On Pseudo Ricci-Symmetric Manifolds

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Abstract

In the present study we consider pseudo Ricci-symmetric manifolds in the sense of M. C. Chak. We show that pseudo Ricci-symmetric manifolds satisfying \( \text{div} R = 0 \) (resp. \( \text{div} C = 0 \)) property are Einstein (resp. Ricci flat) manifolds.

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1 Introduction

Let \( M \) be an \( n \)-dimensional \( (n \geq 3) \) Riemannian manifold. For the vector fields \( X, Y, Z \in \chi(M) \) and the the Levi-Civita connection \( \nabla \) of \( M \) the curvature tensor \( R \) and the Ricci operator \( S \) of \( M \) are defined by

\[
R(X, Y)Z = \nabla_X (\nabla_Y Z) - \nabla_Y (\nabla_X Z) - \nabla_{[X,Y]}Z,
\]

and

\[
S(X, Y) = g(SX, Y)
\]

respectively. Furthermore for the vector field \( W \) the Riemannian Christoffel curvature tensor \( R \) of \( M \) is defined by \( R(X, Y, Z, W) = g(R(X, Y)Z, W) \) \([4]\).

Let \( \Pi \) be a non-degenerate tangent plane to \( M \) at \( p \in M \) given by \( X, Y \in \chi(M) \). Then the sectional curvature \( K(\Pi) \) of \( \Pi \) defined by

\[
K(X, Y)\{g(X, X)g(Y, Y) - g(X, Y)^2\} = g(R(X, Y)Y, X)
\]

which is independent of the choice of the basis \( X, Y \) for \( \Pi \).

A tensor field \( R \) of type \((1, 2)\) on \( M \) is called algebraic curvature tensor field if it has symmetric properties of the curvature tensor field of Riemannian manifolds.

The curvature tensor \( R \) satisfies the second Bianchi identity if

\[
\]

Let \( R \) be an algebraic curvature tensor field which satisfies the second Bianchi identity. If \( S \) is the associated Ricci tensor field then


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(1.2) \[(\text{div} \mathcal{R})(X,Y,Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z).\]

For a Riemannian manifold $M$ if the Ricci tensor $S$ is of the form $S = \lambda g$ then it
is called \textit{Einstein space} [4]. If $S = 0$ then $M$ is called \textit{Ricci-flat}.

The Weyl conformal curvature tensor $C$ is defined by

\[
C(X, Y, Z, W) = R(X, Y, Z, W) - \frac{1}{n-2} \{g(X, W)S(Y, Z) + g(Y, Z)S(X, W) - \\
- \frac{\tau}{(n-1)(n-2)} \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}.
\]

An algebraic curvature tensor field $R$ is harmonic (or Codazzi type in the sense of [7]) if $(\text{div} \mathcal{R})(X, Y, Z) = 0$. A Riemannian manifold $M$ is called $R$-harmonic if its
curvature tensor field $R$ is harmonic [1].

The divergence $\text{div} C$ of the Weyl conformal curvature tensor $C$ is defined by

\[
(\text{div} C)(X, Y, Z) = \frac{n-3}{n-2} \{(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\} - \\
- \frac{1}{2(n-1)} \{g(Y, Z)\nabla_X \tau - g(X, Z)\nabla_Y \tau\}.
\]

In the present study we consider pseudo Ricci-symmetric submanifolds and also
hypersurfaces.

The notion of pseudo Ricci-symmetric (PRS) manifolds were introduced by M.
C. Chaki in 1987. A non-flat Riemannian manifold $(M^n, g)$ ($n > 2$) is called \textit{pseudo
Ricci-symmetric} if its Ricci tensor $S$ is not identically zero and satisfies

\[
(\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(X, Z) + \alpha(Z)S(Y, X),
\]

where $\alpha$ is a 1-form which is non-zero for every $X, Y, Z \in \chi(M)$ and $\nabla$ being operator
covariant differentiation with respect to the metric $g$ [2]. In [3] the authors consid-
ered conformally flat pseudo Ricci-symmetric manifolds, see also [5] for the case $M$
is a contact manifold.

