On the Randers Spaces of Second Order

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Abstract

The geometry of Randers spaces of order one has been investigated since fifties by G. Randers [9], using Riemannian techniques, only. A Finsler approach of this Randers space was given by R. Ingarden [4]. A fully Finslerian treatment was developed by R. Miron in [6,7]. In this paper we define and study the geometrical theory of a Randers space of order two, using the theory of the bundle of accelerations of order two, $T^2M$.

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1 Introduction

The first examples of regular Lagrangians of order greater then one were given by R.Miron. These are the prolongations of order $k$, $(k>1)$ of a Riemannian, or a Finslerian metric. Using the prolongation of order two of a Riemannian metric $\gamma_{ij}$ and a globally defined covector field $A_i$ on the base manifold, we define the fundamental function $F$ of a Randers space $RF^{(2)} = (M, \alpha+\beta)$. Then we determine the fundamental tensor $g_{ij}$ of $RF^{(2)}$ and the relation between $d$-tensors $\gamma_{ij}$ and $g_{ij}$, that is similar with the $k=1$ case. As this Randers space is a Finsler space of order two, we use techniques of Finslerian geometries of higher order to determine the canonical nonlinear connection, the canonical metrical $N$–connection. We may notice here that all the geometrical objects depend on the electromagnetic tensor $F_{ij}$ of the space $RF^{(2)}$.

2 The notion of Randers space of order two

Let $M$ be an $n$-dimensional, smooth manifold and $R^n = (M, \gamma_{ij}(x))$ be a Riemannian space. We denoted by $Prof^2R^n = (\tilde{T^2M},G)$ the prolongation of order two of the space $R^n$, [8]. The nonlinear connection $\tilde{N}$ of the space $Prof^2R^n$, [8], has the dual coefficients

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\[ M_{j}^{(1)} = \gamma_{jk}^{i}(x)y_{j}^{(1)i}, \]
\[ M_{j}^{(2)} = \frac{1}{2}(\Gamma M_{j}^{(1)} + M_{m}^{(1)} M_{j}^{(m)}), \]

where \( \Gamma \) is the operator
\[ \Gamma = y_{j}^{(1)i} \frac{\partial}{\partial x^{j}} + 2y_{j}^{(2)i} \frac{\partial}{\partial y_{j}^{(1)i}} \]

and \( \gamma_{jk}^{i}(x) \) are the Christoffel symbols of the Riemannian space \( \mathcal{R}^{n} \). Between the coefficients of the nonlinear connection \( N_{j}^{(1)} \) and \( N_{j}^{(2)} \) and the dual coefficients \( \dot{\alpha}_{j}^{(1)} \) and \( \dot{\alpha}_{j}^{(2)} \) we have the relations:
\[ \begin{align*}
N_{j}^{(1)} &= M_{j}^{(1)}, \\
N_{j}^{(2)} &= M_{j}^{(2)} - M_{m}^{(1)} M_{j}^{(m)}.
\end{align*} \]

Having the coefficients of the nonlinear connection \( \dot{N} \) we can write its adapted basis, \( \left\{ \frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial y_{j}^{(1)i}}, \frac{\partial}{\partial y_{j}^{(2)i}} \right\} \), given by
\[ \begin{align*}
\frac{\partial}{\partial x^{i}} &= \frac{\partial}{\partial x^{i}} - N_{j}^{(1)} \frac{\partial}{\partial y_{j}^{(1)i}} - N_{j}^{(2)} \frac{\partial}{\partial y_{j}^{(2)i}}, \\
\frac{\partial}{\partial y_{j}^{(1)i}} &= \frac{\partial}{\partial y_{j}^{(1)i}} - N_{j}^{(1)} \frac{\partial}{\partial y_{j}^{(2)i}} \\
\frac{\partial}{\partial y_{j}^{(2)i}} &= \frac{\partial}{\partial y_{j}^{(2)i}}.
\end{align*} \]

The dual basis of the previous adapted basis is given by \( \{dx^{i}, \delta y_{j}^{(1)i}, \delta y_{j}^{(2)i}\} \), where:
\[ \begin{align*}
\delta y_{j}^{(1)i} &= dy_{j}^{(1)i} + M_{j}^{(1)} dx^{i}, \\
\delta y_{j}^{(2)i} &= dy_{j}^{(2)i} + M_{j}^{(2)} dy_{j}^{(1)i} + M_{j}^{(1)} dx^{i}.
\end{align*} \]

We consider the Liouville \( d \)-vector fields, [8]
\[ z^{(1)m} = y_{j}^{(1)m}, \]
\[ z^{(2)m} = y_{j}^{(2)m} + \frac{1}{2} \gamma_{ij}^{m}(x)y_{j}^{(1)i}y_{j}^{(1)j}. \]
The following theorem is known

**Theorem 2.1** The function \( \alpha^2 = \gamma_{ij}(x)z^{(2)i}z^{(2)j} \) is a differential Lagrangian. It has the properties:

1) \( \alpha^2 \) is global defined on \( \widetilde{T^2M} = T^2M \setminus \{0\} \);
2) \( \alpha^2 \) is a regular Lagrangian;
3) \( \alpha^2 \) depends on the metric \( \gamma_{ij} \), only;
4) The Lagrangian \( \alpha^2 \) is homogeneous of order 4 on the fibres of \( T^2M \);
5) The fundamental tensor field of \( \alpha^2 \) is given by

\[
\frac{1}{2} \frac{\partial^2 \alpha^2}{\partial y^{(2)i} \partial y^{(2)j}} = \gamma_{ij}(x).
\]

Let us consider the functions

\[
\beta(x, y^{(1)}, y^{(2)}) = A_i(x)z^{(2)i}
\]

where \( A_i(x) \) are the electromagnetic potentials on the base manifold \( M \). Clearly, the function \( \beta \) has a physical means.

Let us consider the function \( F : T^2M \to \mathbb{R} \) which is given by \( F = \alpha + \beta \), i.e.

\[
F(x, y^{(1)}, y^{(2)}) = \sqrt{\gamma_{ij}(x)z^{(2)i}z^{(2)j} + A_i(x)z^{(2)i}}
\]

and the square of this function

\[
L(x, y^{(1)}, y^{(2)}) = (\alpha + \beta)^2.
\]

We can formulate

**Theorem 2.2** a) The function \( L = (\alpha + \beta)^2 \) is a differentiable Lagrangian of order 2;

b) \( F \) is 2-homogeneous and \( L \) is 4-homogeneous on the fibers of \( T^2M \);

c) The fundamental tensor field of the Lagrangian \( L = (\alpha + \beta)^2 \) is given by

\[
g_{ij} = (\mathcal{L}_l \gamma_{ij} + l_i l_j) - p \overset{\circ}{l}_i \overset{\circ}{l}_j
\]

where \( \overset{\circ}{l}_i = \frac{\partial \alpha}{\partial y^{(2)i}}, \ l_i = \overset{\circ}{l}_i + A_i, \ p = \frac{\alpha + \beta}{\alpha} \).

**Proof.**

a) \( L \) is of \( C^\infty \)-class on \( \widetilde{T^2M} \) and continuous on the null section of the projection \( \pi : T^2M \to M \) because \( \alpha \) and \( \beta \) have these properties.

b) It is known that \( z^{(2)i} \) is 2-homogeneous of the fibres of \( \widetilde{T^2M} \).

c) By a straightforward calculus we obtain the formula (1.9).

\[\square\]

**Theorem 2.3** The pair \( F^{(2)n} = (M, F = \alpha + \beta) \) is a Finsler space of order 2.
Proof. We must show the following properties:
1. \( F \) is of \( C^\infty \)-class on \( T^2 M \) and continuous on the null section;
2. \( F \) is positive on an open set, where \( \beta \geq 0; \)
3. \( F \) is 2-homogeneous on the fibres of \( T^2 M; \)
4. The Hessian of \( F^2 \) with the elements:

\[
g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(2)i} \partial y^{(2)j}}
\]

is positively defined.

Since the properties 1,2,3 holds in virtue of the properties of \( \alpha \) and \( \beta \) we have to prove the property 4. In order to do it, we calculate the contravariant \( g^{ij} \) of the fundamental tensor field of the Lagrangian \( L \).

The contravariant \( g^{ij} \) of \( g_{ij} \) is given by:

\[
g^{ij} = \frac{1}{p} \gamma^{ij} - \frac{1}{p^2} [l^{i'} l^{j'} (1 - \bar{F})^2 + l^i A^j + l^j A^i],
\]

where \( \bar{F} = \frac{1}{p} \gamma^{ij} l^i l^j. \)

Moreover, \( det||g_{ij}|| = p^{n+1} det||\gamma_{ij}|| \) where \( p = \frac{\alpha + \beta}{\beta} > 0. \)

The Finsler space \( RF^{(2)n} = (M, F) \) is called the Randers space of order two.

Theorem 2.4 The nonlinear connection \( ^o_N \) of the space \( Prol^2 \mathbb{R}^n \), from (1.1), is a nonlinear connection for the Randers space of order 2 \( RF^{(2)n} \) determined only by the fundamental function \( F = \alpha + \beta \).

