R-Separated Spaces
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Abstract

In this paper we have generalized the axioms of the separated spaces $T_i$, $(i = 0, \ldots, 4)$, by replacing the equality relation on a topological space $X$, $\Delta_X$, by an arbitrary binary relation, $R$. Many theorems in general topology may be generalized in this way. It will be interesting to study spaces separated by functions, equivalence relations or order relations. In section 1 are presented axioms and characterizing theorems of $R$-separation, in section 2 are presented some properties of spaces separated by equivalence relations and in section 3 we will obtain some results concerning spaces separated by functions.

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Key words: topological space, normal, regular space, quotient space, equivalence relation.

1 Axioms and theorems of $R$ separation

First let us make the following notations: $(X, \mathcal{T})$ is a topological space; $R \subseteq X \times X$ is a binary relation on $X$; $\overline{R}$ is the dual of $R$ (i.e. $xRy \iff (x, y) \notin R$); $R^{-1}$ is the inverse of $R$ (i.e. $xR^{-1}y \iff (y, x) \in R$); $xR_A \iff xRy$, $(\forall) y \in A$: $xR_A \iff xRy$ $(\forall) y \in A$; $ARB \iff xRy$, $(\forall) x \in A$, $(\forall) y \in B$: $ARB \iff xRy$, $(\forall) x \in A$, $(\forall) y \in B$; $R(x) = \{y \mid xRy\}$ and $R^{-1}(x) = \{y \mid yRx\}$; $R(A) = \{y \mid (\exists) x \in A \text{ so that } xRy\}$; $N_x$ is the neighborhood filter of $x \in X$; For any $A \subseteq X$ we note by $V_A = \{B \mid (\exists) D \in \mathcal{T} \text{ so that } A \subseteq D \subseteq B\}$ and we note $CA = X \setminus A$.

We will replace in the classical definitions of the separated spaces $x \neq y$ by $xRy; x \notin A$ by $xR_A$ or $A\overline{R}x$ and $A \cap B$ by $A\overline{RB}$. Replacing in the following considerations the relation $R$ by $\Delta_X$ (i.e. $xRy \iff x = y$), we shall find the classical case of $T_i$ spaces ($i = 0, 4$).

Definition 1. $X$ is $T^R_0$ - space iff $(\forall) x, y \in X$ with $xRy$, $(\exists) V_x \in V_x$ so that $V_x \overline{R}y$ or $(\exists) V_y \in V_y$ so that $x\overline{R}V_y$.

Definition 2. $X$ is $T^R_1$ - space iff $(\forall) x, y \in X$ with $xRy$, $(\exists) V_x \in V_x$ and $(\exists) V_y \in V_y$ so that $x\overline{R}V_y$ and $V_x \overline{R}y$.

Definition 3. $X$ is $T^R_2$ - space iff $(\forall) x, y \in X$ with $xRy$, $(\exists) V_x \in V_x$, $(\exists) V_y \in V_y$ so that $V_x \overline{R}V_y$.

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Remark 1. $T^R_2 \subset T^R_1 \subset T^R_0$.

Remark 2. If in the definitions 1, 2, 3 we take $x R y$ iff $x = y$, then they become those in the classical case.

Theorem 4. $X$ is $T^R_0$ - space iff for each $x, y \in X$ we have: $y \in \overline{R(x)}$ and $x \in R^{-1}(y)$ if $x R y$, where by $\overline{A}$ we note the closure of the subset $A$ of $X$.

Proof. $V_x \cap V_y$ or $x R y \iff V_x \cap R^{-1}(y) = \emptyset$ or $R(x) \cap V_y = \emptyset \iff y \notin \overline{R(x)}$ or $y \notin \overline{R} \iff x \notin \overline{R^{-1}(y)}$ or $y \notin \overline{R(x)}$ for $x R y \iff x \notin \overline{R^{-1}(y)}$ and $y \notin \overline{R(x)}$, then $x R y$. □

Theorem 5. $X$ is $T^R_1$ - space iff $R(x)$ and $R^{-1}(x)$ are closed subsets, for every $x \in X$.

Proof. "⇒". Suppose that $X$ is $T^R_1$ - space and $x \notin \overline{R(x)}$. Then exists $V_x \in \mathcal{V}_x$ an $V_y \in \mathcal{V}_y$ so that $x \notin \overline{R(x)}$ and $x \notin \overline{R^{-1}(x)}$ and $x \notin \overline{R^{-1}(y)}$ and $x \notin \overline{R(x)}$.

"⇒". (a) If $R^{-1}(y)$ is a closed subset of $X$, then for each $x \notin \overline{R(x)}$, we have $x \notin \overline{R^{-1}(y)}$.

