Variational Solutions of Stationary Hamilton-Jacobi Equations

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Abstract

This work deals with the stationary Hamilton-Jacobi equation \( F(B^* \psi_y) = \langle Ay, \psi_y \rangle = g \) in the class of convex continuous functions \( \psi \) on a real Hilbert space \( H \). After obtaining an asymptotic result, the existence, the uniqueness and a Galerkin approximation of the solution to the above equation are established.

Mathematics Subject Classification: 49L20, 49A15
Key words: Hamilton-Jacobi equations, variational solutions

Let \( H \) and \( U \) be two real Hilbert spaces with the scalar products \( \langle \cdot, \cdot \rangle \) and \( < \cdot, \cdot > \), respectively. The norms of \( H \) and \( U \) will be denoted by \( | \cdot | \) and \( | \cdot |_U \), respectively. We define the function \( \psi^\infty : H \to \mathbb{R} \)

\[
\psi^\infty(y) = \inf \left\{ \int_0^\infty (g(x(t)) + h(u(t)))dt; \ x' = Ax + Bu, \ x(0) = y; \ u \in L^1(\mathbb{R}^+; U) \right\}
\]

in the following hypotheses:

(i) \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions on \( H \)
   i.e., \( A \) is \( m \)-dissipative on \( H \);
   \( B \) is a linear continuous operator from \( U \) to \( H \).

(ii) The function \( g : H \to \mathbb{R} \) belongs to \( K \cap C_{lip}^1(H) \) and satisfies the following two conditions:

\[
g(y_n) \to 0 \text{ for some sequences } \{y_n\} \text{ then } y_n \to 0;
\]

\[
g(e^{-(\lambda - A)t}y) \in L^1(\mathbb{R}^+) \text{ for } \lambda > 0 \text{ and all } y \in H.
\]

Here \( e^{-(\lambda - A)t} \) is the semigroup generated by the operator \( -\lambda I + A \), \( K \) is the closed convex cone of \( C(H) \) consisting of all convex functions \( \varphi : H \to \mathbb{R} \) which satisfy the condition \( \varphi(y) \geq \varphi(0) = 0 \) for all \( y \in H \) and \( C_{lip}^1(H) \) is the space of all \( \varphi \in C^1(H) \) such that the Fréchet derivative \( \varphi^{(1)} \) is Lipschitz on \( H \).

(iii) \( F : U \to \mathbb{R} \) is a convex function which is bounded on bounded subsets and belongs to \( C^1(U) \). We associate the function \( h : U \to \mathbb{R}_+ \) defined by

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\[ h(u) = \sup \{ - \langle p, u \rangle + F(p); p \in U \} \text{ i.e., } h(u) = F^*(u), \]

to the function \( F \). Then the assumption (iii) implies \[ \lim_{|u|_H \to \infty} \frac{h(u)}{|u|_H^r} = +\infty. \] Further assume that \( F \) and \( F^* \) are strictly convex.

(iv) \( \psi_0 : H \to \mathbb{R} \) is a function which belongs to \( K \cap C^1_{L^0}(H) \).

Under the above hypotheses, we see that \( \Psi^\infty < +\infty \) on \( H \) and for every \( y \in H \) the infimum defining \( \Psi^\infty(y) \) is attained in a unique pair \((x^*, u^*)\). Moreover, the function \( \Psi^\infty \) is convex and lower semicontinuous on \( H \). Since it is everywhere finite we conclude that it is continuous on \( H \).

Now we shall prove the following asymptotic result.

**Theorem 1.** Let assumptions (i), (ii), (iii) and (iv) be satisfied. Then the solution \( \Psi \) to the problem

\[ \begin{align*}
\psi_t + F(B^* \psi_y) - (Ay, \psi_y) &= g; \quad t \geq 0, \quad y \in D(A) \\
\psi(0, y) &= \psi_0(y), \quad y \in H
\end{align*} \]

satisfies

\[ \lim_{t \to \infty} \psi(t, y) = \psi^\infty(y) \text{ for all } y \in H. \]

We have denoted by \( \Psi_t \) and \( \Psi_y \) the partial derivatives (in a generalized sense) of the function \( \Psi \) with respect to \( t \) and \( y \), respectively. Here \( B^* \) is the adjoint of \( B \).

