Hypersurfaces with Constant Scalar Curvature in a Hyperbolic Space Form

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Abstract

Let \( M^n \) be a complete hypersurface with constant normalized scalar curvature \( \bar{R} \) in a hyperbolic space form \( H^{n+1} \). We prove that if \( \bar{R} = R + 1 \geq 0 \) and the norm square \( |h|^2 \) of the second fundamental form of \( M^n \) satisfies
\[
n\bar{R} \leq \sup |h|^2 \leq \frac{n}{(n-2)(nR-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n],
\]
then either \( \sup |h|^2 = n\bar{R} \) and \( M^n \) is a totally umbilical hypersurface; or
\[
\sup |h|^2 = \frac{n}{(n-2)(nR-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n],
\]
and \( M^n \) is isometric to \( S^{n-1}(r) \times H^1(-1/(r^2 + 1)) \), for some \( r > 0 \).

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1 Introduction

Let \( R^{n+1}(c) \) be an \((n+1)\)-dimensional Riemannian manifold with constant sectional curvature \( c \). We also call it a space form. When \( c > 0 \), \( R^{n+1}(c) = S^{n+1}(c) \) (i.e. \((n+1)\)-dimensional sphere space); when \( c = 0 \), \( R^{n+1}(c) = R^n \) (i.e. \((n+1)\)-dimensional Euclidean space); when \( c < 0 \), \( R^{n+1}(c) = H^{n+1}(c) \) (i.e. \((n+1)\)-dimensional hyperbolic space). We simply denote \( H^{n+1}(-1) \) by \( H^{n+1} \). Let \( M^n \) be an \( n \)-dimensional hypersurface in \( R^{n+1}(c) \), and \( e_1, \ldots, e_n \) a local orthonormal frame field on \( M^n \), \( \omega_1, \ldots, \omega_n \) its dual coframe field. Then the second fundamental form of \( M^n \) is
\[
h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j.
\]

Further, near any given point \( p \in M^n \), we can choose a local frame field \( e_1, \ldots, e_n \) so that at \( p \), \( \sum_{i,j} h_{ij} \omega_i \otimes \omega_j = \sum_i k_i \omega_i \otimes \omega_j \), then the Gauss equation writes
\[
R_{ijij} = c + k_ik_j, \quad i \neq j.
\]


\[ n(n-1)(R-c) = n^2H^2 - |h|^2, \]

where \( R \) is the normalized scalar curvature, \( H = \frac{1}{n} \sum_{i} k_i \) the mean curvature and \( |h|^2 = \sum_{i} k_i^2 \) the norm square of the second fundamental form of \( M^n \).

As it is well known, there are many rigidity results for minimal hypersurfaces or hypersurfaces with constant mean curvature \( H \) in \( \mathbb{R}^{n+1} \) \((c \geq 0)\) by use of J. Simons’ method, for example, see [1], [4], [5], [8], [12] etc., but less were obtained for hypersurfaces immersed into a hyperbolic space form. Walter [13] gave a classification for non-negatively curved compact hypersurfaces in a space form under the assumption that the \( r \)th mean curvature is constant. Morvan-Wu [7], Wu [14] also proved some rigidity theorems for complete hypersurfaces \( M^n \) in a hyperbolic space form \( H^{n+1}(c) \) under the assumption that the mean curvature is constant and the Ricci curvature is non-negative. Moreover, they proved that \( M^n \) is a geodesic distance sphere in \( H^{n+1}(c) \) provided that it is compact.

On the other hand, Cheng-Yau [3] introduced a new self-adjoint differential operator \( \Box \) to study the hypersurfaces with constant scalar curvature. Later, Li [6] obtained interesting rigidity results for compact hypersurfaces with constant scalar curvature in space-forms using the Cheng-Yau’s self-adjoint operator \( \Box \).