From here: (1) $V_x \in \mathcal{V}_x$ with $V_x \cap R^{-1}(y) = \emptyset \Rightarrow V_x \cap \overline{R(x)}$. (b) $V_y \in \mathcal{V}_y$ with $x \notin \overline{R(x)}$ just like in (a). □

Theorem 6. $X$ is $T^R_2$ - space iff $R$ is a closed subset of $X \times X$.

Proof. "⇒". Suppose $X$ is $T^R_2$ - space and $y \in \bigcap_{V \in \mathcal{V}_y} \overline{R(V)} = R(x)$ for every $x \in X$ we have $y \notin \overline{R(V)} = R(x)$ and for every $y \in X$ we have $y \notin \overline{R(V)} = R(x)$.

Proof. "⇒". Suppose $x \notin \overline{R(V)} = R(x)$ and $y \in \bigcap_{V \in \mathcal{V}_y} \overline{R(V)} = y \in \overline{R(V)}$, $(\forall) V \in \mathcal{V}_y$. If $y \notin R(x) \Rightarrow x \notin \overline{R(V)} \Rightarrow (\exists) \mathcal{V}_x \in \mathcal{V}_x$ and $(\exists) \mathcal{V}_y \in \mathcal{V}_y$, so that $V_x \cap \overline{R(V)} \Rightarrow R(V(x)) \cap V_y = \emptyset \Rightarrow y \notin \overline{R(V)}$, contradiction, so $y \notin R(x)$ and from here $\bigcap_{V \in \mathcal{V}_y} \overline{R(V)} \cap R(x) \Rightarrow R(x)$.

In the same way we infer $\bigcap_{V \in \mathcal{V}_y} \overline{R(V)} = R(x)$.

"⇒". Suppose $x \notin \overline{R(V)} \Rightarrow y \notin \overline{R(x)} \Rightarrow y \notin \bigcap_{V \in \mathcal{V}_y} \overline{R(V)} \Rightarrow (\exists) \mathcal{V}_x \in \mathcal{V}_x$ so that $y \notin \overline{R(V)} = (\exists) \mathcal{V}_y \in \mathcal{V}_y$ so that $V_y \cap \overline{R(V)} = \emptyset \Rightarrow V_y \cap \overline{R(V)} \Rightarrow X$ is $T^R_2$. □

Remark 3. It is enough to replace in Theorem 7, $\mathcal{V}_x$ with a neighborhood basis of $x$.

Definition 8. (a) $X$ is a $R_1$ - regular space iff for each $F$, closed subset so that $F \overline{Ry}$, there exists $V_F \in \mathcal{V}_F$ (neighborhood of $F$) and there exists $V_y \in \mathcal{V}_y$ so that $V_F \cap \overline{R(y)}$. (b) $X$ is a $R_2$ - regular space iff for each $F$, closed subset with $x \overline{RF}$, there exists $V_F \in \mathcal{V}_F$ and there exists $V_x \in \mathcal{V}_x$ so that $x \overline{RF}$. (c) $X$ is a $R$ - regular space iff $X$ is a $R_1$ and $R_2$ space.

Remark 4. If $R$ is a symmetric relation (i.e. $R \subset R^{-1}$) then (a) ⇔ (b) ⇔ (c).

Remark 5. If in this definition we take $x \overline{Ry}$ if $x = y$ then they become those in the classical case.
Theorem 9. (a) $X$ is a $R_0$ - regular space iff for each $y \in X$ and $U \in \mathcal{V}_{R^{-1}(y)}$ there exists $V \in \mathcal{V}_y$ so that $R^{-1}(V) \subset U$.

(b) $X$ is a $R_0$ - regular space iff for each $x \in X$ and $U \in \mathcal{V}_{R(x)}$ there exists $V \in \mathcal{V}_x$ so that $R(V) \subset U$.

(c) $X$ is a $R$ - regular space iff (a) and (b) are both true.

Proof. (a) "$\Rightarrow$". Suppose $X$ is $R_0$ regular space. Let $y \in X$ be and $U \in \mathcal{V}_{R^{-1}(y)}$. Suppose that $U$ is an open set. Then $F = CU$ is a closed set $\Rightarrow F \cap R^{-1}(y) = 0 \Rightarrow F \cap y \Rightarrow$ there exists $V_F \in \mathcal{V}_F$ so that $V_F \cap Y = R^{-1}(V_F) = 0$. Without loss the generality we can suppose $V_F$ open set $\Rightarrow CF$ is a closed set $\Rightarrow R^{-1}(V_F) \subset CV_F = CV_F$. But $F \subset V_F \Rightarrow CV_F \subset CF = U \Rightarrow R^{-1}(V_F) \subset U$.

