

# The Minimum Number of Zeros of Lipschitz-Killing Curvature

Dorin Andrica and Cornel Pinte

## Abstract

In the previous paper [11] some sufficient topological conditions, expressed in terms of homotopy groups, of two given differentiable manifolds  $M, N$  in order that the  $\varphi$ -category of the pair  $(M, N)$  be infinite, are given. In the more recent paper [12] it is proved that, in the same topological conditions, the  $\varphi$ -category is actually infinite uncountable. As an application we observe in this paper that a lower bound for the minimum number of points of zeros of the Lipschitz-Killing curvature of some  $m$ -dimensional manifold immersible in  $\mathbf{R}^{m+k}$ , with respect to all this kind of immersions, can be given in terms of  $\varphi$ -category.

**Mathematics Subject Classification:** 53Q99, 53Q05, 55P99, 53P65, 55Q05

**Key words:** Lipschitz-Killing curvature, homotopy group, critical set, immersibility category

## 1 Introduction

Recall that the  $\varphi$ -category of the pair  $(M, N)$  of closed manifolds is defined as

$$\varphi(M, N) = \min\{\#C(f) \mid f \in C^\infty(M, N)\},$$

where  $C(f)$  is the critical set of  $f$ . A survey containing general results concerning  $\varphi$ -category is contained in the paper [1] (see also [2] and [3]).

Let  $M^m$  be a differentiable manifold immersible in  $\mathbf{R}^{m+k}$ ; let  $f : M \rightarrow \mathbf{R}^{m+k}$  be an immersion and

$$v \in (df)_p(T_p(M))^\perp \cap S^{m+k-1},$$

where  $p \in M$  is a given point. Consider the second fundamental form  $\phi_{f,p}^v$  of  $f$  and the projections  $N_f : \mathcal{N}^{f,1} \rightarrow S^{m+k-1}$ ,  $\pi : \mathcal{N}^{f,1} \rightarrow M$ , where

$$\begin{aligned} \mathcal{N}^{f,1} &= \{(x, v) \in M \times S^{m+k-1} : v \perp (df)_p(T_p(M))\} = \\ &= \bigcup_{p \in M} \left( \{p\} \times (S^{m+k-1} \cap (df)_p(T_p(M))^\perp) \right). \end{aligned}$$

Observe that  $\pi$  is a fibration with fibre  $S^{k-1}$ . Recall that the *Lipschitz-Killing curvature* of  $f$  at the point  $p \in M$  in the direction  $v$  is defined as

$$L_{f,v} = \frac{\det(\phi_{f,p}^v)}{\det(g_f(p))}$$

and that  $\phi_{f,p_0}^{v_0}$  is degenerated if and only if  $(p_0, v_0)$  is a critical point of  $N_f$  (for more details we refer to the books [4] and [6]). Therefore

$$C(N_f) = \{(p, v) \in \mathcal{N}^{f,1} : L_{f,v}(p) = 0\},$$

that is the  $\varphi$ -category of the pair  $(\mathcal{N}^{f,1}, S^{m+k-1})$  is a lower bound for the minimum number of points of zeros of the Lipschitz-Killing curvature with respect to all immersions of  $M$  in  $\mathbf{R}^{m+k}$ . This fact can be shortly written by means of the inequality

$$L_k(M) \geq \varphi(\mathcal{N}^{f,1}, S^{m+k-1}),$$

where

$$L_k(M) = \min\{\#C(N_f) : f \in \text{Imm}^k(M)\}$$

is the so called  $L_k$ -category or the *immersibility category* of  $M$ , and  $\text{Imm}^k(M)$  is the set of all immersions of  $M$  into  $\mathbf{R}^{m+k}$ . The immersibility category  $L_k(M)$  is studied in the case of surfaces and  $k = 1$  in the recent second author's paper [12]. To prove that the  $L_k(M)$  is infinite uncountable in certain situations, we will use the following theorem.

**Theorem 1.1** ([12]) *Let  $M, N$  be compact connected differentiable manifolds of the same dimension  $m$ .*

(i) *If  $m \geq 3$  and  $\pi_1(M)$  cannot be embedded as a subgroup in  $\pi_1(N)$ , then  $\varphi(M, N) = \aleph_1$ ;*

(ii) *If  $m \geq 4$  and  $\pi_q(M) \not\cong \pi_q(N)$  for some  $q \in \{2, \dots, m-2\}$ , then  $\varphi(M, N) = \aleph_1$ .*

Let us mention that Theorem 1.1 is improving ([11], Theorem 3.1) which states that, under the same hypothesis, the  $\varphi$ -category of the pair  $(M, N)$  is just  $\geq \aleph_0$ .

