Homogeneous Lorentzian Structures on Some Gödel-Levichev’s Spacetimes, and Associated Reductive Decompositions

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Abstract

For the Levichev homogeneous spacetimes of type 2α on the Gödel group, the homogeneous Lorentzian structures and the associated reductive decompositions are determined.

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1 Introduction and preliminaries

É. Cartan gave in [2] the classical characterization of Riemannian symmetric spaces as the spaces of parallel curvature. This was extended by Ambrose and Singer, who gave in [1] a characterization for a connected, simply connected and complete Riemannian manifold to be homogeneous, in terms of a (1, 2) tensor field \( S \), called by Tricerri and Vanhecke in [7] a homogeneous Riemannian structure, which satisfies certain equations (see (1.1) below). In [3] it is defined a homogeneous pseudo-Riemannian structure on a pseudo-Riemannian manifold \((M, g)\) as a tensor field \( S \) of type (1, 2) such that \( \tilde{\nabla} \) being the Levi-Civita connection and \( R \) its curvature tensor, the connection \( \tilde{\nabla} = \nabla - S \) satisfies the Ambrose-Singer equations

\[
(1.1) \quad \tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0.
\]

In [3] it is proved that if the pseudo-Riemannian manifold \((M, g)\) is connected, simply connected and geodesically complete then it admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold. This means that \( M = G/H \), where \( G \) is a connected Lie group acting transitively and effectively on \( M \) as a group of isometries, \( H \) is the isotropy group at a point \( o \in M \), and the Lie algebra \( \mathfrak{g} \) of \( G \) may be decomposed into a vector space
direct sum of the Lie algebra $h$ oh $H$ and an $\text{Ad}(H)$-invariant subspace $m$, that is $g = h \oplus m$, $\text{Ad}(H)m \subset m$. (If $G$ is connected and $M$ is simply connected then $H$ is connected, and the latter condition is equivalent to $[h, m] \subset m$.)

Let $(M, g)$ be a connected, simply connected, and geodesically complete pseudo-Riemannian manifold, and suppose that $S$ is a homogeneous pseudo-Riemannian structure on $(M, g)$. We fix a point $o \in M$ and put $m = T_o(M)$. If $\tilde{\mathcal{R}}$ is the curvature tensor of the connection $\tilde{\nabla} = \nabla - S$, we can consider the holonomy algebra $\tilde{h}$ of $\tilde{\nabla}$ as the Lie subalgebra of “skew-symmetric” endomorphisms of $(m, g_o)$ generated by the operators $\tilde{\mathcal{R}}_{ZW}$, where $Z, W \in m$. Then, according to the Ambrose-Singer construction [1, 7], a Lie bracket is defined in the vector space direct sum $\tilde{g} = \tilde{h} \oplus m$ by

\begin{align}
[U, V] &= UV - VU, & U, V &\in \tilde{h}, \\
[U, Z] &= U(Z), & U &\in \tilde{h}, Z \in m, \\
[Z, W] &= \tilde{\mathcal{R}}_{ZW} + S_ZW - S_WZ, & Z, W &\in m,
\end{align}

(1.2)

and we say that $(\tilde{g}, \tilde{h})$ is the reductive pair associated to the homogeneous pseudo-Riemannian structure $S$.

Tricerri and Vanhecke [7] have classified the homogeneous Riemannian structures into eight classes, which are defined by the invariant subspaces of certain space $S_1 \oplus S_2 \oplus S_3$. In [4] a similar classification for the pseudo-Riemannian case is given. For more details see below.

On the other hand, Levichev consider in [5] the usual Gödel metric

$$g = - \frac{e^{-2x_4}}{2} dx_1^2 - 2e^{-2x_4} dx_1 dx_2 - dx_2^2 + dx_3^2 + dx_4^2,$$

as a left-invariant metric on the Gödel group $G$, and defines several families of metrics on $G$, thus obtaining several types of homogeneous Lorentz spaces. The ones of type $2a$ are connected, simply connected, and geodesically complete. In the present note we determine the homogeneous Lorentzian structures on these homogeneous space-times and their type in Tricerri-Vanhecke’s classification, and the associated reductive decompositions.

2 Homogeneous Lorentzian structures

The Gödel group is the simply connected Lie group $G$ whose Lie algebra $g$ has four generators $e_1, e_2, e_3, e_4$, with the only nonvanishing bracket

$$[e_4, e_1] = e_1.$$

The group $G$ admits a realization as $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4)\}$ with multiplication $z = x \cdot y$ obtained from the matrix expression

$$x = \begin{pmatrix} e^{x_4} & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
The commutation relations of its Lie algebra in the system of coordinates chosen on $G$ coincide with the brackets above.