"$\Leftarrow$". Let $F$ be a closed set and $y \in X$ so that $F \cap R^{-1}(y) = 0 \Rightarrow R^{-1}(V_F) \subset CF; CF = U$ is an open set $\Rightarrow U \in \mathcal{V}_{R^{-1}(y)} \Rightarrow$ there exists $V \in \mathcal{V}_y$ so that $R^{-1}(V_F) \subset U \Rightarrow CR^{-1}(V_F) \subset CR^{-1}(V_F) \subset V_F \Rightarrow V_F \cap R^{-1}(y) = 0 \Rightarrow V_F \subset R^{-1}(V_F)$.

(b) In the same way as (a).

(c) Is the consequence of (a) and (b). $\square$

Definition 10. $X$ is a $R$ - normal space iff for each $F_1, F_2$ closed sets so that $F_1 \subset F_2$, there exists:

$$V_1 \in \mathcal{V}_{F_1}, V_2 \in \mathcal{V}_{F_2} \Rightarrow V_1 \subset F_2.$$

Theorem 11. $X$ is a $R$ - normal space iff for each $F$ closed set and $U \in \mathcal{V}_{R(F)}$, there exists $V \in \mathcal{V}_F$ so that (a) $R(V) \subset U$ and (b) for each $U \in \mathcal{V}_{R^{-1}(y)}$, there exists $V \in \mathcal{V}_y$ so that $R^{-1}(V) \subset U$.

Proof. "$\Rightarrow$". Suppose $X$ is a $R$ - normal space. Let $F$ be a closed set and $U \in \mathcal{V}_{R(F)} \Rightarrow R(F) \subset U$. Suppose $U$ is a open set $\Rightarrow F_1 = CU$ is a closed set. As $CR(F) \cap F_1 \Rightarrow R(F) \cap F_1 = \emptyset \Rightarrow F \subset F_1 \Rightarrow$ exists $V_F \in \mathcal{V}_F$ and $V_1 \in \mathcal{V}_{F_1}$ so that $V_F \subset V_1$. But $V_1 \cap F_1 \Rightarrow CV_1 \subset CF_1$. Suppose $V_1$ is an open set $\Rightarrow CV_1 \subset CV_1$. But $V_F \cap V_1 \Rightarrow R(V_F) \subset CV_1 \Rightarrow R(V_F) \subset CV_1 \subset CF_1 = U$. So $R(V_F) \subset U$. In the same way for each $U \in \mathcal{V}_{R^{-1}(y)}$, there exists $V \in \mathcal{V}_y$ so that $R^{-1}(V) \subset U$.

"$\Leftarrow$". Let $F_1, F_2$ be closed sets so that $F_1 \subset F_2$, $F_1 \cap F_2 = \emptyset \Rightarrow R(F_1) \cap F_1 = CF_1$; $CF_1 = U$ is an open set $\Rightarrow U \in \mathcal{V}_{R(F_1)} \Rightarrow$ there exists $V_1 \in \mathcal{V}_{F_1}$ so that $R(V_1) \subset U \Rightarrow CR(V_1) = V_2$, is an open set and $V_2 \subset F_2 \Rightarrow V_2 \in \mathcal{V}_{F_2}$. Observe that $CV_2 = R(V_1) \subset CR(V_1) \Rightarrow R(V_1) \cap V_2 = \emptyset \Rightarrow V_1 \subset F_2$.

Remark 6. Definition 5 $\Leftrightarrow$ condition (a) $\Leftrightarrow$ condition (b), as we can see from the proof. The $R$ - separated spaces can be characterized by using sequences. First we define the $T_3^R$ and $T_4^R$ spaces.

Definition 12. (a) $X$ is $T_3^R$ space iff $X$ is a $T_1$ space and $R$ - regular space.

(b) $X$ is $T_4^R$ space iff $X$ is a $T_1$ space and an $R$ - normal space.

Theorem 13. $T_4^R \subset T_3^R \subset T_2^R \subset T_1^R \subset T_0^R$.

Proof. Observe that if $X$ is $T_1$ space then $\{x\} = \{x\}$ for each $x \in X$; using this condition results the first and the second inclusion of theorem 7. $\square$

Remark 7. A naturally condition for $T_3^R$ and $T_4^R$ spaces would to be $T_1^R$ space, but it is not good enough to Theorem 7.
Theorem 14. $X$ is $T_0^R$ space iff for the generalized sequences $(x_\alpha)_{\alpha \in I}$ and $(y_\beta)_{\beta \in J}$ we have

$$
\begin{align*}
& x_\alpha \to x \\
& y_\beta \to y \\
& x_\alpha R y \\
& x R y_\beta \\
& x R y_\beta
\end{align*}
\Rightarrow
\begin{align*}
x R y.
\end{align*}
$$

Proof. We will use Theorem 1.