In the present paper, we use Cheng-Yau’s self-adjoint operator \( \Box \) to study the complete hypersurfaces in a hyperbolic space form with constant scalar curvature, and prove the following theorem:

**Theorem.** Let \( M^n \) be an \( n \)-dimensional \((n \geq 3)\) complete hypersurface with constant normalized scalar curvature \( R \) in \( H^{n+1} \). If

1. \( \bar{R} = R + 1 \geq 0 \),
2. the norm square \(|h|^2\) of the second fundamental form of \( M^n \) satisfies

\[
 n\bar{R} \leq \sup |h|^2 \leq \frac{n}{(n-2)(nR-2)} [n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n],
\]

then either

\[ \sup |h|^2 = n\bar{R} \]

and \( M^n \) is a totally umbilical hypersurface; or

\[ \sup |h|^2 = \frac{n}{(n-2)(nR-2)} [n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n], \]

and \( M^n \) is isometric to \( S^{n-1}(r) \times H^1(-1/r^2 + 1) \), for some \( r > 0 \).

## 2 Preliminaries

Let \( M^n \) be an \( n \)-dimensional hypersurface in \( H^{n+1} \). We choose a local orthonormal frame \( e_1, \ldots, e_{n+1} \) in \( H^{n+1} \) such that at each point of \( M^n \), \( e_1, \ldots, e_n \) span the tangent space of \( M^n \) and form an orthonormal frame there. Let \( \omega_1, \ldots, \omega_{n+1} \) be its dual coframe. In this paper, we use the following convention on the range of indices:
Then the structure equations of $H^{n+1}$ are given by

\begin{equation}
\mathbf{d}\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,
\end{equation}

\begin{equation}
\mathbf{d}\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D,
\end{equation}

\begin{equation}
K_{ABCD} = -(\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}).
\end{equation}

Restricting these forms to $M^n$, we have

\begin{equation}
\omega_{n+1} = 0.
\end{equation}

From Cartan’s lemma we can write

\begin{equation}
\omega_{n+1} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.
\end{equation}

From these formulas, we obtain the structure equations of $M^n$:

\begin{equation}
\mathbf{d}\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,
\end{equation}

\begin{equation}
\mathbf{d}\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,
\end{equation}

\begin{equation}
R_{ijkl} = -(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (h_{ik} h_{jl} - h_{il} h_{jk}),
\end{equation}

where $R_{ijkl}$ are the components of the curvature tensor of $M^n$ and

\begin{equation}
h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j
\end{equation}

is the second fundamental form of $M^n$. We also have

\begin{equation}
R_{ij} = -(n-1)\delta_{ij} + n H h_{ij} - \sum_k h_{ik} h_{kj},
\end{equation}

\begin{equation}
n(n-1)(R + 1) = n^2 H^2 - |h|^2,
\end{equation}

where $R$ is the normalized scalar curvature, and $H$ the mean curvature.

Define the first and the second covariant derivatives of $h_{ij}$, say $h_{ijk}$ and $h_{ijkl}$ by

\begin{equation}
\sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{kj} \omega_{ki} + \sum_k h_{ik} \omega_{kj},
\end{equation}

\begin{equation}
\sum_t h_{ijkt} \omega_t = dh_{ijt} + \sum_m h_{mjk} \omega_{mi} + \sum_m h_{imk} \omega_{mj} + \sum_m h_{ijm} \omega_{mk}.
\end{equation}

Then we have the Codazzi equation

\begin{equation}
\sum_{k,l} h_{ijkl} \omega_l = dh_{ijk} + \sum_m h_{mj} \omega_{mki} + \sum_m h_{im} \omega_{mjk} + \sum_m h_{ijm} \omega_{mk}.
\end{equation}
\[ h_{ijk} = h_{ikj}, \quad (17) \]

and the Ricci identity

\[ h_{ijkl} - h_{ijlk} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}. \quad (18) \]

For a \( C^2 \)-function \( f \) defined on \( M^n \), we defined its gradient and Hessian \( (f_{ij}) \) by the following formulas

\[ df = \sum_i f_i \omega_i, \quad \sum_i f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji}. \quad (19) \]

The Laplacian of \( f \) is defined by \( \Delta f = \sum_i f_{ii} \).