Consider the subspaces $L_1, L_2, L_3$ of $g$ generated respectively by $e_1, e_2, e_3$ and $e_1, e_2, e_3$. Then the homogeneous Lorentz group of type $2a$ is defined by the conditions: $L_2, L_3$ are timelike, and $L_1$ is spacelike (for more details see [5]). Then, for each couple of real numbers $p, q$ with $0 \leq p < 1, q > 0$, the left-invariant Lorentzian metric $g_{p, q}$ on $G$ obtained by left translations from the scalar product at the origin with matrix given, with respect to the above basis of $g$, by

$$
\langle \cdot, \cdot \rangle_{p, q} = \begin{pmatrix}
1 & p & 0 & 0 \\
p & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix},
$$

is given by

$$
g_{p, q} = \begin{pmatrix}
e^{-2x_4} & e^{-2x_4}p & 0 & 0 \\
e^{-2x_4}p & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}.
$$

As causal spacetimes, the Lorentz Lie groups corresponding to the Gödel group with the metric of type $2a$ are homogeneously globally hyperbolic, which is a strong causality condition. We recall that: A causal curve in a Lorentz manifold $M$ is a curve whose velocity vectors are all nonspacelike; if $M$ is globally hyperbolic then any pair of points that can be joined by a causal curve can be joined by a (longest) causal geodesic; a solvable Lorentz Lie group $G$ is said to be homogeneously globally hyperbolic if it is globally hyperbolic and has a Cauchy surface $S$ passing through the identity element $e \in G$ and containing the center of $G$ (for more details see [5, 6]); a Cauchy surface of a spacetime is a subset that is met exactly once by every inextendible timelike curve in the spacetime.

On account of Koszul’s formula for the Levi-Civita connection for a left-invariant metric $g$ on a Lie group,

$$
2g(\nabla_{e_i}e_j, e_k) = g([e_i, e_j], e_k) - g([e_j, e_k], e_i) + g([e_k, e_i], e_j),
$$

we obtain that the non-null covariant derivatives between generators are

$$
\nabla_{e_1}e_1 = \frac{1}{q} e_4, \quad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \frac{p}{2q} e_4, \\
\nabla_{e_1}e_4 = \frac{p^2 - 2}{2(1 - p^2)} e_1 + \frac{p}{2(1 - p^2)} e_2, \\
\nabla_{e_2}e_4 = \nabla_{e_4}e_2 = -\frac{p}{2(1 - p^2)} e_1 + \frac{p^2}{2(1 - p^2)} e_2, \\
\nabla_{e_4}e_1 = -\frac{p^2}{2(1 - p^2)} e_1 + \frac{p}{2(1 - p^2)} e_2.
$$
So, the nonvanishing components of the curvature tensor, with the convention $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$, are, putting $R_{e_i e_j e_k}$ for $R(e_i, e_j) e_k$,

\[
R_{e_1 e_2 e_1} = \frac{p^3}{4q(1-p^2)} e_1 - \frac{p^2}{4q(1-p^2)} e_2,
R_{e_1 e_2 e_2} = \frac{p^3}{4q(1-p^2)} e_1 - \frac{p^2}{4q(1-p^2)} e_2,
R_{e_1 e_4 e_1} = \frac{p(2-p)}{4q(1-p^2)} e_1,
R_{e_1 e_4 e_2} = -\frac{p^3}{4q(1-p^2)} e_2,
R_{e_2 e_4 e_1} = \frac{p^2 - 4}{4(1-p^2)} e_1 + \frac{p}{1-p^2} e_2,
R_{e_2 e_4 e_2} = -\frac{p^3}{4q(1-p^2)} e_2,
\]

and the nonvanishing components of the Riemann-Christoffel curvature tensor, with the convention $R(X, Y, Z, W) = g(R(Z, W)Y, X)$, putting $R_{e_i e_j e_k e_l}$ for $g(R(e_k, e_l) e_j, e_i)$, are

\[
R_{e_1 e_2 e_1 e_2} = \frac{p^2}{4q},
R_{e_1 e_4 e_1 e_4} = \frac{5p^2 - 4}{4(1-p^2)},
R_{e_2 e_4 e_2 e_4} = \frac{p^2}{4(1-p^2)}.
\]

We shall now determine the homogeneous Lorentzian structures on these spaces. For this, we must solve the Ambrose-Singer equations 1.1. The first Ambrose-Singer equation amounts to $S_{XY} = -S_{XY}$ for any homogeneous pseudo-Riemannian structure $S$. One can write the second Ambrose-Singer equation $\nabla R = 0$ as

\[
R_{\nabla U, XYZW} + R_{X \nabla U, YZW} + R_{XYZ \nabla U, ZW} + R_{XYZ, UZW} = S_{UXR(Z,W)Y} - S_{UYR(Z,W)X} + S_{UZR(X,Y)W} - S_{UWR(X,Y)Z}.
\]