"$\Rightarrow$". Suppose $y \in \overline{R(x)}$ and $x \in \overline{R^{-1}(y)} \Rightarrow x R y$. See that:

$$
\begin{align*}
x_\alpha \to x \\
x_\alpha R y \Rightarrow x \in \overline{R^{-1}(y)}
\end{align*}
\Rightarrow
\begin{align*}
x \in \overline{R^{-1}(y)}.
\end{align*}
$$

Also:

$$
\begin{align*}
y_\beta \to y \\
x R y_\beta \Rightarrow y \in \overline{R(x)}
\end{align*}
\Rightarrow
\begin{align*}
y \in \overline{R(x)}.
\end{align*}
$$

By using Theorem 4 we have $x R y$.

"$\Leftarrow$". $y \in \overline{R(x)} \Rightarrow$ there exists $(y_\beta)_{\beta \in J}$ so that $y_\beta \to y$ and $y_\beta \in \overline{R(x)} \Rightarrow x R y_\beta$, $x \in \overline{R^{-1}(y)} \Rightarrow$ there exists $(x_\alpha)_{\alpha \in I}$ so that $x_\alpha \to x$ and $x \in \overline{R^{-1}(y)} \Rightarrow x_\alpha R y$. From here it follows $x R y$. $\Box$

Theorem 15. $X$ is $T_1^R$ space iff for the generalized sequences $(x_\alpha)_{\alpha \in I}$ and $(y_\beta)_{\beta \in J}$ we have

$$
\begin{align*}
x_\alpha \to x \\
x_\alpha R y
\end{align*}
\Rightarrow
\begin{align*}
x R y \
\end{align*}
$$

Proof. We use Theorem 5.

"$\Rightarrow$". $x_\alpha \to x$, $x_\alpha R y \Rightarrow x_\alpha \in \overline{R^{-1}(y)} = \overline{R^{-1}(y)} \Rightarrow x \in \overline{R^{-1}(y)} \Rightarrow x R y$, $y_\beta \to y$, $x R y_\beta \Rightarrow y_\beta \in \overline{R(x)} = \overline{R(x)} \Rightarrow y \in \overline{R(x)} \Rightarrow x R y$.

"$\Leftarrow$". $y \in \overline{R(x)} \Rightarrow$ there exists $(y_\beta)_{\beta \in J}$ a generalized sequence so that $y_\beta \in \overline{R(x)}$ and $y_\beta \to y$. Observe that $x R y_\beta \Rightarrow x R y \Rightarrow y \in \overline{R(x)} \Rightarrow R(x) = \overline{R(x)}$, $x \in \overline{R^{-1}(y)} \Rightarrow$ there exists $(x_\alpha)_{\alpha \in I}$ a generalized sequence so that $x_\alpha \in \overline{R^{-1}(y)}$ and $x_\alpha \to x$. Observe that $x_\alpha R y \Rightarrow x R y \Rightarrow x \in \overline{R^{-1}(y)} \Rightarrow x R y$. $\Box$

Theorem 16. $X$ is $T_2^R$ space iff for each generalized sequences $(x_\alpha)_{\alpha \in I}$ and $(y_\alpha)_{\alpha \in I}$ so that $x_\alpha R y_\alpha$, $x_\alpha \to x$ and $y_\alpha \to y$, we have $x R y$.

Proof. We use Theorem 6.

$X$ is $T_2^R$ space iff $R \subseteq X \times X$ is a closed set $\iff$ for each $(x_\alpha, y_\alpha) \in R$ so that $(x_\alpha, y_\alpha) \to (x, y)$ we have $(x, y) \in R \iff$ for each $x_\alpha \to x; y_\alpha \to y$, $x_\alpha R y_\alpha$ it follows $x R y$. $\Box$

Example 17. If "$\leq$" is an order relation on $X$ and if $X$ is a $T_1^R$ space then

$x_\alpha \leq y; x_\alpha \to x \Rightarrow x \leq y; x \leq y_\beta; y_\beta \to y \Rightarrow x \leq y$.

If $X$ is a $T_2^R$ space, then from $x_\alpha \leq y_\alpha; x_\alpha \to x; y_\alpha \to y$ results $x \leq y$.

Example 18. Let "$<$" be an order relation on $R$: $x < y$ iff $y - x \in \mathbb{N}$. $R$ is $T_1^<$ space for $i \in \{0, 1, 2\}$ but it is not $T_i^<$ space for $i \in \{3, 4\}$.