Let \( \phi = \sum_{ij} \phi_{ij} \omega_i \otimes \omega_j \) be a symmetric tensor defined on \( M^n \), where

\[ \phi_{ij} = nH \delta_{ij} - h_{ij}. \quad (20) \]

Following Cheng-Yau [3], we introduce an operator \( \Box \) associated to \( \phi \) acting on any \( C^2 \)-function \( f \) by

\[ \Box f = \sum_{ij} \phi_{ij} f_{ij} = \sum_{ij} (nH \delta_{ij} - h_{ij}) f_{ij}. \quad (21) \]

Since \( \phi_{ij} \) is divergence-free, it follows [3] that the operator \( \Box \) is self-adjoint relative to the \( L^2 \) inner product of \( M^n \), i.e.

\[ \int_{M^n} f \Box g = \int_{M^n} g \Box f. \quad (22) \]

We can choose a local frame field \( e_1, \ldots, e_n \) at any point \( p \in M^n \), such that \( h_{ij} = k_i \delta_{ij} \) at \( p \), by use of (21) and (14), we have

\[ \Box (nH) = nH \Delta (nH) - \sum_i k_i (nH)_{ii} = \]

\[ = \frac{1}{2} \Delta (nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i (nH)_{ii} = \]

\[ = \frac{1}{2} n(n - 1) \Delta R + \frac{1}{2} \Delta |h|^2 - n^2 |\nabla H|^2 - \sum_i k_i (nH)_{ii}. \quad (23) \]

On the other hand, through a standard calculation by use of (17) and (18), we get

\[ \frac{1}{2} \Delta |h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i k_i (nH)_{ii} + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2. \quad (24) \]

Putting (24) into (23), we have

\[ \Box (nH) = \frac{1}{2} n(n - 1) \Delta R + |\nabla h|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2. \quad (25) \]
From (11), we have \( R_{ijij} = -1 + k_i k_j, \ i \neq j, \) and by putting this into (25), we obtain

\[
\Box (nH) = \frac{1}{2} n(n-1) \Delta R + |\nabla h|^2 - n^2 |\nabla H|^2 - n|h|^2 + n^2 H^2 - |h|^4 + nH \sum_i k_i^3.
\]

(26)

Let \( \mu_i = k_i - H \) and \( |Z|^2 = \sum_i \mu_i^2, \) we have

\[
\sum_i \mu_i = 0, \quad |Z|^2 = |h|^2 - nH^2,
\]

(27)

\[
\sum_i k_i^3 = \sum_i \mu_i^3 + 3H |Z|^2 + nH^3.
\]

(28)

From (26)-(28), we get

\[
\Box (nH) = \frac{1}{2} n(n-1) \Delta R + |\nabla h|^2 - n^2 |\nabla H|^2 + |Z|^2 (-n + nH^2 - |Z|^2) + nH \sum_i \mu_i^3.
\]

(29)

We need the following algebraic lemma due to M. Okumura [9] (see also [1]).

**Lemma 2.1.** Let \( \mu_i, \ i = 1, \ldots, n, \) be real numbers such that \( \sum_i \mu_i = 0 \) and \( \sum_i \mu_i^2 = \beta^2, \) where \( \beta = \text{constant} \geq 0. \) Then

\[
- \frac{n-2}{\sqrt{n(n-1)}} \beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} \beta^3,
\]

(30)

and the equality holds in (30) if and only if at least \( (n-1) \) of the \( \mu_i \) are equal.

By use of Lemma 2.1, we have

\[
\Box (nH) \geq \frac{1}{2} n(n-1) \Delta R + |\nabla h|^2 - n^2 |\nabla H|^2 + (|h|^2 - nH^2) \left( -n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{|h|^2 - nH^2} \right).
\]

(31)

### 3 Umbilical hypersurface in a hyperbolic space form

In this section, we consider some special hypersurfaces in a hyperbolic space form which we will need in the following discussion.