**Proof:** The solution \( \Psi \) to the Cauchy problem (4) is given by (see [1], p.54)

\[ \begin{align*}
\psi(t, y) &= \inf \left\{ \int_0^t (g(x(s)) + h(u(s)))ds + \psi_0(x(t)); \quad x' = Ax + Bu, \right. \\
x(0) &= y; \quad u \in L^1(0, t; U) \right\}
\]

for \( t \geq 0, \ y \in H \). This yields

\[ \int_0^t (g(x'(s)) + h(u'(s)))ds + \psi_0(x'(t)) = \psi(t, y) \leq \int_0^t (g(x^*(s)) + h(u^*(s)))ds + \psi_0(x^*(t)) \]

where \((x^*, u^*)\) is the optimal pair in Problem (1) and \((x', u')\) in (6), i.e.,

\[ \begin{align*}
(x')' &= Ax' + Bu' \text{ on } [0, t] \\
x'(0) &= y.
\end{align*} \]

Letting \( t \to +\infty \), we see by (7) that

\[ \lim \sup_{t \to \infty} \psi(t, y) \leq \psi^\infty(y). \]

We set

\[ \hat{u}'(s) = \begin{cases} u'(s) & \text{for } 0 \leq s \leq t \\ 0 & \text{for } s > t \end{cases} \]
and
\[ \tilde{x}'(s) = e^{As}y + \int_0^s e^{A(s-\tau)}B\tilde{u}(\tau)d\tau. \]
Let some sequence \( t_n \to \infty \) and let \( \{\tilde{x}^n\} \) and \( \{\tilde{u}^n\} \) be such that
\[ \psi_\infty(y) \leq \int_0^\infty (g(\tilde{x}^n(s)) + h(\tilde{u}^n(s)))ds \leq \psi_\infty(y) + \frac{1}{n}, \]
\[ (\tilde{x}^n)' = A\tilde{x}^n + B\tilde{u}^n \quad \text{on} \quad R^+ \]
\[ \tilde{x}^n(0) = y. \]
Since \( |\tilde{x}^n(s)| \leq C \left(|y| + \int_0^s \|\tilde{u}^n(\tau)\|d\tau\right), \forall s \geq 0 \) and \( g(\tilde{x}^n(s)) \geq g(0) - |\partial g(0)||\tilde{x}^n(s)|, \)
\( \forall s \geq 0 \) by (9) it follows that \( \int_0^\infty h(\tilde{u}^n(s))ds \leq C \left(\int_0^\infty \|\tilde{u}^n(s)\|ds + 1\right) \) where \( C \) is independent of \( n \). The latter implies that \( \{\tilde{u}^n\} \) is weakly compact in \( L^1(R^+; U) \) (by virtue of the Dunford-Pettis theorem). Hence on some sequence \( t_n \to \infty \), \( \tilde{u}^n \to \tilde{u} \)
weakly in \( L^1(R^+; U) \) and \( \tilde{x}^n(s) \to \tilde{x}(s) \) weakly in \( H \) for every \( s \geq 0 \), where \( \tilde{x} \) is the mild solution to
\[ \begin{cases} 
    \tilde{x}' = Ax + Bu & \text{on} \quad R^+ \\
    x(0) = y.
\end{cases} \]
Since the functional \((x, u) \to \int_0^T (g(x) + h(u))dt\) is weakly lower semicontinuous on every \( C([0, T]; H) \times L^1(0, T; U) \) (because it is convex and lower semicontinuous), it follows by (7) and (8) that
\[ \int_0^\infty (g(\tilde{x}(s)) + h(\tilde{u}(s)))ds \leq \limsup_{t \to \infty} \psi(t, y) \leq \psi_\infty(y) \]
which obviously implies (5) as claimed. \( \Box \)

Now let us consider the stationary Hamilton-Jacobi equation
\[ F(B^*\psi_y) - (Ay, \psi_y) = g, \quad \forall y \in D(A) \]
in the real Hilbert space \( H \).

By a solution to (10) we shall mean a function \( \psi : H \to R \) which is Gâteaux differentiable and satisfies Eq. (10) for all \( y \in D(A) \).

Equation (10) is relevant in the calculus of variations with infinite horizon and in nonlinear analysis. The main result is

**Theorem 2.** Let assumptions (i), (ii) part (3) and (iii) be satisfied. Then Eq. (10) has at least one solution \( \psi_\infty \in K \). If also condition (2) holds then the solution \( \psi_\infty \) to Eq. (10) is unique.