Proof. (a) It is obvious that "$<$" is an order relation on $R$ and more: $< (x) = x + \mathbb{N}$, $<^{-1} (y) = y - \mathbb{N}$, for each $x, y \in \mathbb{R}$. $x_n < y_n \Rightarrow y_n - x_n \in \mathbb{N}$, where $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are real sequences. $x_n \to x, y_n \to y \Rightarrow y_n - x_n \to y - x$. From here exists $n_0 \in \mathbb{N}$ so that $y_n - x_n \in (y - x - 1/2, y - x + 1/2)$ for each $n \geq n_0$. Because $y_n - x_n \in \mathbb{N}$ it follows $y_n - x_n = m \in \mathbb{N}$, so $y_n - x_n$ is constant for...
\[ n \geq n_0 \Rightarrow y_n - x_n = y - x \in \mathbb{N} \Rightarrow x < y. \] Using Theorem 16 results \( R \) is \( T^*_\varepsilon \) space, so \( T^*_\varepsilon \) space for every \( i \in \{0, 1, 2\}. \)

(b) \( R \) is not \( \prec \) regular space. From Theorem 9 we have that for \( x = 0 \), if \( V = \bigcup_{n=0}^{\infty} \left( n - \frac{1}{2^n}, n + \frac{1}{2^n} \right) \in V_{\prec(0)} \), because \( \prec (0) = \mathbb{N} \). \( R \) is \( \prec \) regular space \( \Rightarrow \) exists \( U \in V_0 \) so that \( \prec (U) \subset V \). Let \( \varepsilon > 0 \) be so that \( U_0 = (-\varepsilon, \varepsilon) \subset U \). We have \( \prec (U_0) \subset \prec (U) \subset \prec (U) \subset V \) and then \( U_0 + \mathbb{N} \subset V \). So \( \bigcup_{n=0}^{\infty} \left( n - \frac{1}{2^n}, n + \frac{1}{2^n} \right) \). It follows \( 0 < \varepsilon \leq \frac{1}{2^n} \) for each \( n \in b \Rightarrow \varepsilon = 0 \), contradiction. Therefore \( R \) is not \( T^*_\varepsilon \) space for \( i \in \{3, 4\} \). \( \square \)

2 \ Spaces separated by equivalence relations

In this section we will consider the case of an equivalence relation \( R = \rho \). First we define a notion of continuity of a binary relation.

Definition 19. (a) A binary relation \( \rho \) on a topological space \( (X, \mathcal{T}) \) is continuous iff for every \( D \in \mathcal{T} \) an open set, \( \rho^{-1}(D) \in \mathcal{T} \) is an open set.

(b) \( \rho \) is an open relation iff for each \( D \in \mathcal{T}, \rho(D) \in \mathcal{T} \) is an open set.

Remark 8. If \( \rho = \rho \) is an equivalence relation, then \( \rho = \rho^{-1} \) so (a) \( \iff \) (b).

Theorem 20. Let \( \rho \) be an equivalence relation on \( X \) and \((\hat{X}, \hat{\mathcal{T}})\) be the quotient space. If \( \hat{X} \) is \( T_i \) space then \( X \) is \( T^\rho_i \) space for each \( i \in \{0, 4\} \).

Proof. Let \( \hat{x} = p(x) \) be the equivalence class of each \( x \in X \), and \( p : X \to \hat{X}, p(x) = \hat{x} \) be the canonical projection. Suppose \( \hat{X} \) is \( T_2 \) space and let \( x, y \in X \) be so that \( x \sim y \Rightarrow \hat{x} \neq \hat{y} \), more, \( p(x) \cap p(y) = \emptyset \). There exists \( V_x \in V_x \) and \( V_y \in V_y \) neighborhoods of \( \hat{x}, \hat{y} \) so that \( V_x \cap V_y = \emptyset \).

If \( V_x = p^{-1}(V_x) \) and \( V_y = p^{-1}(V_y) \), then, as \( p \) is continuous we have \( V_x \in V_x \) and \( V_y \in V_y \) and more because \( V_x \cap V_y = \emptyset \Rightarrow V_x \cap V_y = \emptyset \Rightarrow X \) is \( T^\rho_2 \) space. The cases of \( i \in \{0, 1, 2, 3, 4\} \) can be proved analogously. \( \square \)

This theorem has a converse given by:

Theorem 21. If \( \rho \) is an equivalence relation on \( X \) and if \( \rho \) is continuous then \( X \) is \( T^\rho_i \) space \( \Rightarrow \hat{X} \) is \( T_i \) space, for each \( i = \{0, 4\} \).

