On the Stationary, Potential, Subsonic Flow

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Abstract

In this communication we are concerned with the problem of the stationary, potential, subsonic flow. Firstly, we formulate the mechanical problem and the associated minimization problem with constraints, in a functional space $W$ endowed with a certain norm.

Section 2 contains the main original results. A density Lemma for the space $W$ is presented, then is introduced a stable, convergent, internal approximation of $W$. We state the approximate minimization problem and prove a convergence theorem. A minimax problem corresponds to the approximate minimization problem with constraints $(P_k)$. For this one, we deduce the existence of the saddle point, using a separation Hahn-Banach theorem. In the end is presented an iterative algorithm for determining the minimax point.

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1 Theoretical background

Let $\Omega$ be an open, bounded set in $\mathbb{R}^n (n = 2, 3)$ with the boundary $\partial \Omega$ Lipschitz continuous. The governing equations of the potential, subsonic, stationary flow are [1]

\begin{equation}
\text{div} \left[ \left( k - \frac{1}{2} |\nabla \varphi|^2 \right)^{1/\gamma - 1} \nabla \varphi \right] = 0 \text{ in } \Omega,
\end{equation}

\begin{equation}
\left( k - \frac{1}{2} |\nabla \varphi|^2 \right)^{1/\gamma - 1} \frac{\partial \varphi}{\partial n} = g \text{ on } \Gamma,
\end{equation}

\begin{equation}
|\nabla \varphi| < v_{cr},
\end{equation}

where

$$
\rho(\nabla \varphi) = \rho_0 \left( 1 - \frac{\gamma - 1}{2} \frac{|\nabla \varphi|^2}{c_0^2} \right)^{1/\gamma - 1} = \rho_0 \left( \frac{\gamma - 1}{c_0^2} \right)^{1/\gamma - 1} \left( k - \frac{|\nabla \varphi|^2}{2} \right)^{1/\gamma - 1}
$$

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is the density,
\[ k = \frac{\gamma}{\gamma - 1}, \]
\[ p = p_0 \left( \frac{\rho}{\rho_0} \right)^\gamma \] is the pressure,
\[ \bar{v} = \nabla \varphi \text{ is the velocity of the flow}, \]
\[ g \in H^{1/2}(\Gamma), \]
\[ n \text{ is the external unit normal.} \]

**Remark 1.1.** For \( |\nabla \varphi| < \sqrt{2k\frac{\gamma - 1}{\gamma + 1}} = v_{cr} \), the flow is subsonic. We denote \( \| \|_0 = \text{the norm in } L^2(\Omega), \| \|_1 = \text{the norm in } H^1(\Omega). \) We shall use the following result ([2]):

**Lemma 1.1.** Let be \( P_k \) the set of polynomials of degree less than or equal to \( k \) in the variables \( x_1, \ldots, x_n. \) The seminorm

\[ W^{k+1,p}(\Omega)/P_k \ni \bar{v} \mapsto \| \bar{v} \|_{k+1,p} = \| v \|_{k+1,p} = \left( \sum_{|\alpha| = k+1} \int_\Omega |D^\alpha v|^p \, dx \right)^{1/p} \]

is a norm over the quotient space \( W^{k+1,p}(\Omega)/P_k. \)

Further, introduce the space \( W = \left\{ \psi \in H^1(\Omega) : \int_\Omega \psi(x) \, dx = 0 \right\}. \)

**Lemma 1.2.** The mapping \( W \ni u \mapsto \| u \|_W = \| \text{grad } u \|_0 \) is a norm on \( W, \) equivalent to the norm \( \| \|_1. \)

We need a result due to Pironneau ([1]),

**Theorem 1.1.** Let be \( b < v_{cr}. \) The problem (1.1)-(1.3) is equivalent to the minimisation problem:

find \( \varphi \in K_b = \{ \psi \in W | |\nabla \varphi| \leq b \} \) so that \( J_0(\varphi) \leq J_0(\psi), \forall \psi \in K_b, \)

with

\[ J_0(\psi) = -\int_\Omega \left( k - \frac{1}{2}|\nabla \psi|^2 \right)^{\gamma/\gamma - 1} \, dx - \frac{\gamma}{\gamma - 1} \int_\Gamma g \gamma_0(\psi) \, d\sigma, \]

where we denoted by \( \gamma_0 \) the trace application. Moreover,

\[ J_0(\psi)v^2 \geq c^2 \| v \|_W^2 \quad (\forall)v \in W. \]

## 2 Main results

Let be \( T_h \) a family of regular triangulations of the polygonal (polyhedral) domain \( \Omega. \) This means

\[ \sigma(h) = \sup_{s \in T_h} \frac{\rho_s}{d_s} \leq \alpha \quad (\forall)h, \]

\[ (\forall)h, \]
where

$S$ is a simplex in $\mathbb{R}^n$, $n = 2, 3$ (triangle or tetrahedron),

$\rho_S$ = the diameter of the smallest ball containing $S$,

$\rho'_S$ = the diameter of the largest ball contained in $S$.

