Control of the Flux Substrate
Entering an Enzymatic Membrane

Vasile Iftode

Abstract

In this paper, we shall study an optimal control problem governed by a nonlinear partial differential equation where the system’s state is a differentiable function of the control parameter.

AMS Subject Classification: 34K05, 35A99

Key words: optimal control, PDE, dual system, weak solution

Let us consider a membrane which contains a certain enzyme and which separates two compartments containing solutions of the respective substrate of enzyme. In membrane occurs the reaction substrate plus enzyme to product, i.e., \( E + S \Leftrightarrow ES \rightarrow E + P \). If \( S(x,t) \) is the concentration of the substrate at the moment \( t \) in the point \( x \) of the unidimensional membrane then, according to Fick’s law, the evolution of \( S \) in membrane is described by the equation

\[
\frac{\partial S}{\partial t} - D \frac{\partial^2 S}{\partial x^2} + \frac{VS}{K + S} = 0, \quad x \in (a,b), \quad t \in [0,T]
\]

where \( K \) is the constant of Michaelis and \( V \) is the reaction speed.

If in the two compartments is present an inhibitor by concentration \( i = i(x,t) \) which is free to spread through membrane, then the evolution of system is described by the normalized equations (see [1], [3])

\[
\begin{align*}
\frac{\partial S}{\partial t} - D \frac{\partial^2 S}{\partial x^2} + \frac{\sigma S}{1 + ai + S} &= 0 \quad \text{in} \quad (a,b) \times (0,T) \\
\frac{\partial i}{\partial t} - C \frac{\partial^2 i}{\partial x^2} &= 0 \quad \text{in} \quad (a,b) \times (0,T)
\end{align*}
\]

with the initial conditions

\[
(2) \quad S(x,0) = 0, \quad i(x,0) = 0, \quad x \in (a,b)
\]

and the boundary conditions

\begin{equation}
\begin{cases}
S(a, t) = \alpha, \ S(b, t) = \beta, \ \forall t \in [0, T] \\
i(a, t) = u(t), \ i(b, t) = v(t).
\end{cases}
\end{equation}

This is a nonlinear parameter distributed system on \((a, b)\) with the boundary controls \(u\) and \(v\) which can be manipulated for to modify the concentration \(S\) in a certain aim.

Let us consider an enzymatic membrane \(M\) which separates two compartments I and II. The membrane is made of an inactive protein and contains an enzyme \(E\). In the compartments I and II there are a solution which contains a substrate \(S\) (corresponding of enzyme \(E\)) and an inhibitor \(I\), respectively, which both will spread in \(M\). In membrane, \(S\) will begin to react (favoured by \(E\) which has a catalyst assignment), reaction which is slowed down by the inhibitor \(I\). If we consider that the membrane has the thickness equal to the unit and denote by \((y(x, y), i(x, t))\) the concentration of the substrate and of the inhibitor, respectively, in the point \(x \in [0, 1]\) of the membrane at the moment \(t\), we have for \((y, i)\) the system (see Eqs. (1)-(3))

\begin{equation}
\begin{cases}
y_t - y_{xx} + \frac{\sigma y}{y + i + 1} = 0 \quad \text{in} \quad (0, 1) \times (0, T) \\
i_t - i_{xx} = 0 \quad \text{in} \quad (0, 1) \times (0, T)
\end{cases}
\end{equation}

with the initial conditions

\begin{equation}
y(x, 0) = 0, \ i(x, 0) = 0, \ x \in (0, 1)
\end{equation}

and the boundary conditions

\begin{equation}
\begin{cases}
y(0, t) = \alpha, \ y(1, t) = \beta, \ \forall t \in [0, T] \\
i(0, t) = u_1(t), \ i(1, t) = u_2(t)
\end{cases}
\end{equation}

where \(\alpha, \beta, \sigma\) are positive constants and \(u(t) = (u_1(t), u_2(t))\) are the control functions of the process subject to the constraints \(u = (u_1, u_2) \in U\), where

\begin{equation}
U = \{(u_1, u_2) \in L^2(0, T) \times L^2(0, T); \ 0 \leq u_1(t) \leq L, \ 0 \leq u_2(t) \leq L, \ \text{a.e.} \ t \in [0, T]\}.
\end{equation}

The experimenter can modify the parameters \(u_1, u_2\) to his liking (as part of restriction (7)) for to obtain a certain flux substrate entering to membrane. If the desired flux is defined by the functions \((y_1^0(t), y_2^0(t))\) then the problem can be mathematical formulated in this way

\begin{equation}
\min \left\{ \int_0^T ((-y_x(0, t) - y_1^0(t))^2 + (y_x(1, t) - y_2^0(t))^2) dt; \ (u_1, u_2) \in U \right\}
\end{equation}

where \((y, i)\) satisfy the state system (4), (5), (6).

By a \textit{weak solution} of equation

\begin{equation}
\begin{cases}
i_t - i_{xx} \quad \text{in} \quad (0, 1) \times (0, T) \\
i(x, 0) = 0
\end{cases}
\end{equation}
with the Dirichlet boundary conditions

\[ i(0, t) = u_1(t), \quad i(1, t) = u_2(t) \]

we shall mean a function \( i \in L^2((0, 1) \times (0, T)) \) such that

\[ \int_0^T \int_0^1 i(z_t - z_{xx}) \, dx \, dt = \int_0^T (u_1(t)z_x(1, t) - u_2(t)z_x(0, t)) \, dt, \quad \forall z \in Y \]

where

\[ Y = \{ z \in L^2((0, 1) \times (0, T)); \ z_t, z_x, z_{xx} \in L^2((0, 1) \times (0, T)); \ z(0, t) = z(1, t) = 0; \ z(x, 0) = 0 \}. \]

For each \( q \in L^2((0, 1) \times (0, T)) \) there exists one and only one solution \( z^q \in Y \) to the equation \( z_t - z_{xx} = q \) with the initial and Dirichlet conditions equal to zero and the application \( q \to z^q \) is continuous from \( L^2((0, 1) \times (0, T)) \) to \( Y \). It results from this that the functional

\[ q \to \int_0^T (u_1(t)z^q(1, t) - u_2(t)z^q(0, t)) \, dt \]

is continuous on \( L^2((0, 1) \times (0, T)) \) and therefore, for every \( u \in L^2(0, T) \times L^2(0, T) \) there exists an unique \( i \in L^2((0, 1) \times (0, T)) \) satisfying (10). Thus, we proved the existence and uniqueness of a weak solution for the problem (9).