First we want to give a description of the real hyperbolic space-form \( H^{n+1}(c) \) of constant curvature \( c \ (< 0) \). For any two vectors \( x \) and \( y \) in \( R^{n+2} \), we set

\[
g(x, y) = \sum_{i=1}^{n+1} x_i y_i - x_{n+2} y_{n+2}.
\]

\( (R^{n+2}, g) \) is the so-called Minkowski space-time. Denote \( \rho = \sqrt{-1/c} \). We define
Then $H^{n+1}(c)$ is a connected simply-connected hypersurface of $R^{n+2}$. It is not hard to check that the restriction of $g$ to the tangent space of $H^{n+1}(c)$ yields a complete Riemannian metric of constant curvature $c$. Here we obtain a model of a real hyperbolic space form.

We are interested in those complete hypersurfaces with at most two distinct constant principal curvatures in $H^{n+1}(c)$. This kind of hypersurfaces was described by Lawson [5] and completely classified by Ryan [11].

**Lemma 3.1** [11]. Let $M^n$ be a complete hypersurface in $H^{n+1}(c)$. Suppose that, under a suitable choice of a local orthonormal tangent frame field of $TM^n$, the shape operator over $TM^n$ is expressed as a matrix $A$. If $M^n$ has at most two distinct constant principal curvatures in $H^n(c)$, then it is congruent to one of the following:

1. $M_1 = \{ x \in H^{n+1}(c) \mid x_1 = 0 \}$. In this case, $A = 0$, and $M_1$ is totally geodesic. Hence $M_1$ is isometric to $H^n(c)$;

2. $M_2 = \{ x \in H^{n+1}(c) \mid x_1 = r > 0 \}$. In this case, $A = \frac{1}{\rho^2} I_n$, where $I_n$ denotes the identity matrix of degree $n$, and $M_2$ is isometric to $H^n(-1/(r^2 + \rho^2))$;

3. $M_3 = \{ x \in H^{n+1}(c) \mid x_{n+2} = x_{n+1} + \rho \}$. In this case, $A = \frac{1}{\rho} I_n$, and $M_3$ is isometric to a Euclidean space $E^n$;

4. $M_4 = \{ x \in H^{n+1}(c) \mid \sum_{i=1}^{n+1} x_i^2 = r^2 > 0 \}$. In this case, $A = \sqrt{1/r^2 + 1/\rho^2} I_n$, and $M_4$ is isometric to a round sphere $S^n(r)$ of radius $r$;

5. $M_5 = \{ x \in H^{n+1}(c) \mid \sum_{i=1}^{k+1} x_i^2 = r^2 > 0, \sum_{j=k+2}^{n+1} x_j^2 - x_{n+2}^2 = -\rho^2 - r^2 \}$. In this case, $A = \lambda I_k \oplus \nu I_{n-k}$, where $\lambda = \sqrt{1/\rho^2 + 1/r^2}$, and $\nu = \frac{1}{\sqrt{1/r^2 + 1/\rho^2}}$, $M_5$ is isometric to $S^k(r) \times H^{n-k}(-1/(r^2 + \rho^2))$.

**Remark 3.1.** $M_1, \ldots, M_5$ are often called the standard examples of complete hypersurfaces in $H^{n+1}(c)$ with at most two distinct constant principal curvatures. It is obvious that $M_1, \ldots, M_4$ are totally umbilical. In the sense of Chen [2], they are called the hyperspheres of $H^{n+1}(c)$. $M_5$ is called the horosphere and $M_4$ the geodesic distance sphere of $H^{n+1}(c)$.

**Remark 3.2.** Ryan [11] stated that the shape operator of $M_2$ is $A = \sqrt{1/r^2 - 1/\rho^2} I_n$, and $M_2$ is isometric to $H^n(-1/r^2)$, where $r \leq \rho$. This is incorrect and we have corrected it here.