We proceed with the concept of stable, convergent, internal approximation $(W_h, p_h, r_h)$ for a normed space $W$ ([6]).

**Definition 2.1.** Let be $(W, \| \|)$ a normed space and $(W_h, p_h, r_h)$ a family of triples so that:

$W_h$ is a normed space,

$p_h : W_h \to W$ is a linear, continuous application,

$r_h : W \to W_h$.

The approximation $(W_h, p_h, r_h)$ of $W$ is called internal, stable and convergent if:

1. $W_h \subset W$, ($\forall) h$,

2. $\|p_h\|_{L(W_h, W)} \leq M$ independently of $h$,

3. $\lim_{\rho(h) \to 0} \|p_h r_h u - u\|_W = 0$, ($\forall) u \in W$ where $\rho(h) = \sup_{S \in T_h} \rho_S$.

We introduce the following notations: $(a_i)_{i=1,n+1}$ are the vertices of $S$;

$E_h$ the set of vertices of all simplices $S \in T_h$;

$(\lambda_i)_{i=1,n+1} = \text{the barycentric coordinates};$

$$V_h = \left\{ u_h : \Omega \to \mathbb{R} \mid u_h|_{S} = \sum_{i=1}^{n+1} u_h(a_i) \lambda_i \right\}.$$  

We denote by $(u_{h,M})_{M \in E_h}$ a basis in $V_h$, which verifies $u_{h,M}(M) = 1$, ($\forall) M \in E_h$ and $u_{h,M}(P) = 0$, ($\forall) P \neq M$, $P \in E_h$.

Then the set of functions $(w_{h,M})_{M \in E_h}$, defined by

$$w_{h,M}(x) = u_{h,M}(x) - \frac{1}{\text{meas}(\Omega)} \int_{\Omega} u_{h,M}(x)dx$$

is a basis in $W_h = \left\{ v_h \in V_h \mid \int_{\Omega} v_h(x)dx = 0 \right\}$.

**Lemma 2.1.** The set $\bar{W} = \left\{ u \in C^2(\bar{\Omega}) \mid \int_{\Omega} u(x)dx = 0 \right\}$ is dense in $(W, \| \|)$.

**Lemma 2.2.** Let be $(T_h)$ a regular triangulation of the domain $\Omega$. 
\[ \begin{align*}
\phi_h : W_h &\rightarrow W, \quad \phi_h u_h = u_h, \\
\gamma_h : W &\rightarrow W_h, \quad \gamma_h \nu(x) = \sum_{M \in E_h} w_{h, M}(x) v(M).
\end{align*} \]

Then the approximation \((W_h, \phi_h, \gamma_h)\) of \(W\) is internal, stable and convergent.

Now, we are able to approximate the minimisation problem \((1.4)\).

**Theorem 2.1.** Let \(K_{h \delta} = \{ \psi_h \in W_h \mid |\nabla \psi_h| \leq \delta \}\). The minimisation problem \((P_h)\):

find \(\varphi_h \in K_{h \delta}\) so that \(J_0(\varphi_h) \leq J_0(\psi_h), \quad \forall \psi_h \in K_{h \delta}\) admits a unique solution, for any \(h\). Moreover, \(\lim_{\rho(h) \rightarrow 0} \| \varphi_h - \varphi \|_W = 0\) where \(\varphi\) denotes the unique solution of the problem \((1.4)\).

We denote \(X_h = \left\{ \mu_h \in L^2(\Omega) \mid \mu_h = \sum_{S \in T_h} \mu_{h, S} \chi_S \right\}\), where \(\chi_S\) is the characteristic function of the set \(S\) and \(\Lambda_h = \{ \mu_h \in X_h \mid \mu_h \geq 0 \}\).