If we redenote the unknown \( y \), we shall can suppose that \( \alpha = \beta = 0 \) in the boundary conditions (6). The function \( i = i^u \) being previous determined, the equation in \( y \) of the system (4) with the homogeneous boundary conditions admits an unique solution \( y^u \in L^2(0, T; H_0^1(0, 1)) \) such that \( y^u \in L^2(0, T; H^{-1}(\Omega)) \) (see [2], p.140).

In particular, it follows that \( y^u \in C([0, T]; L^2(0, 1)) \). The existence in the problem (8) results by standard methods.

Now we shall deduce the optimality conditions (the maximum principle). We denote by \( (z, q) = D_u(y^u, i^u)(v) \) the Gâteaux derivative of the application \( u \to (y^u, i^u) \) from \( L^2(0, T) \times L^2(0, T) \) to \( (L^2((0, 1) \times (0, T)))^2 \) at

\[ v = (v_1, v_2) \in L^2(0, T) \times L^2(0, T). \]

Let us observe that \( z \) is the solution of the system in variation

\[
\begin{align*}
z_t - z_{xx} + \frac{\sigma(i^u + 1)z}{(y^u + i^u + 1)^2} & = -\frac{\sigma y^u}{(y^u + i^u + 1)^2} q = 0 & \text{in} & & (0, 1) \times (0, T) \\
g_t - q_{xx} & = 0 & \text{in} & & (0, 1) \times (0, T) \\
z(x, 0) & = 0, \quad q(x, 0) = 0, \quad x \in (0, 1) \\
zie q(0, t) = 0, \quad z(1, t) = 0, \quad t \in [0, T] \\
g(0, t) & = v_1(t), \quad q(1, t) = v_2(t), \quad t \in [0, T].
\end{align*}
\]

If \( (u^*, y^*, i^*) \) is optimal in Problem (8), then obviously we have
\[
\int_0^T (z_x^*(0, t)(y_x^*(0, t) + y_1^0(t)) + z_x^*(1, t)(y_x^*(1, t) - y_2^0(t)))dt \geq 0
\]

where \(z^*\) is the solution of the system (11) with \(u = u^*, v \in T_U(u^*)\). (Here \(T_U(u^*)\) is the cone of tangents to \(U\) at \(u^* \in U\).

Let us consider the dual system (suggested by the system in variation (11))

\[
(12) \quad \begin{cases}
  p_1^1 + p_{xx}^1 - \frac{\sigma(t^* + 1)}{(y^* + i^* + 1)^2} p^1 = 0 & \text{in } (0, 1) \times (0, T) \\
  p_2^1 + p_{xx}^2 + \frac{y^*}{(y^* + i^* + 1)^2} p^1 = 0 & \text{in } (0, 1) \\
  p^1(x, T) = 0, p^2(x, T) = 0 & \text{in } (0, 1) \\
  p^1(0, t) = -(y_x^*(0, t) + y_1^0(t)), p^1(1, t) = (y_x^*(1, t) - y_2^0(t)) & \text{in } (0, T) \\
  p^2(0, t) = 0, p^2(1, t) = 0 & \text{in } (0, T)
\end{cases}
\]

which admits an unique solution \(p^1 \in L^2((0, 1) \times (0, T)), p^2 \in L^2((0, T); H^1_0(0, 1))\) with \(p_2^1 \in L^2((0, 1) \times (0, T))\).

If we multiply the first equation of (11) (where \(u = u^*\)) with \(p^1\), the second with \(p^2\) and integrate on \((0, 1) \times (0, T)\), finally, it follows that (the formally integration by parts is justified by the definition of weak solution \(p^1 \in L^2((0, 1) \times (0, T))\) of the equation (12))

\[
\int_0^T (z_x^*(0, t)(y_x^*(0, t) + y_1^0(t)) + z_x^*(1, t)(y_x^*(1, t) - y_2^0(t)))dt =
\]

\[
= \int_0^T (p_2^2(1, t)v_2(t) - p_2^2(0, t)v_1(t))dt \geq 0, \quad \forall v \in T_U(u^*).
\]

Hence

\[
(p_2^2(0, t), -p_2^2(1, t)) \in N_U(u^*).
\]

(Here \(N_U(u^*)\) is the cone of normals to \(U\) at \(u^* \in U\).) If follows, by definition of \(N_U(u^*)\), that

\[
(13) \quad u_1^*(t) = \begin{cases}
  0 & \text{if } p_2^2(0, t) > 0 \\
  L & \text{if } p_2^2(0, t) < 0
\end{cases}
\]

\[
(14) \quad u_2^*(t) = \begin{cases}
  0 & \text{if } p_2^2(1, t) < 0 \\
  L & \text{if } p_2^2(1, t) > 0.
\end{cases}
\]

The system (4)~(6), (12)~(14) can be integrated by numerical methods.

**References**


University Politehnica of Bucharest  
Department of Mathematics I  
Splaiul Independenței 313  
77206 Bucharest, Romania