Pseudo-Umbilical Spacelike Submanifolds in the Indefinite Space Form

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Abstract

We prove an integral inequality for compact pseudo-umbilical spacelike submanifolds in the indefinite space form. As an application of the inequality, we give a necessary and sufficient condition for such submanifolds to be totally geodesic.

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1 Introduction

Let $M^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature $c$ whose index is $p$, which is called as indefinite space form of index $p$. Let $M^n$ be an $n$-dimensional Riemannian manifold immersed in $M^{n+p}(c)$. As the semi-Riemannian metric of $M^{n+p}(c)$ induces the Riemannian metric of $M^n$, $M^n$ is called a spacelike submanifold. Let $h$ be the second fundamental form of the immersion, and $\xi$ the mean curvature vector. Denote by $\langle \cdot , \cdot \rangle$ the scalar product of $M^{n+p}(c)$. If there exists a function $\lambda$ on $M^n$ such that

\[ \langle h(X,Y), \xi \rangle = -\lambda \langle X,Y \rangle \]

for any tangent vector $X,Y$ on $M^n$, then $M^n$ is called a pseudo-umbilical spacelike submanifold of $M^{n+p}(c)$. It is clear that $\lambda \geq 0$. If the mean curvature vector $\xi$ vanishes identically, then $M^n$ is called a maximal spacelike submanifold of $M^{n+p}(c)$. Every maximal spacelike submanifold of $M^{n+p}(c)$ is itself a pseudo-umbilical spacelike submanifold of $M^{n+p}(c)$.

Maximal and pseudo-umbilical spacelike submanifolds have been studied by many researchers. For example in 1988 Ishihara [1] proved that if $M^n$ is a complete and maximal spacelike submanifold of $M^{n+p}(c)$, then either $M^n$ is totally geodesic (when $c \geq 0$) or $0 \leq S \leq -np\omega$ (when $c < 0$), where $S$ is the square length of the second fundamental form of $M^n$. In 1995 Sun [2] first proved that the mean curvature $H$ of

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the pseudo-umbilical submanifolds $M^n$ in $M^{n+p}(c)$ is constant, then he generalized Ishihara’s result to pseudo-umbilical submanifolds, obtaining the inequality

$$nH^2 \leq S \leq \frac{1}{2}n(p - \sqrt{(H^2 - c)^2 + 4H^2c/p}).$$

In this paper, we prove an integral inequality for compact pseudo-umbilical spacelike submanifolds in the indefinite space form and as an application of the inequality, we give a necessary and sufficient condition for such submanifolds to be totally geodesic. We will prove the following

**Theorem 1.** Let $M^n$ be an $n$-dimensional compact pseudo-umbilical spacelike submanifold in $M^{n+p}(c)$, then

$$\int_{M^n} \left\{ \frac{1}{2} \sum R^2_{mi,jk} + \sum R^2_{mj,i} - ncR + nH^2R \right\} * 1 \leq 0.$$  

**Theorem 2.** Let $M^n$ be an $n$-dimensional compact pseudo-umbilical spacelike submanifold in $M^{n+p}(c)$, then

$$\int_{M^n} \left\{ \frac{1}{2} \sum R^2_{mi,jk} + (n - 2)cS - n(n - 1)c^2 + n^2(n - 1)cH^2 + nH^2S \right\} * 1 \leq 0,$$

and equality holds if and only if $M^n$ is totally geodesic.

In the above Theorem, $\sum R^2_{mi,jk}$ is the square length of the Riemannian curvature tensor of $M^n$, $\sum R^2_{mj,i}$ the square length of the Ricci curvature tensor and $R$ the scalar curvature. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \cdots \leq n + p; \quad 1 \leq i, j, k, \cdots \leq n, \quad n + 1 \leq \alpha, \beta, \gamma, \cdots \leq n + p,$$

and we shall agree that repeated indices are summed over the respective ranges.

