion of $M$ on a quaternionic Kähler manifold $M'$. Then $M'$ is a quaternionic Kähler manifold with constant quaternionic sectional curvature $c > 0$.

**Proof.** By (4), (6) and Lemma 3.1 we have

$$R'(X', Y')Z' = \|H\|^2 (g'(Y', Z')X' - g'(X', Z')Y') + \sum_{a=1}^{3} (g'(J'_a Y', Z')J'_a X' - g(J'_a X', Z')J'_a Y') + 2g'(X', J'_a Y')J'_a Z').$$

where $\|H\|$ is a constant on $M'$ and $X', Y', Z' \in \Gamma(TM')$. \hfill \Box

**Remark** There exist no proper totally umbilical QR-submanifolds in positively or negatively curved quaternionic Kähler manifolds (see [3]).

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References


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