## 2 The implications of Theorem 1.1 on the $L_k$ -category

Because the conditions of Theorem 1.1 are expressed in terms of homotopy groups of the given manifolds and our lower bound for the minimum number of points of zeros of the Lipschitz-Killing curvature is  $\varphi(\mathcal{N}^{f,1}, S^{m+k-1})$ , we will try to relate in what follows the homotopy groups of  $\mathcal{N}^{f,1}$  with those of  $M$ . We denote by  $k(M)$  the smallest natural number such that  $M$  is immersible in  $\mathbf{R}^{m+k(M)}$ . By means of Whitney's theorem, observe that  $k(M) \leq 2m + 1$  and if  $M^m$  is a compact manifold, then  $k(M) \leq 2m - 1$ . Using [9], Theorem 1 and the proof of [9], Theorem 2 one can see some lower bounds for  $k(G_{2,n})$  and  $k(G_{3,n})$ . For some lower bounds of  $k(P^n(\mathbf{R}))$  and of  $k(P^n(\mathbf{C}))$ ,  $k(P^n(\mathbf{H}))$  see [8] and [7] respectively.

**Theorem 2.1** *Let  $M^m$  be a connected closed manifold.*

(i) *If  $m \geq 1$  and  $\pi_1(M)$  is not trivial, then  $L_k(M) = \aleph_1$ ,  $\forall k \geq \max\{3, k(M)\}$ ;*

(ii) *If  $m \geq 2$  and  $\pi_q(M)$  is not trivial for some  $q \in \{2, \dots, k-1\}$ , then  $L_k(M) \geq \aleph_1$ ,  $\forall k \geq \max\{3, k(M)\}$ .*

**Proof.** The homotopy sequence of the fibration  $S^{k-1} \hookrightarrow \mathcal{N}^{f,1} \xrightarrow{\pi} M$  is

$$\dots \rightarrow \pi_r(S^{k-1}) \rightarrow \pi_r(\mathcal{N}^{f,1}) \rightarrow \pi_r(M) \rightarrow \pi_{r-1}(S^{k-1}) \rightarrow \dots$$

Because the homotopy groups  $\pi_q(S^{k-1}), \pi_{q-1}(S^{k-1})$  of the sphere are trivial for  $q \in \{1, \dots, k-2\}$  and  $\pi_{k-1}(S^{k-1}) \simeq \mathbf{Z}$  it follows that  $\pi_q(\mathcal{N}^{f,1}) \simeq \pi_q(M)$  for all  $q \in \{1, \dots, k-2\}$  and the morphism  $\pi_{k-1}(\mathcal{N}^{f,1}) \rightarrow \pi_{k-1}(M)$ , induced by the projection  $\pi : \mathcal{N}^{f,1} \rightarrow M$  is surjective. Therefore, to prove (i) apply Theorem 1.1 (i), taking into account that  $\pi_1(M) \not\simeq 0 \simeq \pi_1(S^{m+k-1})$ . Analogously, to prove (ii) apply theorem 1.1 (ii), taking into account that for some  $q \in \{2, \dots, k-1\}$  we have  $\pi_q(M) \not\simeq 0 \simeq \pi_q(S^{m+k-1})$ .

**Theorem 2.2** *Let  $M^m$  be a connected closed manifold immersible in  $\mathbf{R}^{m+2}$ .*

(i) *If  $m \geq 2$  and  $\pi_1(M)$  is not trivial, then  $L_2(M) = \aleph_1$ ;*

(ii) *If  $m \geq 3$  and  $\pi_2(M) \not\simeq \mathbf{Z}$ , then  $L_2(M) = \aleph_1$ ;*

(iii) *If  $m \geq 4$  and  $\pi_q(M)$  is not trivial for some  $q \in \{3, \dots, m-1\}$ , then  $L_2(M) = \aleph_1$ .*

**Proof.** The homotopy sequence of the fibration  $S^1 \hookrightarrow \mathcal{N}^{f,1} \xrightarrow{\pi} M$  is

$$\dots \rightarrow \pi_r(S^1) \rightarrow \pi_r(\mathcal{N}^{f,1}) \rightarrow \pi_r(M) \rightarrow \pi_{r-1}(S^1) \rightarrow \dots$$