Proof. We will prove this result only in the case of \( i = 2 \). Suppose \( X \) is \( T^\rho_2 \) space. First, observe that \( p \) is an open map \( \iff \rho \) is an open relation. Let be \( \hat{x}, \hat{y} \in \hat{X} \) so that \( \hat{x} \neq \hat{y} \Rightarrow x \not\sim y \Rightarrow \) there exists \( V_x \in V_x \) and \( V_y \in V_y \) so that \( V_x \not\supseteq V_y \). Because \( p \) is open map we have that \( V_x = p(V_x) \) and \( V_y = p(V_y) \) are neighborhoods of \( \hat{x} \) and \( \hat{y} \) and more \( V_x \cap V_y = \emptyset \Rightarrow \hat{x} \neq \hat{y} \Rightarrow \hat{X} \) is \( T_2 \) space. \( \square \)

Example 22. 1) Let \( X = \mathbb{R} \) be with usualy topology and \( \rho \) the equivalence on \( \mathbb{R} \) defined by: \( x \not\sim y \iff x - y \in \mathbb{Q} \). Observe that for each \( D \) open set, \( \rho(D) = D + \mathbb{Q} = \mathbb{R} \), so \( \rho \) is a continuous relation. From here we can see that \( R \) is not \( T^\rho_2 \) space because for each \( x \not\sim y \), \( V_x \in V_x, V_y \in V_y \) results \( \rho(V_x) = \rho(V_y) = \mathbb{R} \) so \( y \not\in \rho(V_y) \) and \( x \not\in \rho(V_x) \).

If \( \mathbb{R} = \mathbb{R}/\rho \) would be \( T_0 \) space, then \( \mathbb{R} \) would be \( T^\rho_2 \) space, so \( \mathbb{R}/\rho \) is not \( T_2 \) space, \( i = 0, 4 \).

2) Some surfaces can be obtained as quotient spaces by identifying points of the border of a plane quadrat \( P \subset \mathbb{R}^2 \). For example the 2-sphere, \( S^2 \). Construction of
3 Function relations

Let \( f : X \to X \) be a function and \((X, \mathcal{T})\) be a topological space. We will establish some properties of \( T^i \) spaces, \( i = 0, 1, \ldots \).

**Theorem 23.** If \( X \) is \( T_0 \) space and \( f \) is continuous, then \( X \) is \( T^i_0 \) space.

**Proof.** Observe that if \( f \) is continuous then \( f(A) \subset f(A) \) and \( f^{-1}(B) \subset f^{-1}(B) \) for each \( A, B \) subsets of \( X \). \( X \) is \( T_0 \) space iff:

\[
\begin{align*}
&\text{(1)} & \quad x \in \overline{\{ y \}} \text{ and } y \in \overline{\{ x \}} \Rightarrow x = y.
\end{align*}
\]

We shall prove:

\[
\begin{align*}
&\text{(2)} & \quad y \in \{ f(x) \} \text{ and } x \in \{ f^{-1}(y) \} \Rightarrow f(x) = y.
\end{align*}
\]

\( x \in \{ f^{-1}(y) \} \) and \( \{ f^{-1}(y) \} \subset f^{-1}(\overline{\{ y \}}) \Rightarrow x \in f^{-1}(\overline{\{ y \}}) \), because \( f \) is continuous. From here \( f(x) \in \overline{\{ y \}} \) and \( f(x) \in \{ f^{-1}(y) \} \) \( \Rightarrow y = f(x) \), so \( X \) is \( T^i_0 \) space (2). \( \Box \)

**Theorem 24.** If \( X \) is \( T^i_0 \) space and if \( f \) is bijection, having the inverse \( f^{-1} \) continuous, then \( X \) is \( T^i_0 \) space.

**Proof.** Let \( x, y \in X \) be. We shall prove that: \( x \in \overline{\{ y \}} \text{ and } y \in \overline{\{ x \}} \Rightarrow x = y \), i.e. \( X \) is \( T^i_0 \) space. Since \( f \) is bijection \( \Rightarrow \) exists \( z \in X \) so that \( y = f(z) \). Suppose that \( x \in \{ f(z) \} \) and \( f(z) \in \{ x \} \Leftrightarrow x \in \overline{\{ y \}} \) and \( y \in \overline{\{ x \}} \). Because \( f^{-1} \) is continuous we have: \( z \in f^{-1}(\overline{\{ x \}}) \subset f^{-1}(\overline{\{ y \}}) \) and so \( z \in \{ f^{-1}(x) \} \). From here \( x \in \{ f(z) \} \) and \( x \in \overline{\{ f^{-1}(x) \}} \Rightarrow x = f(z) \), because \( X \) is \( T^i_0 \) space. So we have \( x = f(z) = y \), and then \( X \) is \( T^0 \) space. \( \Box \)