Solving, we obtain that the nonvanishing components of $S$ are

\[
S_{e_1 e_2 e_1 e_2} = 1-p^2, \quad S_{e_4 e_1 e_4} = \frac{p}{2},
\]

except for $S_{e_i e_1 e_4}$, $i = 1, \ldots, 4$, for which we must use the third Ambrose-Singer equation. In our case, since we are considering left-invariant differential forms, the forms involved in this equation are linear combinations with constant coefficients of the basis $\{e^1, e^2, e^3, e^4\}$ of left-invariant forms on $G$ dual to the basis $\{e_1, e_2, e_3, e_4\}$. Moreover, since for a constant function $f$, one has $\nabla_X f = 0$ and $\nabla_X f = 0$, we also have $S_{X,Y} = 0$. Thus, the third Ambrose-Singer equation $\tilde{S} = 0$ can be written as

\[
S_{\nabla X, YZW} + S_{Y \nabla X, ZW} + S_{YZ \nabla X, W} = S_{S_X Y, ZW} + S_{Y S_X, ZW} + S_{Y Z S_X, W},
\]

for $X, Y, Z, W \in \mathfrak{g}$.

Solving, we obtain the nonzero components

\[
S_{e_1 e_2 e_1} = 1, \quad S_{e_2 e_1 e_4} = \frac{p}{2}.
\]
Consequently, the non-null components $S_{c_i}e_j$ are
\[ S_{c_1}e_1 = \frac{1}{q}e_4, \quad S_{c_1}e_2 = \frac{1-p^2}{q}e_4, \]
\[ S_{c_1}e_4 = \frac{-p^3 + p - 1}{1 - p^2}e_1 + \frac{p^2 - 1 - p^2}{1 - p^2}e_2, \quad S_{c_2}e_1 = \frac{p}{2q}e_4, \]
\[ S_{c_2}e_1 = -\frac{p^2}{2(1-p^2)}e_1 + \frac{p}{2(1-p^2)}e_2, \quad S_{c_2}e_2 = -\frac{p}{2(1-p^2)}e_1 + \frac{p^2}{2(1-p^2)}e_2. \]

Then, with the convention $v \wedge w = v \otimes w - w \otimes v$ for the exterior product, we have proved the following

**Theorem 1** The homogeneous Lorentzian structures on the Gödel-Levichev space $(G, g_{p,q})$ of type $(2)\alpha$ are given by
\[ \theta^1 \otimes \theta^1 \wedge \theta^4 + (1-p^2)\theta^1 \otimes \theta^2 \wedge \theta^4 + \frac{p}{2}(\theta^2 \otimes \theta^1 \wedge \theta^4 + \theta^4 \otimes \theta^1 \wedge \theta^2). \]

We recall some definitions and a result from Tricerri and Vanhecke [7] (see also [4]). Let $E$ be a real vector space of dimension $n$ endowed with an inner product $(\cdot, \cdot)$ of signature $(k, n-k)$. The space $(E, \langle \cdot, \cdot \rangle)$ will be the model for each tangent space $T_x M$, $x \in M$, of a reductive homogeneous pseudo-Riemannian manifold of signature $(k, n-k)$. Consider the vector space $S(E)$ of tensors of type $(0,3)$ on $(E, \langle \cdot, \cdot \rangle)$ satisfying the same symmetries as those of a homogeneous pseudo-Riemannian structure $S$, that is, $S(E) = \{ \sigma \in \otimes^3 E^* : S_{XYZ} = -S_{XYZ}, X, Y, Z \in E \}$, where $S_{XYZ} = \langle S_X Y, Z \rangle$. Let $c_{12} : S(E) \rightarrow V^*$ be the map defined by $c_{12}(\sigma)(Z) = \sum_{i=1}^{n} \varepsilon_i S_{c_i e_i} Z$, $Z \in E$, where $\{ e_i \}$ is an orthonormal basis of $E$, $\langle e_i, e_i \rangle = \varepsilon_i = \pm 1$. Then we have that if $\dim E \geq 3$, then $S(E)$ decomposes into the orthogonal direct sum of subspaces which are invariant and irreducible under the action of the pseudo-orthogonal group $O(k, n-k) : S(E) = S_1(E) \oplus S_2(E) \oplus S_3(E)$, where

- $S_1(E) = \{ \sigma \in S(E) : S_{XYZ} = \langle X, Y \rangle \omega(Z) - \langle X, Z \rangle \omega(Y), \omega \in E^* \}$,
- $S_2(E) = \{ \sigma \in S(E) : S_{XYZ} = 0, c_{12}(\sigma) = 0 \}$,
- $S_3(E) = \{ \sigma \in S(E) : S_{XYZ} + S_{YXZ} = 0 \}$,
- $S_1(E) \oplus S_2(E) = \{ \sigma \in S(E) : \langle X, Y \rangle \omega(Z) - \langle X, Z \rangle \omega(Y) \}$,
- $S_2(E) \oplus S_3(E) = \{ \sigma \in S(E) : \langle S_{XYZ} + S_{YXZ} = 2\langle X, Y \rangle \omega(Z) - \langle X, Z \rangle \omega(Y), \omega \in E^* \}$.