3 Nonlinear connection of the space \( RF^{(2)n} \)

Let us consider the mixed form \( F^j_j(x) = \gamma^m(x) F_{m}^j(x) \) of the electromagnetic tensor field \( F_{m}^j(x) = \frac{\partial A_j}{\partial x^n} - \frac{\partial A_m}{\partial x^n}. \)

Theorem 3.1 The Randers space \( RF^{(2)n} = (M, \alpha + \beta) \) has a nonlinear connection \( N \), whose dual coefficients are given by

\[
\begin{align*}
M_j^j &= M_j^j - F_j^j ||y^{(1)}||,
M_j^j &= M_j^j - \frac{1}{2} F_j^j ||y^{(1)}|| + \frac{1}{2} (M_j^m F_m^j + M_j^m F_j^m) ||y^{(1)}|| - \frac{1}{2} ||y^{(1)}||^2 F_j^m F_m^j.
\end{align*}
\]

where \( ||y^{(1)}|| = \sqrt{\gamma_{ij} (x) y^{(1)}(x)} y^{(1)j}. \)

Indeed, \( M_j^j \) has the same rule of transformation with respect to changes of coordinates on \( T^2 M \) like \( \tilde{M}_j^j \) and \( \tilde{M}_j^j = \frac{1}{2} \left( \Gamma^{ij}_k \tilde{M}_j^k + \tilde{M}_m^j \tilde{M}_m^k \right). \)
The nonlinear connection (2.1) will be called canonical nonlinear connection of the Randers space \( RF^{(2)n} \).

The relations between coefficients \((N^i_j, N^j_i)\) and dual coefficients of the nonlinear connection \(N\) are given by:

\[
\begin{align*}
N^j_i &= M^j_i, \\
N^i_j &= \frac{1}{2} \left( \Gamma^i_j_k M^k_j + M^i_j M^m_j - M^k_i M^j_k \right) - \frac{1}{2} \left( \Gamma^i_j_k N^k_j - N^i_k N^k_j \right).
\end{align*}
\]

(2.2)

We use this canonical nonlinear connection in study of the canonical metrical \(N\)-connection.

4 The canonical \(N\)-linear connection

Let us consider \(d\)-tensor field

\[
T^i_j = \Gamma(F^i_j ||y^{(1)}||) + F^i_j ||y^{(1)}|| \left( M^i_j + M^i_m M^m_j \right) - M^i_j F^m_j - \frac{1}{2} \left( \Gamma^i_j_k N^k_j - N^i_k N^k_j \right).
\]

(3.1)

The adapted basis \(\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}} \right\}\) of the canonical nonlinear connection can be expressed in the following form

\[
\begin{align*}
\frac{\delta}{\delta x^i} &= \frac{\delta}{\delta x^i} + F^i_j ||y^{(1)}|| \frac{\delta}{\delta y^{(1)i}} + \frac{1}{2} T^i_j \frac{\partial}{\partial y^{(2)i}}, \\
\frac{\delta}{\delta y^{(1)i}} &= \frac{\delta}{\delta y^{(1)i}} + F^i_j ||y^{(1)}|| \frac{\partial}{\partial y^{(2)i}}, \\
\frac{\partial}{\partial y^{(2)i}} &= \frac{\partial}{\partial y^{(2)i}}.
\end{align*}
\]

(3.2)

Of course, this adapted basis depend only on the fundamental function \(F = \alpha + \beta\) of the Randers spaces of order two, \( RF^{(2)n} \).

Using a very known method we can determine the canonical \(N\)-linear connection \(D\) on \(T^2 M\) metrical with respect to fundamental tensor \(g_{ij}\), which depend only on the fundamental function \(F\) of the space \( RF^{(2)n} \).

We have:

**Theorem 4.1** 1) There exists an unique \(N\)-linear connection \(D\) on \(\widehat{T}^2 M\) which verifies the following axioms:

1° The nonlinear connection \(N\) is specified by (2.2).

2° \(g_{ij} = 0,\quad g_{ij}^{(1)} = 0,\quad g_{ij}^{(2)} = 0,\)

3° \(T^i_j = S^i_{jk} = S^i_{jk} = 0.\)
2) This connection has the coefficients given by the generalized Christoffel symbols:

\[
\begin{align*}
\Gamma_{jm}^i &= \frac{1}{2} \frac{\partial}{\partial x^s} \left[ \frac{\partial g_{jm}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^m} - \frac{\partial g_{jm}}{\partial x^s} \right], \\
\Theta_{jm}^{(1)i} &= \frac{1}{2} \frac{\partial}{\partial y^{(1)}j} \left[ \frac{\partial g_{jm}}{\partial y^{(1)m}} + \frac{\partial g_{sj}}{\partial y^{(1)s}} - \frac{\partial g_{jm}}{\partial y^{(1)s}} \right], \\
\Theta_{jm}^{(2)i} &= \frac{1}{2} \frac{\partial}{\partial y^{(2)}j} \left[ \frac{\partial g_{jm}}{\partial y^{(2)m}} + \frac{\partial g_{sj}}{\partial y^{(2)s}} - \frac{\partial g_{jm}}{\partial y^{(2)s}} \right].
\end{align*}
\]

(3.3)

3) This connection depend only on Randers space $RF^{(2)n}$.

The metrical connection from the previous Theorem is the canonical connection of the Randers space $RF^{(2)n} = (M, \alpha + \beta)$.

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References


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