**Theorem 25.** If \( X \) is \( T^i_1 \) space and \( f \) is continuous, then \( X \) is \( T^i_1 \) space.

**Proof.** Observe that \( X \) is \( T^i_1 \) space iff \( \{ f(x) \} \) and \( \{ f^{-1}(y) \} \) are closed sets, for each \( x, y \in X \). \( X \) is \( T^i_1 \) space \( \Rightarrow \) \( \{ f(x) \} \) is closed set (1). Because \( \{ y \} \) is a closed set and \( f \) is continuous, we have that \( f^{-1}(y) \) is a closed set (2). From (1) and (2) we have that \( X \) is \( T^i_1 \) space. \( \Box \)

**Theorem 26.** If \( X \) is \( T^i_1 \) space and \( f \) is onto, then \( X \) is \( T^i_1 \) space.

**Proof.** \( X \) is \( T^i_1 \) space \( \Rightarrow \) \( \{ f(x) \} \) is a closed set for each \( x \in X \). Because for each \( y \in X \), exists \( x \in X \) so that \( y = f(x) \), we have that \( \{ y \} = \{ f(x) \} \) is a closed set so \( X \) is \( T^i_1 \) space. \( \Box \)

**Theorem 27.** If \( X \) is \( T^i_2 \) space and \( f \) is continuous, then \( X \) is \( T^i_2 \) space.

**Proof.** Note \( G_f \) the graph of \( f \). We shall prove that \( CG_f \) is an open subset of \( X \times X \). Let be \( (x, y) \in CG_f \). Then \( f(x) \neq y \). Because \( X \) is \( T^i_2 \) space, there exists \( V \in \mathcal{V}_y \). We note by \( \text{Int} \) \( P \) the interior of \( P \) and by \( bP \) the border of \( P \). We consider \( P \) as a topological subspace of \( \mathbb{R}^2 \). See that if \( D \) is an open set of \( P \), then \( D \cup bP \) is also an open set of \( P \). We define on \( P \) the relation \( \rho : xPy \Leftrightarrow x = y \) or \( x, y \in bP \). We see that \( \rho \) is a continuous equivalence relation, since if \( D \) is an open subset of \( P \):

a) If \( D \cap bP = \emptyset \) then \( \rho(D) = D \) is an open subset of \( P \).

b) If \( D \cap bP \neq \emptyset \) then \( \rho(D) = D \cup bP \) is also an open subset of \( P \).

We define \( S^2 = P/\rho \) the quotient space of \( P \). It is not difficult to see that \( S^2 \) is homeomorphic with any sphere of \( \mathbb{R}^3 \). \( P \) is a \( T^2 \) space, using Theorem 16 of Section 1.
and $V_1 \in \mathcal{V}_{f(x)}$ so that $V \cap V_1 = \emptyset$. But $f$ is continuous and from here! there exists $U \in \mathcal{V}_x$ so that $f(U) \subset V \implies f(U) \cap V = \emptyset \implies U \times V \subset CG_f \implies CG_f$ is a open set $\implies G_f$ is a closed set $\implies X$ is $T^f_2$ (see Theorem 4 Section 1).

**Remark 9.** Theorem 27 states that each continuous function on a $T_2$ space has a closed graphical.

**Theorem 28.** If $X$ is $T^f_2$ space and $f$ is bijection having the inverse $f^{-1}$ continuous, then $X$ is $T^f_2$ space.

**Proof.** Let $y \in X$ be $\Rightarrow$ there exists $x \in X$ so that $y = f(x) \implies x = f^{-1}(y)$. Because $f^{-1}$ is continuous we have: For each $U \in \mathcal{V}_x$ there exists $V \in \mathcal{V}_y$ so that $f^{-1}(V) \subset U \implies V \subset f(U) \Rightarrow \overline{V} \subset \overline{f(U)}$. From here we infer

$$y \in \bigcap_{V \in \mathcal{V}_y} \overline{V} = \bigcap_{V \in \mathcal{V}_{f(x)}} \overline{V} \subset \bigcap_{V \in \mathcal{V}_y} \overline{f(U)} = \{ f(x) \} = \{ y \},$$

because $X$ is $T^f_2$ (Theorem 4 Section 1). So we have: $\bigcap_{V \in \mathcal{V}_y} \overline{V} = \{ y \}$ for each $y \in X$, and then $X$ is $T^f_2$ space.