### 4 The proof of Theorem

The following lemma essentially due to Cheng-Yau [3].
Lemma 4.1. Let $M^n$ be an $n$-dimensional hypersurface in $H^{n+1}$. Suppose that the normalized scalar curvature $R = \text{constant}$ and $R \geq -1$. Then $|\nabla h|^2 \geq n^2|\nabla H|^2$.

**Proof.** From (14),
\[ n^2H^2 - \sum_{i,j} h^2_{ij} = n(n - 1)(R + 1). \]

Taking the covariant derivative of the above expression, and using the fact $R = \text{constant}$, we get
\[ n^2HH_k = \sum_{i,j} h_{ij}h_{ijk}. \]

By Cauchy-Schwarz inequality, we have
\[
\sum_k n^4H^2(H_k)^2 = \sum_k (\sum_{i,j} h_{ij}h_{ijk})^2 \leq (\sum_{i,j} h^2_{ij}) \sum_{i,j,k} h^2_{ijk},
\]
that is
\[ n^4H^2|\nabla H|^2 \leq |h|^2|\nabla h|^2. \]

On the other hand, from $R + 1 \geq 0$, we have $n^2H^2 - |h|^2 \geq 0$. Thus
\[ H^2|\nabla h|^2 \geq n^2H^2|\nabla H|^2 \]
and Lemma 4.1 follows.

From the assumption of the Theorem that $R$ is constant and $R = \bar{R} + 1 \geq 0$ and Lemma 4.1 we have
\[
\square(nH) \geq (|h|^2 - nH^2) \left(-n + 2nH^2 - |h|^2 - \frac{n(n-2)}{n(n-1)}H \sqrt{|h|^2 - nH^2}\right).
\]

By Gauss equation (14) we know that
\[
|Z|^2 = |h|^2 - nH^2 = \frac{n-1}{n}(|h|^2 - n\bar{R}).
\]

From (32) and (33) we have
\[
\square(nH) \geq \frac{n-1}{n}(|h|^2 - n\bar{R})\phi_H(|h|),
\]
where
\[
\phi_H(|h|) = -n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H \sqrt{|h|^2 - nH^2}.
\]

By (33) we can write $\phi_H(|h|)$ as
\[
\phi_{\bar{R}}(|h|) = -n + 2(n-1)\bar{R} - \frac{n-2}{n} |h|^2 - \frac{n-2}{n} \sqrt{(n(n-1)\bar{R} + |h|^2)(|h|^2 - n\bar{R})}.
\]

Therefore (34) becomes
\[
\square(nH) \geq \frac{n-1}{n}(|h|^2 - n\bar{R})\phi_{\bar{R}}(|h|),
\]
It is a direct check that our assumption
\[ \sup |h|^2 \leq \frac{n}{(n - 2)(nR - 2)} [n(n - 1)\bar{R}^2 - 4(n - 1)\bar{R} + n] \]
is equivalent to
\[ (-n + 2(n - 1)\bar{R} - \frac{n - 2}{n} \sup |h|^2)^2 \geq \frac{(n - 2)^2}{n^2} (n(n - 1)\bar{R} + \sup |h|^2)(\sup |h|^2 - n\bar{R}). \]
But it is clear from (37) that (38) is equivalent to
\[ -n + 2(n - 1)\bar{R} - \frac{n - 2}{n} \sup |h|^2 \geq \frac{n - 2}{n} \sqrt{(n(n - 1)\bar{R} + \sup |h|^2)(\sup |h|^2 - n\bar{R})}. \]
So under the hypothesis that
\[ \sup |h|^2 \leq \frac{n}{(n - 2)(nR - 2)} [n(n - 1)\bar{R}^2 - 4(n - 1)\bar{R} + n], \]
we have
\[ \phi_{\bar{R}}(\sqrt{\sup |h|^2}) \geq 0. \]

On the other hand,
\[ \Box(nH) = \sum_{i,j} (nH \delta_{ij} - nh_{ij})(nH)_{ij} = \sum_i (nH - nh_{ii})(nH)_{ii} = \]
\[ = n \sum_i H(nH)_{ii} - n \sum_i k_i(nH)_{ii} \leq (|H|_{max} - C)\Delta(nH), \]
where $|H|_{max}$ is the maximum of the mean curvature $H$ and $C = \min k_i$ is the minimum of the principal curvatures of $M^n$.