Because the homotopy groups  $\pi_q(S^1)$  of the circle are trivial for  $q = 0$  and  $q \geq 2$  and  $\pi_1(S^1) \simeq \mathbf{Z}$ , it follows that the morphism  $\pi_1(\mathcal{N}^{f,1}) \rightarrow \pi_1(M)$  induced by the projection  $\pi : \mathcal{N}^{f,1} \rightarrow M$  is surjective, meaning that  $\pi_1(M) \not\simeq 0$  implies  $\pi_1(\mathcal{N}^{f,1}) \not\simeq 0 \simeq \pi_1(S^{m+1})$ . Therefore using Theorem 1.1 (i), it follows that  $L_2(M) = \aleph_1$  and the statement (i) is proved. To prove the second statement, observe that the morphism

$$\pi_2(\mathcal{N}^{f,1}) \rightarrow \pi_2(M)$$

induced by the projection  $\pi : \mathcal{N}^{f,1} \rightarrow M$  is injective and

$$\pi_2(M)/\pi_2(\mathcal{N}^{f,1}) \simeq \mathbf{Z}.$$

Combining this observation with the hypothesis of (ii) it follows that

$$\pi_2(\mathcal{N}^{f,1}) \neq 0 \simeq \pi_2(S^{m+1}).$$

Therefore using Theorem 1.1 (ii) conclude that  $L_2(M) = \aleph_1$ .

The statement (iii) follows immediately from Theorem 1.1 (ii) taking into account that for  $q \in \{3, \dots, m-1\}$  we have  $\pi_q(\mathcal{N}^{f,1}) \simeq \pi_q(M) \not\simeq 0 \simeq \pi_q(S^{m+1})$ .

### 3 Some manifolds with infinite uncountable $L_k$ -category

In this section we are going to use the results of the previous section to obtain some manifolds with infinite  $L_k$ -category.

**Proposition 3.1** (i) For  $m \geq 1$  we have that

$$L_k(P^m(\mathbf{R})) = \aleph_1, \forall k \geq \max\{3, k(P^m(\mathbf{R}))\};$$

(ii) For  $m \geq 1$  we have that

$$L_k(L^{2m-1}(q_1, \dots, q_p)) = \aleph_1, \forall k \geq \max\{3, k(L^{2m-1}(q_1, \dots, q_p))\};$$

(iii) For  $m \geq 2$  we have that  $L_k(SO_m) \geq \aleph_1, \forall k \geq \max\{3, k(SO_m)\}$ .

**Proof.** Indeed, for  $m \geq 3$  we have

$$\pi_1(P^2(\mathbf{R})) \simeq \pi_1(P^m(\mathbf{R})) \simeq \pi_1(SO_m) \simeq \mathbf{Z}_2 \neq 0,$$

while  $P^1(\mathbf{R}) \sim_{Top} SO_2 \sim_{Top} S^1$  so that

$$\pi_1(P^1(\mathbf{R})) \simeq \pi_1(SO_2) \simeq \mathbf{Z} \neq 0,$$

and

$$\pi_1(L^{2m-1}(q_1, \dots, q_p)) \simeq \mathbf{Z}_m \neq 0, \forall m \geq 1.$$

Therefore using Theorem 2.1 (i), proposition 3.1 is completely proved.

**Proposition 3.2** (i)  $L_k(G_{p,l}) = \aleph_1, \forall k \geq \max\{3, k(G_{p,l})\}$ ;

(ii) If  $m \geq 3$ , then  $L_k(Spin_m) = \aleph_1, \forall k \geq \max\{4, k(Spin_m)\}$ .

**Proof.** (i) Let us assume that  $p \geq l$ . In this case it follows, using [13], Theorem 10.16, pp. 204, that

$$\pi_q(G_{p,l}) \simeq \pi_q(V_{p+l,l}) \oplus \pi_{q-1}(O_l), \forall q.$$

On the other hand the inclusion  $SO_l \hookrightarrow O_l$  induces isomorphism of homotopy groups  $\pi_i(SO_l) \rightarrow \pi_i(O_l)$ , so that for  $l = 2$  we have  $\pi_1(O_2) \simeq \pi_1(SO_2) \simeq \mathbf{Z}$  and for  $l \geq 3$  we have  $\pi_1(O_l) \simeq \pi_1(SO_l) \simeq \mathbf{Z}_2$ . In any case  $\pi_2(G_{p,l})$  has as a subgroup either  $\mathbf{Z}$  or  $\mathbf{Z}_2$ , that is,  $\pi_2(G_{p,l}) \neq 0$ , meaning by Theorem 2.1 (ii), that  $L_k(G_{p,l}) = \aleph_1, \forall k \geq \max\{3, k(G_{p,l})\}$ .