In the present study we consider pseudo Ricci-symmetric manifolds. We show that
pseudo Ricci-symmetric manifolds satisfy $\text{div} \mathcal{R} = 0$ (resp. $\text{div} C = 0$) property are
Einstein (resp. Ricci flat) manifolds.

2 \textbf{Pseudo Ricci-Symmetric manifolds}

Let $(M, g)$, ($n \geq 3$), be an $n$-dimensional Riemannian manifold and $e_i, e_j$ ($1 \leq i, j \leq n$)
othornormal vector fields tangent to $M$ and $K_{ij}$ is the sectional curvature of a
plane spanned by the vectors $e_i$ and $e_j$. Then by definition of $S$ we have

\[
S(e_i, e_i) = \sum_{k=1}^{n} g(\mathcal{R}(e_k, e_i)e_i, e_k) = \sum_{k=1}^{n} K_{ik},
\]
(2.2) \[ S(e_j,e_j) = \sum_{k=1}^{n} K_{jk}, \quad S(e_i,e_j) = 0. \]

First we prove the following result.

**Proposition.** Let \( M^n \) be a \( n \)-dimensional Riemannian manifold. If \( M \) is pseudo Ricci-symmetric then

(2.3) \[ \sum_{k=1}^{n} (K_{ik} - K_{jk}) g(e_j, \nabla_{e_i} e_i) = \alpha(e_j) \sum_{k=1}^{n} K_{ik}, \]

(2.4) \[ e_i \left[ \sum_{k=1}^{n} K_{ik} \right] = 4 \alpha(e_i) \sum_{k=1}^{n} K_{ik}, \]

(2.5) \[ e_i \left[ \sum_{k=1}^{n} K_{jk} \right] = 2 \alpha(e_i) \sum_{k=1}^{n} K_{jk}. \]

**Proof.** Let \( e_i, e_j \) be orthonormal vector fields tangent to \( M \). Combining (2.1)-(2.2) and (1.4) we find

(2.6) \[ (\nabla_{e_i} S)(e_i,e_j) = \alpha(e_j) S(e_i, e_i), \]

(2.7) \[ (\nabla_{e_i} S)(e_i,e_i) = 4 \alpha(e_i) S(e_i,e_i), \]

(2.8) \[ (\nabla_{e_i} S)(e_j,e_j) = 2 \alpha(e_i) S(e_j,e_j). \]

Moreover, from the covariant differentiation of \( S \) we have

(2.9) \[ (\nabla_{e_i} S)(e_i,e_j) = -S(\nabla_{e_i} e_i, e_j) - S(e_i, \nabla_{e_i} e_j), \]

(2.10) \[ (\nabla_{e_i} S)(e_i,e_i) = \nabla_{e_i} S(e_i,e_i), \]

(2.11) \[ (\nabla_{e_i} S)(e_j,e_j) = \nabla_{e_i} S(e_j,e_j) - 2S(\nabla_{e_i} e_j, e_j). \]

By the use of (2.1)-(2.2) we get

(2.12) \[ S(\nabla_{e_i} e_i, e_j) = \sum_{k=1}^{n} g(R(e_k, e_i) e_i, e_j) = \sum_{k=1}^{n} K_{jkg}(e_j, \nabla_{e_i} e_i), \]

(2.13) \[ S(e_i, \nabla_{e_i} e_j) = \sum_{k=1}^{n} K_{jkg}(e_j, \nabla_{e_i} e_i), \quad S(\nabla_{e_i} e_i, e_i) = 0. \]

Combining (2.13), (2.13) and (2.9) we obtain
(2.14) \[ (\nabla_{e_i} S)(e_i, e_j) = \left( \sum_{k=1}^{n} (K_{ik} - K_{jk}) \right) g(e_j, \nabla_{e_i} e_i). \]

Furthermore differentiating (2.1) and (2.2) covariantly we have

(2.15) \[ (\nabla_{e_i} S)(e_i, e_i) = e_i \left[ \sum_{k=1}^{n} K_{ik} \right]. \]