**Proof.** For every \( t \geq 0 \), the unique optimal pair \((x^*, u^*)\) in Problem (1) is also an optimal pair for the finite horizon control problem
\[ \psi_\infty(y) = \inf \left\{ \int_0^t (g(x(s)) + h(u(s)))ds + \psi_\infty(x(t)); \ x' = Ax + Bu, \ x(0) = y \right\} \]
\[ = \int_0^t (g(x^*(s)) + h(u^*(s)))ds + \psi_\infty(x^*(t)). \]
By the maximum principle ([1], p. 13), for every \( t \geq 0 \) there exists \( p^t \in C([0,t]; H) \) which satisfies the system

\[
\begin{cases}
(p^t)' + A^*p^t \in \partial g(x^t), & 0 \leq s \leq t \\
B^*p^t \in \partial h(u^t), & 0 \leq s \leq t \\
p^t(t) \in -\partial \psi^\infty(x^t(t))
\end{cases}
\]

(Here \( \partial \) is the subdifferential symbol). Next by (11) we see that

\[
\psi^\infty(x^t(t)) = \int_t^\infty (g(x^s(s)) + h(u^s(s)))ds, \quad \forall t \geq 0.
\]

Assume that \( y \in D(A) \). Then \( x^t \) is right differentiable at \( t = 0 \) and \( \frac{d^+x^t}{dt}(0) = Ay + Bu^t(0) \). Since

\[
\lim_{t \to 0} \frac{\psi^\infty(x^t(t)) - \psi^\infty(y)}{t} \to \left( \eta, \frac{d^+x^t}{dt}(0) \right)
\]

where \( \eta \) is an element of \( \partial \Psi^\infty(y) \), it follows by (12) that

\[
(\eta, Ay + Bu^t(0)) + g(y) + h(u^t(0)) = 0
\]

where \( u^t(0) = \nabla h^*(-B^*\eta) \) and \( \eta \in \partial \psi^\infty(y) \) (we may take \( \eta = -p^t(0) \)). By using (13) and the conjugacy formula, we get

\[
F(B^*\eta) - (Ay, \eta) = g(y).
\]

To conclude the proof of existence it suffices to show that \( \Psi^\infty \) is Gâteaux differentiable, i.e., \( \partial \Psi^\infty \) is single valued.

To this end we define the operator \( \Gamma : H \to H \), \( \Gamma y = -p(0) \) where \( p \in C([0, T]; H) \) is the solution to the system (\( T \) is fixed)

\[
\begin{cases}
x' = Ax + Bu, & 0 \leq t \leq T \\
p' + A^*p \in \partial g(x), & 0 \leq t \leq T \\
x(0) = y, \quad p(T) \in -\partial \psi^\infty(x(T))
\end{cases}
\]

which, by virtue of the maximum principles is equivalent with the control problem

\[
\inf \left\{ \int_0^T (g(x) + h(u))dt + \Psi^\infty(x(T)); \ x' = Ax + Bu, \ x(0) = y \right\}.
\]

Since \( F \) and \( F^* \) are strictly convex, so is \( h^* \) and therefore there exists a unique optimal pair \( (x, u) \) for the problem (15). Since \( F \) is Gâteaux differentiable, \( \partial F^* \) and therefore \( \partial h \) is single valued. Thus there exists a unique \( B^*p = \partial h(u) \) which satisfies the system (14) and therefore \( \Gamma \) is single valued. On the other hand, we see that \( \Gamma y \in \partial \psi^\infty(y) \). To prove that \( \Gamma = \partial \psi^\infty \) it suffices to show that the range \( R(I + \Gamma) \) is all of \( H \). To this end we consider the equation \( y + \Gamma y = w \) which is equivalent to
\[ \begin{aligned}
&x' = Ax + Bu, \quad 0 \leq t \leq T \\
p' + A^*p \in \partial g(x), \quad 0 \leq t \leq T \\
x(0) = y, \quad p(0) = y - w \\
p(T) \in -\partial \Psi^\infty(x(T)) \\
B^*p \in \partial h(u), \quad 0 \leq t \leq T
\end{aligned} \]

(16)

and again by virtue of the maximum principle it is equivalent to the control problem

\[
\inf \left\{ \int_0^T (g(x) + h(u))dt + \Psi^\infty(x(T)) + \frac{1}{2}|x(0) - w|^2; \quad x' = Ax + Bu \right\}
\]

which clearly admits at least one solution which is also a solution to (16). Hence \( \Gamma = \partial \Psi^\infty \), and \( \partial \Psi^\infty = (\Psi^\infty)' \) is single valued as claimed.

As for uniqueness we consider the differential equation

\[
\begin{aligned}
x' &= Ax - B\nabla h^*(B^*\Psi^0_x(x)), \quad t \geq 0 \\
x(0) &= y
\end{aligned}
\]