(ii) Indeed  $\pi_q(Spin_m) \simeq \pi_q(SO_m), \forall k \geq 2$ ,  $Spin_m$  being the universal cover of  $SO_m$ . Hence

$$\pi_3(Spin_m) \simeq \pi_3(SO_m) \simeq \begin{cases} \mathbf{Z} & \text{if } m = 3 \text{ and } m \geq 5 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } m = 4. \end{cases}$$

For the above mentioned homotopy groups, see for instance [5], pp. 224. Therefore in the given conditions,  $\pi_3(Spin_m) \neq 0$ , meaning by Theorem 2.1 (ii), that  $L_k(Spin_m) = \aleph_1, \forall k \geq \max\{4, k(Spin_m)\}$ .

**Remark 3.3** For  $k \geq \max\{3, p + 1, k(G_{p,l})\}$  the first point of Proposition 3.2 can be also proved by means of [13], Theorem 10.13, pp. 203, from which it follows that  $\pi_p(V_{p+l,l})$  is either infinite cyclic or cyclic of order two.

**Acknowledgements.** This paper is supported by the European Research and Training Network "GEOMETRIC ANALYSIS". It was written in a period when the second author had a research appointment position at "Paul Sabatier" University, Toulouse, France (November 1, 2000-May 1, 2001)

## References

- [1] Andrica D., *Functions with minimal critical set: new results and open problems*, Mathematical Analysis and Applications, Hadronic Press, Palm Harbor, FL 34682-1577, 2000, pp 1-10.
- [2] Andrica D., Pinte C., *Critical points of vector-valued functions*, Proc. 24<sup>th</sup> Conf. Geom. Top., Univ. of Timișoara, **Part 2**(1996), 15-24.
- [3] Andrica D., Pinte C., *Elements of Homotopy. Theory with Applications to the Study of Critical Points*, (Romanian), Ed. Mirton, Timișoara, 2002.
- [4] Burghilea D. et al, *Introduction in Differential Topology (Romanian)*, Ed. Științifică, Bucharest, 1973.
- [5] Doubróvine B., Novikov S., Fomenko A., *Géométrie Contemporaine. Méthodes et Applications*, 2<sup>e</sup> partie. Géométrie e Topologie des Variétés, Editions MIR Moscou, 1982.
- [6] Jost J., *Riemannian Geometry and Geometric Analysis*, Second Edition, Springer, 1998.
- [7] Lawson H.B. Jr., Michelshon M.L., *Clifford bundles of immersions of manifolds and the vector field problem*, J. Diff. Geom. 15 (1980), 237-267.
- [8] Milnor J., Stasheff J.D., *Characteristic Classes*, Ann. of Math. Studies 76, Princeton University Press, Princeton, 1974.
- [9] Oproiu V., *Some non-embedding theorems for the Grassmann manifolds  $G_{2,n}$  and  $G_{3,n}$* , Proc. Edinburgh Math. Soc. (1976-77), **20**, pp. 177-185.
- [10] Pei D., *Singularities of  $\mathbf{R}P^2$ -valued Gauss maps of surfaces in the Minkowski 3-space*, Hokkaido Mathematical Journal, vol XXXVIII, No. 1 February, (1999), pp. 97-115.
- [11] Pinte C., *Differentiable mappings with an infinite number of critical points*, Proceedings of the A.M.S., vol. 128, Nr. 11, (2000), pp. 3435-3444.

- [12] Pinte C., *Some pairs of manifolds with infinite uncountable  $\varphi$ -category*, to appear in *Topological Methods in Nonlinear Analysis*.
- [13] Whitehead G.W., *Elements of Homotopy Theory*, Springer-Verlag, 1978.

"Babeş-Bolyai" University, Faculty of Mathematics,  
Str. M. Kogălniceanu 1, 3400 Cluj-Napoca, Romania  
E-mail addresses: dandrica@math.ubbcluj.ro  
cpinte@math.ubbcluj.ro