Now we need the following maximum principle at infinity for complete manifolds due to Omori [10] and Yau [15]:

**Lemma 4.2.** Let $M^n$ be an $n$-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $f : M^n \to \mathbb{R}$ a smooth function bounded from below. Then for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M^n$ such that

(i) $|\nabla f|(p_\varepsilon) < \varepsilon$,

(ii) $\Delta f(p_\varepsilon) > -\varepsilon$,

(iii) $\inf f \leq f(p_\varepsilon) \leq \inf f + \varepsilon$.

From the hypothesis of the Theorem and Gauss equation, we know that the Ricci curvature is bounded below. So we may apply Lemma 4.2 to the following smooth function $f$ on $M^n$ defined by

\[ f = \frac{1}{\sqrt{1 + (nH)^2}}. \]

It is immediate to check that
Let \( n \) have either (48) or (47) then equalities hold in (30) and Lemma 4.1, we follow that hypersurfaces with constant scalar curvature.

By Lemma 4.2 we can find a sequence of points \( p_k, k \in N \) in \( M^n \), such that

\[
(44) \quad \lim_{k \to \infty} f(p_k) = \inf f, \quad \Delta f(p_k) > -\frac{1}{k}, \quad |\nabla f|^2(p_k) < \frac{1}{k^2}.
\]

Using (44) in the equations (42) and (43) and the fact that

\[
(45) \quad \sup_{p \in M_n}(nH)(p) = \sup_{p \in M_n}(nH)(p),
\]

we get

\[
(46) \quad -\frac{1}{k} \leq -\frac{1}{2} \frac{\Delta(nH)^2}{(1 + (nH)^2)^{3/2}}(p_k) + \frac{3}{k^2} (1 + (nH)^2(p_k))^{1/2}.
\]

Hence we obtain

\[
(47) \quad \frac{\Delta(nH)^2}{(1 + (nH)^2)^2}(p_k) < \frac{2}{k} \left( \frac{1}{\sqrt{1 + (nH)^2(p_k)}} + \frac{3}{k} \right).
\]

On the other hand, by (36) and (41), we have

\[
(48) \quad \frac{n-1}{n} (|h|^2 - n\bar{R}) \bar{\phi}_R(|h|) \leq \Delta(nH) \leq n(|H|_{\text{max}} - C) \Delta(nH).
\]

At points \( p_k \) of the sequence given in (44), this becomes

\[
(49) \quad \frac{n-1}{n} (|h|^2(p_k) - n\bar{R}) \bar{\phi}_R(|h|(p_k)) \leq \Delta(nH(p_k)) \leq n(|H|_{\text{max}} - C) \Delta(nH)(p_k).
\]

Let \( k \to \infty \) and use (47) we have that the right hand side of (49) goes to zero, so we have either \( \frac{n-1}{n} (\sup |h|^2 - n\bar{R}) = 0 \), i.e. \( \sup |h|^2 = n\bar{R} \) or \( \bar{\phi}_R(\sqrt{\sup |h|^2}) = 0 \).

If \( \sup |h|^2 = n\bar{R} \), by (33) \( |Z|^2 = \frac{n-1}{n} (|h|^2-n\bar{R}) \) we have \( \sup |Z|^2 = \frac{n-1}{n} (\sup |h|^2-n\bar{R}) = 0 \), then \( |Z|^2 = 0 \) and \( M^n \) is totally umbilical.