We state the following two problems:

**Primal problem.** Find \(\varphi_h \in K_{h \delta}\), so that

\[ J_0(\varphi_h) = J_0(\psi_h), \quad (\forall) \psi_h \in K_{h \delta}. \]

**The minimax problem.** Find \((\varphi_h, \lambda_h) \in W_h \times \Lambda_h\), such that

\[ L(\varphi_h, \mu_h) \leq L(\varphi_h, \lambda_h) \leq L(\psi_h, \lambda_h), \quad (\forall) \psi_h \in W_h \times \Lambda_h, \]

where

\[ L(\psi_h, \mu_h) = J_0(\psi_h) + \int_{\Omega} \mu_h (|\nabla \psi_h|^2 - \delta^2) \, dx \]

is the Lagrangean associated to the primal problem.

**Remark 2.1.** According to [4], if \((\varphi_h, \lambda_h)\) is a saddle point for the Lagrangean \(L\), then \(\varphi_h\) is solution for the primal problem.

**Theorem 2.2.** The minimax problem \((2.5)\) has a solution.

For the proof, is used the following separation theorem of Hahn-Banach type [5]:

Let be \(V\) a topological vector space. Suppose \(T\) and \(S\) are two convex sets in \(V\) such that \(T\) has at least one interior point and \(S\) does not contain any interior point of \(T\).

Then there exists a functional \(F \in V^*, F \neq 0\) and \(\alpha \in \mathbb{R}\) such that

\[ F(x) \leq \alpha \leq F(y), \quad (\forall) x \in T, \quad (\forall) y \in S. \]

**Remark 2.2.** The relation \(L(\varphi_h, \mu_h) \leq L(\varphi_h, \lambda_h), \quad (\forall) \mu_h \in \Lambda_h\) can be rewritten as:

\[ \int_{\Omega} (\mu_h - \lambda_h) J_1(\varphi_h) \, dx \leq 0, \quad (\forall) \mu_h \in \Lambda_h, \]

where \(J_1(\varphi_n) = |\nabla \varphi_n|^2 - \delta^2\).

Taking in consideration the variational characterization of the projection on a convex set in a Hilbert space, we infer
\[ \lambda_h = P_{\lambda_h}(\lambda_h + \rho J_1(\varphi_h)), \quad (\forall) \rho > 0, \]

where \( P_{\lambda_h} \) projection operator on \( \Lambda_h \). We proceed with an iterative algorithm for determining the saddle point \((\varphi_h, \lambda_h) \in W_h \times \Lambda_h \).

**Theorem 2.3.** Let be \((\varphi_{h,n})_n \subset W_h, (\lambda_{h,n}) \subset \Lambda_h \) the sequences computed by the following steps: \( \lambda_{h,0} \in \Lambda_h \) is arbitrary.

\[
L(\varphi_{h,n}, \lambda_{h,n}) \leq L(\psi_h, \lambda_{h,n}), \quad (\forall) \psi_h \in W_h,
\]

\[
\lambda_{h,n+1} = P_{\lambda_h}(\lambda_{h,n} + \rho_n J_1(\varphi_{h,n})).
\]

Then, for \( \rho_h > 0 \) suitable chosen, \( \varphi_{h,n} \xrightarrow{n \to \infty} \varphi_h \) in \( W_h \).

**Remark 2.3.** The inequality (2.7) is equivalent to the variational equation

\[
\frac{\gamma}{\gamma - 1} \int_{\Omega} \left( k - \frac{1}{2} |\nabla \varphi_{h,n}|^2 \right) \frac{\partial}{\partial \nu} \nabla \varphi_{h,n} \cdot \nabla \psi_h dx + 2 \int_{\Omega} \lambda_{h,n} \nabla \varphi_{h,n} \cdot \nabla \psi_h dx =
\]

\[
= \frac{\gamma}{\gamma - 1} \int_{\Gamma} g \psi_h d\sigma, \quad (\forall) \psi_h \in W_h
\]

**Remark 2.4.** By the variational characterization of the projection, from eq. (2.8) infer

\[
< \lambda_{h,n+1}, \mu_h - \lambda_{h,n+1} >_{L^2(\Omega)} \leq < \lambda_{h,n} + \rho_n J_1(\varphi_{h,n}), \mu_h - \lambda_{h,n+1} >_{L^2(\Omega)}, \quad (\forall) \mu_h \in \Lambda_h,
\]

which is a variational inequality of the form

\[
a(u, v - u) \leq < f, v - u >, (\forall) v \in K,
\]

where \( a : X_h \times X_h \to \mathbb{R} \) is a bilinear, symmetric, coercive form and \( K = \Lambda_h \) is a convex set. The variational inequality has an unique solution and is equivalent to the minimization problem:

find \( u \in K \), so that \( J(u) \leq J(v), \quad (\forall) v \in K, \)

where \( J(v) = \frac{1}{2} a(u, v) - < f, v >. \)

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**References**


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