Cylindrical Tzitzeica Curves Implies Forced Harmonic Oscillators

Dedicated to Acad. Radu Miron on the occasion of his 75 birthday

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Abstract
Elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition are obtained via the solution of the forced harmonic equation.

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Introduction
Gheorghe Tîţeica (1873-1939), writing in French under the name of Georges Tzitzeica, was a student of Gaston Darboux and thus a member of the second generation of classical differential geometers, after Gauss and Riemann.

There are several Tzitzeica notions:
(1) surfaces ([1], [32])
(2) curves ([19])
(3) hypersurfaces ([7], [23])
(4) submanifolds ([6], [8], [13], [14], [21], [27])
(5) algebras ([20], [22])
(6) connections ([9], [10], [11], [16], [18])
(7) equations ([2], [4], [5], [12], [25], [26]).

Tzitzeica has introduced a class of surfaces, nowadays called Tzitzeica surfaces, in 1907 ([29]) and a class of curves, called Tzitzeica curves, in 1911 ([30]). The relation between these objects is the following: for a Tzitzeica surface with negative Gaussian curvature, the asymptotic lines are Tzitzeica curves. Since their appearance, these notions are a permanent subject of research, suitable for fruitful generalizations.

(Elliptic) cylindrical curves are studied by Gheorghe Vrâncneanu in connection with Levi-Civita and Fenchel theorems from surfaces theory in [33].

We consider two types of cylindrical curves: an usual one, which we prefer to call elliptic, and another one, called hyperbolic, according to the type of circular functions (cosine and sine) used. So, our results are divided in two sections.
In this paper we are interested in Tzitzeica cylindrical curves, more precisely we ask in what conditions a cylindrical curve is a Tzitzeica one, namely the function \( t \rightarrow \frac{\tau(t)}{d^2(t)} \) is constant, where \( d(t) \) is the distance from origin to the osculating plane of curve. The Tzitzeica condition yields a third-order ODE which in our framework admits a direct integration. Therefore the final answer of main problem is given via a second order ODE which in the elliptic case is exactly the equation of a forced harmonic oscillator! In both cases, elliptic and hyperbolic, the solution depends of four real constants: one defining the Tzitzeica condition and other three obtained by integration.

1 Elliptic cylindrical curves

Let in \( \mathbb{R}^3 \) a curve \( C \) given in vectorial form \( C : \mathbf{r} = \mathbf{r}(t) \). This curve is called elliptic cylindrical if has the expression

\[
(1.1) \quad \mathbf{r}(t) = (\cos t, \sin t, f(t))
\]

for some \( f \in C^\infty(\mathbb{R}) \).

The torsion function is

\[
\tau(t) = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''') \times |\mathbf{r}' \times \mathbf{r}''|^2}{\|\mathbf{r}' \times \mathbf{r}''\|^2} = \frac{f' + f'''}{1 + f'^2 + f''^2}.
\]

Then the distance from origin to the osculating plane is

\[
d(t) = \pm \frac{(f + f''')}{\sqrt{1 + f'^2 + f''^2}}.
\]

Let us suppose that the curve is Tzitzeica with the constant \( K \neq 0 \), because the curve is not contained in a plane

\[
\frac{\tau(t)}{d^2(t)} = K = \frac{f'(t) + f'''(t)}{(f(t) + f''(t))^2}.
\]

Integration gives

\[
\frac{1}{f(t) + f''(t)} = -\left(kt + C\right)
\]

with \( C \) a real constant. The last relation reads

\[
(1.2) \quad f''(t) + f(t) = \frac{-1}{Kt + C}.
\]

But the ODE (1.2) is exactly of forced harmonic oscillator type,

\[
\ddot{x} + x = g(t)
\]

solved by the formula ([17, p. 60-61])

\[
x(t) = x(0) \cos t - \dot{x}(0) \sin t + \int_0^t g(u) \sin (u - t) \, du
\]
Therefore we have

**Proposition 1.1.** An elliptic cylindrical curve (1.1) is Tzitzeica if and only if

\[ f(t) = f(0) \cos t - \dot{f}(0) \sin t - \int_0^t \frac{\sin (u-t)}{Ku+C} \, du \]

with \( f(0), \dot{f}(0), K \neq 0, C \) real constants.

### 2 Hyperbolic cylindrical curves

A space curve is called *hyperbolic cylindrical* if

\[ \tau(t) = (\cosh t, \sinh t, f(t)). \]

Straightforward computation gives

\[ \tau(t) = \frac{f' - f'''}{1 + (f'^2 + f''^2)(\cosh^2 t + \sinh^2 t) - 4f'f'' \cosh t \sinh t}. \]

Also

\[ d(t) = \pm \frac{f' - f''}{\sqrt{1 + (f'^2 + f''^2)(\cosh^2 t + \sinh^2 t) - 4f'f'' \cosh t \sinh t}}. \]

For a Tzitzeica hyperbolic curve

\[ \frac{\tau(t)}{d^2(t)} = K = \frac{f' - f'''}{(f - f'')^2} \]

and integration gives

\[ \frac{1}{f - f''} = -(Kt + C) \iff f'' - f = \frac{1}{Kt + C}. \]

Using the same formula (3) from [17, p. 60] and the identity

\[ e^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \]

it results

**Proposition 2.1.** A hyperbolic curve (2.1) is Tzitzeica if and only if:

\[ f(t) = f(0) \cosh t + \dot{f}(0) \sinh t + \int_0^t \frac{\sinh (t+u)}{Ku+C} \, du \]

with \( f(0), \dot{f}(0), K \neq 0 \) and \( C \) real constants.
References


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