**Theorem 29.** If $X$ is $T^f_2$ space and compact, then $f$ is continuous.

**Proof.** $X$ is $T^f_2$ space $\Rightarrow$ $\bigcap_{U \in \mathcal{V}_x} \overline{f(U)} = \{ f(x) \}$ for each $x \in X$. Hence $\bigcup_{U \in \mathcal{V}_x} \overline{f(U)} = X \setminus \{ (f(x)) \}$, for an arbitrary point $x$ of $X$. Let $V \in \mathcal{V}_{f(x)}$; then $X = \bigcup_{U \in \mathcal{V}_x} \overline{f(U)} \cup V$. Because $X$ is a compact space there exist $U_1, U_2, ..., U_n \in \mathcal{V}_x$ such that

$$X = \overline{f(U_1)} \cup \overline{f(U_2)} \cup ... \cup \overline{f(U_n)} \cup V = C \left[ \overline{f(U_1)} \cup \overline{f(U_2)} \cup ..., \cup \overline{f(U_n)} \right] \cup V.$$

Let $U \in \mathcal{V}_x$ be so that $U \subset \bigcap_{i=1}^{n} U_i$. We have $f(U) \subset \bigcap_{i=1}^{n} f(U_i)$. Hence

$$C \overline{f(U)} \supset C \left[ \bigcap_{i=1}^{n} \overline{f(U_i)} \right] \implies \overline{C \overline{f(U)}} \cup V = X \Rightarrow C \overline{f(U)} \cap CV = \emptyset \Rightarrow f(U) \subset V.$$

We have proved that $f$ is continuous in $x$. So $f$ is continuous on $X$.

**Remark 10.** This theorem is in fact an alternative of the principle of the closed graphical.

**Theorem 30.** If $X$ is $f_r$-regular space, then $f$ is continuous.

**Proof.** It is a consequence of Theorem 9 Section 1.

**Theorem 31.** If $X$ is $f$-regular space and $f$ is bijection, then $f$ is homeomorphism.

**Proof.** It is a consequence of Theorem 9 Section 1.

**Theorem 32.** If $X$ is a regular space and $f$ is continuous, then $X$ is $f_r$-regular space.

**Proof.** Let $x \in X$ be and $V \in \mathcal{V}_{f(x)}$ be $\Rightarrow$ there exists $V_1 = \overline{V_1} \in \mathcal{V}_{f(x)}$ so that $V_1 \subset V$, because $X$ is regular space. As $f$ is continuous there exists $U \in \mathcal{V}_x$ so that

$$f(U) \subset V_1 \Rightarrow \overline{f(U)} \subset \overline{V_1} \Rightarrow \overline{f(U)} \subset V.$$

Using now Theorem 9 Section 1, it follows that $X$ is $f_r$-regular space.
Theorem 33. If $X$ is regular space and $f$ is homeomorphism, then $X$ is $f$ regular space.

Proof. It is a consequence of Theorem 9 Section 1. \hfill \Box

Theorem 34. If $X$ is $T_1'$ space (i.e. $f$-normal and $T_1$) then $f$ is continuous. If $f$ is bijection, then it is a homeomorphism.

Proof. It is a consequence of Theorem 11 Section 1, if we remark that $X$ is $T_1$ space $\Rightarrow\{x\}$ is a closed set for each $x \in X$. \hfill \Box

Theorem 35. (a) If $f$ is continuous and $X$ is regular space, then $X$ is $f_r$-regular.

(b) If $f$ is homeomorphism and $X$ is regular space, then $X$ is $f$-regular.

Proof. (a) Let $x \in X$ be; for each $V \in \mathcal{V}_f(x)$, there exists $U \in \mathcal{V}_x$ so that $f(U) \subset V$. Suppose $V = \overline{V}$, because $X$ is regular space. From here $\overline{f(U)} \subset \overline{V}$; because $V = \overline{V} \Rightarrow X$ is $f_r$-regular space.

(b) It is a consequence of (a), observing that $f^{-1}$ is continuous. \hfill \Box

Theorem 36. If $f$ is homeomorphism and $X$ is normal space, then $X$ is $f$-normal.

Proof. Let $F$ be a closed subset of $X \Rightarrow f(F)$ is a closed set, because $f$ is homeomorphism. If $V \in \mathcal{V}_f(F)$ then $V \in \mathcal{V}_f(x)$ for each $x \in F$. But $f$ is continuous $\Rightarrow$ there exists $U_x \in \mathcal{V}_x$ so that $f(U_x) \subset V$. Let $U = \bigcup_{x \in F} U_x$ be; notice $U$ is neighborhood of $F$ it follows $f(U) = \bigcup_{x \in F} f(U_x) \subset V$. Because $X$ is normal space, suppose $V = \overline{V} \Rightarrow \overline{f(U)} \subset V$ (1).

In the same way replacing $f$ with $f^{-1}$ that is continuous too, we will have: for each $F$ a closed subset of $X$, and for each $U \in \mathcal{V}_{f^{-1}}(x)$ there exists $V \in \mathcal{V}_F$ so that $f^{-1}(V) \subset U$, supposing $U = \overline{U}$ (2) From (1), and (2), using Theorem 11 Section 1 we have that $X$ is $f$-normal space. \hfill \Box

References


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