In the present case we deduce

**Corollary 1** The homogeneous Lorentzian structures on $(G, g_{p,q})$ belong to
\[ S_1 \oplus S_2 \oplus S_3 = \{ (S_1 \oplus S_2) \cup (S_1 \oplus S_3) \cup (S_2 \oplus S_3) \}. \]

In particular none of the associated reductive homogeneous spaces is either Lorentzian symmetric, or naturally reductive or cotorsionless.
Proof. Take the orthonormal basis
\[ \tilde{e}_1 = \frac{1}{\sqrt{2(1 + p)}} (e_1 + e_2), \quad \tilde{e}_2 = \frac{1}{\sqrt{2(1 - p)}} (e_1 - e_2), \quad \tilde{e}_3 = e_3, \quad \tilde{e}_4 = \frac{1}{\sqrt{q}} e_4. \]

As a calculation with respect to this basis shows, the condition \( c_{12}(S) = 0 \) is not satisfied. On the other hand, since for instance \( S_{e_1 e_2 e_4} + S_{e_2 e_4 e_1} + S_{e_3 e_1 e_2} \neq 0 \), no structure belong to \( S_1 \oplus S_2 \). Moreover, since for instance \( S_{e_1 e_2 e_4} \neq -S_{e_2 e_1 e_4} \), no structure belong to \( S_3 \); not even to \( S_2 \oplus S_3 \), as the sum \( S_{e_1 e_2 e_4} + S_{e_2 e_1 e_4} \) shows. The Lorentzian symmetric spaces correspond to the class \( \{0\} \), and in [4] it has been proved the equivalence of the third class with the naturally reductive spaces, and of the class \( S_1 \oplus S_2 \) with the cotorsionless spaces. For more details see [4].

3 Associated reductive decompositions

Consider now the Ambrose-Singer connection \( \tilde{\nabla} = \nabla - S \). Then, the non-null covariant derivatives between generators are
\[ \tilde{\nabla}_{e_1} e_2 = \frac{2p^2 + p - 2}{2q}, \quad \tilde{\nabla}_{e_1} e_4 = \frac{p}{2(1 - p^2)} e_1 - \frac{2p^2 + p - 2}{2(1 - p^2)} e_2, \]
and, as a calculation shows, the only nonvanishing curvature operator is
\[ \tilde{R}_{e_1 e_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{p}{2(1 - p^2)} \\ (2p^2 + p - 2) & 0 & 0 & -\frac{1}{2(1 - p^2)} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2q} & 0 & 0 \end{pmatrix}. \]

According to Ambrose-Singer’s Theorem on holonomy, the algebra of holonomy of a connection is generated by the curvature operators. In the present case, the holonomy algebra \( \mathfrak{h} \) has the only generator \( V = \tilde{R}_{e_1 e_4} \). Putting \( m \) for \( g \), and taking \( T = V + e_1 \) we have

**Theorem 2** The reductive pairs \((\tilde{g}, \tilde{h})\) associated to the reductive decompositions \( \tilde{g} = \tilde{h} \oplus m \) corresponding to the homogeneous Lorentzian structures on \((G, g_{p,q})\) given in Theorem 1, are given in terms of the basis \( \{e_1, e_2, e_3, e_4, T\} \) by the (nonvanishing) Lie brackets
\[ [T, e_4] = 2e_1 - T, \quad [e_1, e_2] = -\frac{2p^2 + p - 2}{2q} e_4, \]
\[ [e_1, e_4] = T - \frac{2p^3 - 3p^2 - 2p + 4}{2(1 - p^2)} e_1 + \frac{2p^2 + p - 2}{2(1 - p^2)} e_2. \]

**Proof.** On account of the expressions (1.2), we obtain that
\[ [V, e_2] = \frac{2p^2 + p - 2}{2q} e_4, \quad [V, e_4] = \frac{p(2p^2 + p - 2)}{2(1 - p^2)} e_1 - \frac{2p^2 + p - 2}{2(1 - p^2)} e_2, \]

\[ [e_1, e_2] = -\frac{2p^2 + p - 2}{2q} e_4, \quad [e_1, e_4] = V - \frac{2p^3 - p^2 - 2p + 2}{2(1 - p^2)} e_1 + \frac{2p^2 + p - 2}{2(1 - p^2)} e_2. \]

Then, making the change \( T = V + e_1 \) we conclude.

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