## 2 Local formulas

We choose a local field of semi-Riemannian orthonormal frames $e_1, \cdots, e_{n+p}$ in $M^{n+p}(c)$ such that, restricted to $M^n$, $e_1, \cdots, e_n$ are tangent to $M^n$. Let $\omega_1, \cdots, \omega_{n+p}$ be its dual frame field such that the semi-Riemannian metric of $M^{n+p}(c)$ is given by

$$ds^2 = \sum (\omega_i)^2 - \sum (\omega_\alpha)^2 = \sum \varepsilon_A(\omega_A)^2,$$

where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Then the structure equations of $M^{n+p}(c)$ are given by

$$d\omega_A = -\sum \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum K_{ABCD} \omega_C \wedge \omega_D,$$

$$K_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AD} \delta_{BC} - \delta_{AC} \delta_{BD}).$$
We restrict these forms to $M^n$, then

\[(2.4) \quad \omega_\alpha = 0, \quad \omega_\alpha = \sum h^\alpha_{ij} \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}\]

(2.5) \qquad d\omega_{ij} = -\sum \omega_k \wedge \omega_{kj} - \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l,

(2.6) \qquad R_{ijkl} = c(\delta_{ij} \delta_{kj} - \delta_{ik} \delta_{jl}) - \sum (h^\alpha_{ij} h^\alpha_{kj} - h^\alpha_{ik} h^\alpha_{jl}),

(2.7) \qquad R_{jk} = \sum R_{ijkl} = c(n - 1) \delta_{jk} + \sum h^\alpha_{ik} h^\alpha_{jl},

(2.8) \qquad R = \sum R_{ij} = n(n - 1)c + S,

(2.9) \qquad d\omega_{\alpha\beta} = -\sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2} \sum R_{\alpha\beta ij} \omega_i \wedge \omega_j,

(2.10) \qquad R_{\alpha\beta ij} = -\sum (h^\beta_{ij} h^\alpha_{ji} - h^\alpha_{ij} h^\beta_{ij}).

For indefinite Riemannian manifolds, refer to O’Neill [3].

We call $h = \sum h^\alpha_{ij} \omega_i \omega_j e_\alpha$ the second fundamental form of the immersed manifold $M^n$. Denote by $S = \sum (h^\alpha_{ij})^2$ the square length of $h$, $\xi = \frac{1}{n} \sum \text{tr}H_a e_\alpha$ the mean curvature vector and

\[H = \frac{1}{n} \sqrt{\sum (\text{tr}H_a)^2}\]

the mean curvature of $M^n$ respectively. Here $\text{tr}$ is the trace of the matrix $H_a = (h^\alpha_{ij})$. Now let $e_{n+1}$ be parallel to $\xi$. Then we have

\[(2.11) \quad \text{tr}H_{n+1} = nH, \quad \text{tr}H_a = 0, \quad \alpha \neq n + 1.\]

Let $h^\alpha_{ijk}$ and $h^\alpha_{ijkl}$ denote the covariant derivative and the second covariant derivative of $h^\alpha_{ij}$ respectively, defined by

\[(2.12) \quad \sum h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} - \sum h^\beta_{ik} \omega_{kj} - \sum h^\gamma_{jk} \omega_{ki} - \sum h^\delta_{ij} \omega_{\beta\alpha},\]

(2.13) \qquad \sum h^\alpha_{ijk} \omega_i = dh^\alpha_{jk} - \sum h^\beta_{ik} \omega_{kj} - \sum h^\gamma_{jk} \omega_{ki} - \sum h^\delta_{ij} \omega_{\beta\alpha},

then we have

\[(2.14) \quad h^\alpha_{ijk} = h^\alpha_{kji} = 0,\]

\[(2.15) \quad h^\alpha_{ijk} - h^\alpha_{ijk} = -\sum h^\alpha_{im} R_{mjkl} - \sum h^\alpha_{jm} R_{milk} - \sum h^\alpha_{ij} R_{\alpha\beta kl}.\]

The Laplacian $\Delta h^\alpha_{ij}$ of $h^\alpha_{ij}$ is defined by $\Delta h^\alpha_{ij} = \sum h^\alpha_{ij}$ by $h^\alpha_{ijk}$. By a direct calculation we have (cf. [4])

\[(2.16) \quad \sum h^\alpha_{ij} \Delta h^\alpha_{ij} = \sum h^\alpha_{ij} h^\alpha_{kkij} - \sum h^\alpha_{ij} h^\alpha_{km} R_{mijk} - \sum h^\alpha_{ij} R_{mjkl} - \sum h^\alpha_{ij} h^\beta_{kl} R_{\alpha\beta kl}.\]
3 Proof of Theorem