(2.16) \[ (\nabla_{e_i} S)(e_j, e_j) = e_i \left[ \sum_{k=1}^{n} K_{jk} \right] \left( \text{or } (\nabla_{e_j} S)(e_i, e_i) = e_j \left[ \sum_{k=1}^{n} K_{ik} \right] \right). \]

Since the left hand sides of the equations (2.6)-(2.8) are equal to the left hand sides of (2.14)-(2.16) we get the result. □

**Theorem 2.2.** Let M be a n-dimensional pseudo Ricci-symmetric manifold. If M is R-harmonic (i.e \( \text{div} R = 0 \)) then it is Ricci-flat.

**Proof.** Let M is R-harmonic so \( \text{div} R = 0 \). Using (1.2) we have

(2.17) \[ (\nabla_{e_i} S)(e_i, e_j) - (\nabla_{e_j} S)(e_i, e_i) = 0. \]

Making use of (2.7), (2.8) and (2.1) the equation (2.17) reduces to \( \alpha(e_j) S(e_i, e_i) = 0. \) Since \( \alpha \) is a non-zero one form then \( S(e_i, e_i) = 0. \) Thus M is Ricci flat this complements the proof. □

**Theorem.** Let M be a n-dimensional pseudo Ricci-symmetric manifold. If \( \text{div} C = 0 \) then \( M^n \) is an Einstein manifold.

**Proof.** Suppose \( \text{div} C = 0 \). Then by (1.3) we have

(2.18) \[ (\nabla_{e_i} S)(e_i, e_j) - (\nabla_{e_j} S)(e_i, e_i) + \frac{1}{2} \frac{n - 2}{(n - 1)(n - 3)} e_j[\tau] = 0. \]

where \( \nabla_{e_j} \tau = e_j[\tau] \). Substituting (2.14) and (2.16) into (2.18) we obtain

(2.19) \[ \sum_{k=1}^{n} (K_{ik} - K_{jk}) g(e_j, \nabla_{e_i} e_i) - e_j \left[ \sum_{k=1}^{n} K_{ik} \right] + \frac{1}{2} \frac{n - 2}{(n - 1)(n - 3)} e_j[\tau] = 0. \]

Moreover, substituting (2.3) and (2.5) into (2.19) we get

\[ -\alpha(e_j) \sum_{k=1}^{n} K_{ik} + \frac{1}{2} \frac{n - 2}{(n - 1)(n - 3)} e_j[\tau] = 0. \]

By the use of (2.1) the above equation becomes

(2.20) \[ \alpha(e_j) S(e_i, e_i) = \frac{1}{2} \frac{n - 2}{(n - 1)(n - 3)} e_j[\tau]. \]

On the other hand

(2.21) \[ \sum_{i=1}^{n} \alpha(e_j) S(e_i, e_i) = \frac{1}{2} \frac{n - 2}{(n - 1)(n - 3)} e_j[\tau] \sum_{i=1}^{n} g(e_i, e_i), \]
which implies

\begin{equation}
(2.22) \quad e_j[\tau] = \frac{2(n - 1)(n - 3)}{n(n - 2)} \alpha(e_j) \tau.
\end{equation}

However combining (2.22) and (2.21) one can get \( S(e_i, e_i) = \frac{1}{n} \tau \). Thus \( M \) is an Einstein manifold. □

**Theorem 2.4.** Let \( M \) be a \( n \)-dimensional pseudo Ricci-symmetric manifold whose one of family of curvature lines consists of geodesic (i.e. \( \nabla_{e_i} e_i = 0 \)). Then \( M^n \) is Ricci flat.

**Proof.** Putting \( \nabla_{e_i} e_i = 0 \) into (2.3) we get \( \alpha(e_i) \sum_{k=1}^{n} K_{ik} = 0 \). Since the one form \( \alpha \) is non-zero then by (2.1) one can get \( S(e_i, e_i) = 0 \), which means that \( M \) is Ricci flat. □

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**References**


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