(17)

where \( \Psi^0 \) is any solution to Eq. (10). For each \( y \in D(A) \), Eq. (17) has a unique solution \( \hat{x} \in W^{1,\infty}(R^+;H) \). We set \( \hat{u} = -\nabla h^*(B^*\Psi^0_x(x)) \) and take the inner product of (17) by \( -\Psi^0_{xx}(\hat{x}) \). Since \( \Psi^0 \) is a solution to Eq. (10) we get

\[
\frac{d}{dt}\Psi^0(\hat{x}(t)) - F(B^*\Psi^0_x(\hat{x}(t))) + g(\hat{x}(t)) + (\hat{u}(t), -B^*\Psi^0_x(x^*(t))) = 0
\]

a.e. \( t \geq 0 \). By using the conjugacy formula, the latter becomes

\[
\frac{d}{dt}\Psi^0(\hat{x}(t)) + g(\hat{x}(t)) + h(\hat{u}(t)) = 0 \quad \text{a.e.} \quad t \geq 0
\]

and therefore

\[
\Psi^0(y) = \Psi^0(\hat{x}(t)) + \int_0^t (g(\hat{x}(s)) + h(\hat{u}(s)))ds, \quad \forall t \geq 0.
\]

Finally

\[
\Psi^0(y) = \int_0^\infty (g(\hat{x}(s)) + h(\hat{u}(s)))ds \geq \Psi^\infty(y)
\]

where \( \Psi^\infty \) is given by (1). (To get (19) we have used the fact that \( \hat{x}(t_n) \to 0 \) for some \( t_n \to \infty \) because \( g(\hat{x}) \in L^1(R^+) \) and \( g \) satisfies condition (2)).

Now let \((x^*, u^*)\) be the optimal pair for Problem (1) \((y \in D(A))\). Again by Eq. (10) we have

\[
\begin{aligned}
\frac{d}{dt}\Psi^0(x^*(t)) &= (x^*(t), \Psi^0_x(x^*(t))) = (Ax^*(t) + Bu^*(t), \Psi^0_x(x^*(t))) = \\
&= F(B^*\Psi^0_x(x^*(t))) - g(x^*(t)) - (u^*(t), -B^*\Psi^0_x(x^*(t))) \\
&\geq -h(u^*(t)) - g(x^*(t)) \quad \text{a.e.} \quad t > 0
\end{aligned}
\]
and therefore
\[ \Psi^0(x^*(t)) + \int_0^t (h(u^*(s)) + g^*(x^*(s)))ds \geq \Psi^0(y), \quad \forall t \geq 0. \]
Since \( g(x^*) \in L^1(\mathbb{R}^+) \) it follows that \( \lim_{t_n \to \infty} x^*(t_n) \to 0 \) for some sequence \( t_n \to \infty \)
and therefore \( \lim_{t_n \to \infty} \Psi^0(x^*(t_n)) = \Psi^0(0) = 0. \) Hence \( \Psi^0(y) \leq \Psi^\infty(y) \) for all \( y \in D(A) \).
Along with (19) the latter implies that \( \Psi^\infty = \Psi^0 \) as claimed. \( \square \)

Now we shall briefly discuss a Galerkin approximation of the stationary Hamilton-Jacobi equation (10). Assume that the hypotheses (i)-(iii) hold. It will be more convenient to regard \( A \) as a linear continuous operator from the space \( V \subset H \) to its dual \( V' \subset H \). Following a well-known scheme (see for instance [1], [2], [3]), the internal approximations of the spaces \( V \) and \( U \), of the operators \( A \) and \( B^* \) and of the functions \( F \) and \( g \) will be defined. Thus on the finite dimensional space \( V_h \), Eq. (10) becomes
\[ (20) \quad F_h(B_h^*\Psi_h^y(y_h)) - (A_h y_h, \Psi_h^y(y_h))_h = g_h(y_h) \quad \text{for all} \quad y_h \in V_h. \]
By Theorem 2, Eq. (20) has a generalized solution \( \Psi_h : V_h \to R \) given by
\[ \Psi_h^y(y_h) = \inf \left\{ \int_0^\infty (g_h(x_h(t)) + F_h^*(\Psi_h^y(t)))dt \right\} \]
where \( x_h, u_h \) satisfy the system
\[
\begin{cases}
  x'_h = A_h x_h + B_h u_h \ a.e. \ t > 0 \\
  x_h(0) = y_h.
\end{cases}
\]
As regards the convergence of the solutions \( \Psi_h \) to a solution \( \Psi \) to Eq. (10), we have the following result
\textbf{Theorem 3.} For \( h \to 0 \) we have \( \Psi_h^y(y_h) \to \Psi^y(y) \) for all \( y \in V \), where \( \Psi \) is the \textit{variational solution} to Eq. (10) given by (1).
\textbf{The proof} is similar (making obvious changes) to the proofs of the analogous results in [1] and [3].
\textbf{Acknowledgements.} A version of this paper was presented at the Third Conference of Balkan Society of Geometers, Workshop on Electromagnetic Flows and Dynamics, July 31 - August 3, 2000, University POLITEHNICA of Bucharest, Romania.

\textbf{References}


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