Proof of Theorem 1. From (1.1) and (2.11)

\[ \langle h(e_i, e_j), H e_{n+1} \rangle = -H^2 \delta_{ij} \quad \text{i.e.} \quad h_i^{n+1} = -H \delta_{ij}, \]

therefore

\[ \sum h_i^a h_k^a = -nH \triangle H. \]

Since \( H \) is constant (see [2]),

\[ \sum h_i^a h_k^a = 0. \]

On the other hand, from (2.6)

\[ -\sum h_i^a h_k^a R_{mijk} = \sum \left( \frac{1}{2} \sum (h_i^a h_k^a - h_m^a h_{i}^a) R_{mijk} \right) \]

\[ = \frac{1}{2} \sum \{R_{mijk} - c(\delta_{ij} \delta_{mk} - \delta_{mj} \delta_{ik})\} R_{mijk} = \frac{1}{2} \sum R_{mijk}^2 - cR, \]

\[ -\sum h_i^a h_k^a R_{mjk} = \sum (h_i^a h_k^a - h_m^a h_{m+1}) R_{mijk} = \]

\[ = \sum \{c(\delta_{ij} \delta_{mi} - \delta_{mj} \delta_{im}) - R_{mij} + nH^2 \delta_{m}\} R_{mijk} = \]

\[ = \sum R_{mijk}^2 - (n - 1)cR + nH^2 R. \]

From (2.10)

\[ -\sum h_i^a h_k^a R_{\alpha \beta jk} = \sum h_i^a h_k^a h_i^\beta h_k^\beta - \sum h_i^a h_k^a h_i^\beta h_i^\beta. \]

Since

\[ \sum h_i^a h_k^a h_{ik}^a h_{ik}^a = \sum \text{tr}(H^a H^\beta)^2 \leq \]

\[ \leq \sum \text{tr}(H^a)^2 (H^\beta)^2 = \sum h_i^a h_k^a h_i^\beta h_k^\beta, \]

therefore

\[ -\sum h_i^a h_k^a R_{\alpha \beta jk} \geq 0. \]

From (2.14), (3.3-3.8)

\[ \sum h_i^a \triangle h_i^a \geq \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mjk}^2 - ncR + nH^2 R. \]

Since \( \int_{M^n} \{ \sum h_i^a \triangle h_i^a \} * 1 \leq 0 \) (see [4]), we have

\[ \int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mjk}^2 - ncR + nH^2 R \right\} * 1 \leq 0. \]
Theorem 1 is proved.

**Proof of Theorem 2.** From (2.7)

\[(3.11) \sum R^2_{m,j} = n(n-1)c^2 + 2(n-1)cS + \sum_{m,j} \left( \sum_{i,a} h^a_{m,n} h^a_{ij} \right)^2,\]

\[(3.12) \sum_{m,j} \left( \sum_{i,a} h^a_{m,n} h^a_{ij} \right)^2 \geq \sum_j \left( \sum_{i,a} (h^a_{ij})^2 \right)^2 \geq \frac{1}{n} \left( \sum_{i,a} (h^a_{ij})^2 \right)^2 = \frac{1}{n} S^2,\]

therefore from (3.10),

\[(3.13) \int_{M^n} \left\{ \frac{1}{2} \sum R^2_{mijk} + (n-2)S - n(n-1)c^2 + n^2(n-1)cH^2 + nH^2 S \right\} * 1 \leq 0.\]

If \(M^n\) is totally geodesic, i.e., \(S = 0, h^a_{ij} = 0\), then from (2.6)

\[(3.14) \frac{1}{2} \sum R^2_{mijk} = n(n-1)c^2,\]

in this case, (3.13) becomes an equality; Inversely, if (3.13) becomes an equality, then from (3.12) \(S = 0\), i.e. \(M^n\) is totally geodesic. Theorem 2 is proved.

**References**


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