If \( \bar{\phi}_R(\sqrt{\sup |h|^2}) = 0 \), it is easy to prove that

\[
\sup H^2 = \frac{1}{n^2} \left[ (n-1)^2 \frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2} \right],
\]

then equalities hold in (30) and Lemma 4.1, we follow that \( k_i = \text{constant} \) for all \( i \) and \( n-1 \) of the \( k_i \) are equal. After renumberation if necessary, we can assume

\[
k_1 = k_2 = \cdots = k_{n-1}, \quad k_1 \neq k_n.
\]
Therefore, from Lemma 3.1, we know that $M^n$ is a hypersurface in $H^{n+1}$ with two distinct principal curvatures, and $M^n$ is isometric to $S^{n-1}(r) \times H^1(-1/(r^2 + 1))$, for some $r > 0$. This completes the proof of Theorem.

When $M^n$ is compact, we can prove

**Corollary 1.** Let $M^n$ be an $n$-dimensional ($n \geq 3$) compact hypersurface with constant normalized scalar curvature $R$ in $H^{n+1}$. If

1. $\bar{R} = R + 1 \geq 0$,
2. the norm square $|h|^2$ of the second fundamental form of $M^n$ satisfies

$$n\bar{R} \leq |h|^2 \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n],$$

then $M^n$ is a totally umbilical hypersurface.

**Proof.** From (36) we have

$$\Box(nH) \geq \frac{n}{n-1}(h^2 - n\bar{R})[-n + 2(n-1)\bar{R} - \frac{n-2}{n}|h|^2 - \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |h|^2)(|h|^2 - n\bar{R})}],$$

(51)

It is a direct check that our assumption condition (50) is equivalent to

$$-n + 2(n-1)\bar{R} - \frac{n-2}{n}|h|^2 \geq \frac{(n-2)^2}{n^2}(n(n-1)\bar{R} + |h|^2)(|h|^2 - n\bar{R}).$$

But it is clear from (50) that (52) is equivalent to

$$-n + 2(n-1)\bar{R} - \frac{n-2}{n}|h|^2 \geq \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |h|^2)(|h|^2 - n\bar{R})},$$

therefore the right hand side of (51) is non-negative. Because $M^n$ is compact and the operator $\Box$ is self-adjoint, we have $\int_{M^n}\Box(nH)dv = 0$. Thus either

$$|h|^2 = n\bar{R},$$

(54)

that is, $|h|^2 = nH^2$, $M^n$ is a totally umbilical hypersurface; or

$$|h|^2 = \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n].$$

(55)

In the latter case, equalities hold in (30) and Lemma 4.1, and it follows that $M^n$ has at most two distinct constant principal curvatures. We conclude that $M^n$ is totally umbilical from the compactness of $M^n$. This completes the proof of Corollary 1.

**Corollary 2.** Let $M^n$ be an $n$-dimensional compact hypersurface with constant normalized scalar curvature $R$ and $R + 1 \geq 0$ in $H^{n+1}$. If $M$ has non-negative sectional curvature, then $M$ is a totally umbilical hypersurface.

**Proof.** Because $M^n$ is compact and the operator $\Box$ is self-adjoint, form (25), we have
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\[ \int_{M^n} \left[ |\nabla h|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2 \right] = 0. \]  

(56)

If \( M^n \) has constant normalized scalar curvature \( R \) and \( R \geq -1 \), from Lemma 4.1, we have \( |\nabla h|^2 \geq n^2 |\nabla H|^2 \). So if \( M \) has non-negative sectional curvature, form (56) we have \( |\nabla h|^2 = n^2 |\nabla H|^2 \) and \( R_{ijij} = 0 \), when \( k_i \neq k_j \) on \( M^n \). Since \( R_{ijij} = -1 + k_i k_j \), then either \( M^n \) is totally umbilical, or \( M^n \) has two different principal curvatures, in the latter case, \( M^n \) is still totally umbilical from the compactness of \( M^n \). This completes the proof of Corollary 2.

References


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