Example of Extrinsically Homogeneous Real Hypersurface in $H_3(\mathbb{C})$

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Abstract

The purpose of this paper is to give an example of extrinsically homogeneous real hypersurface in a complex hyperbolic space $H_3(\mathbb{C})$, which is an orbit under a solvable Lie subgroup of the isometry group of $H_3(\mathbb{C})$ and not a Hopf hypersurface.

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Introduction

Let $H_n(\mathbb{C})$ be the hyperbolic complex space form of complex dimension $n(\geq 2)$ endowed with the metric of constant holomorphic sectional curvature $c$. A submanifold in $H_n(\mathbb{C})$ is said to be extrinsically homogeneous if it is an orbit under a closed subgroup of the group of isometries on $H_n(\mathbb{C})$. If the structure vector field of a real hypersurface $M$ in $H_n(\mathbb{C})$ is principal, then $M$ is called a Hopf hypersurface.

As proposed also in R. Niebergall and P. J. Ryan ([3]), the following is an open problem: Classify all extrinsically homogeneous real hypersurfaces in $H_n(\mathbb{C})$.

As a partial answer of this problem, there is a theorem of J. Berndt ([1]) to the effect that if $M$ is an extrinsically homogeneous real hypersurface in $H_n(\mathbb{C})$ and $M$ is a Hopf hypersurface, then $M$ is congruent to one of well-known homogeneous model spaces of $A_0$, $A_1$, $A_2$ and $B$ type.

In this paper, we shall give an example of extrinsically homogeneous real hypersurface in $H_3(\mathbb{C})$ which is not a Hopf hypersurface.

1 A construction of an example

At first we construct an example of extrinsically homogeneous real hypersurface in $H_3(\mathbb{C})$. Basically we shall adopt the notations in S. Helgason ([2]).
Let $GL(4, \mathbb{C})$ be the general linear group of degree 4 over $\mathbb{C}$, and $E_{jk}$ the element
$(\delta_{ij}\delta_{kl})_{1 \leq i, j, k, l \leq 4}$ of $GL(4, \mathbb{C})$, where $1 \leq j, k \leq 4$. For $I = E_{11} - E_{22} - E_{33} - E_{44}$, we put
$G = \{ \sigma \in GL(4, \mathbb{C})| \sigma I \bar{\sigma} = I, \ det \sigma = 1 \}$ and

$$K = \left\{ \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix} | \sigma \in U(1), \ \tau \in U(3), \ det \sigma det \tau = 1 \right\}.$$ 

Then $K$ is a closed subgroup of $G$, and the homogeneous space $G/K$ is just the
hyperbolic complex space form of complex dimension 3, which is denoted by $H_3$. The
Riemannian metric and the complex structure on $H_3$ will be stated later.

In the following, given a Lie group (e.g. $G$), we denote the associated Lie algebra of $G$ by the corresponding bold character (e.g. $g$). Conversely, given a subalgebra (e.g. $l$) of $g$, we denote by the corresponding Roman character (e.g. $L$) the connected Lie
subgroup of $G$ whose Lie algebra is $l$.

We put

$A_1 = iE_{11} - iE_{33}, \ A_2 = iE_{11} - iE_{22}, \ A_3 = iE_{11} - iE_{44},$

$Y_1 = iE_{23} + iE_{32}, \ Y_2 = E_{23} - E_{32}, \ Y_3 = iE_{24} + iE_{42},$

$Y_4 = E_{24} - E_{42}, \ Y_5 = iE_{24} + iE_{43}, \ Y_6 = E_{34} - E_{43},$

$X_1 = E_{12} - iE_{21}, \ X_2 = iE_{13} - iE_{31}, \ X_3 = E_{13} + E_{31},$

$X_4 = E_{12} + E_{21}, \ X_5 = iE_{14} - iE_{41}, \ X_6 = E_{14} + E_{41}.$

Then the set of the above eight vectors (resp. the set $\{A_1, A_2, Y_1, Y_2\}$) forms basis for $g$ (resp. $k$). By a simple computation of bracket product operation in $g$, we have the
following table :

| $[A_1, A_2]$ | 0 | $[A_1, A_3]$ | 0 | $[A_1, Y_1]$ | $-Y_2$ |
| $[A_1, Y_2]$ | $Y_1$ | $[A_1, Y_3]$ | 0 | $[A_1, Y_4]$ | 0 |
| $[A_1, Y_5]$ | $Y_6$ | $[A_1, Y_6]$ | $-Y_5$ | $[A_1, X_1]$ | $-X_4$ |
| $[A_1, X_2]$ | $-X_3$ | $[A_2, X_1]$ | $2X_1$ | $[A_2, X_2]$ | $-X_3$ |
| $[A_2, X_3]$ | $X_1$ | $[A_3, Y_1]$ | 0 | $[A_3, Y_2]$ | 0 |
| $[A_4, X_1]$ | $X_1$ | $[A_4, X_2]$ | $X_1$ | $[A_4, X_3]$ | $-X_1$ |

(1.1)
\[
\begin{align*}
[Y_4, X_2] &= 0, \quad [Y_4, X_3] = 0, \quad [Y_4, X_4] = -X_6,
[Y_4, X_5] &= X_1, \quad [Y_4, X_6] = X_4, \quad [Y_5, Y_6] = 2A_1 - 2A_3,
[Y_5, X_1] &= 0, \quad [Y_5, X_2] = X_6, \quad [Y_5, X_3] = -X_5,
[Y_5, X_4] &= 0, \quad [Y_5, X_5] = X_3, \quad [Y_5, X_6] = -X_2,
[Y_6, X_1] &= 0, \quad [Y_6, X_2] = -X_5, \quad [Y_6, X_3] = -X_6,
[Y_6, X_4] &= 0, \quad [Y_6, X_5] = X_2, \quad [Y_6, X_6] = X_3,
[X_1, X_2] &= Y_2, \quad [X_1, X_3] = -Y_1, \quad [X_1, X_4] = 2A_2,
[X_1, X_5] &= Y_4, \quad [X_1, X_6] = -Y_3, \quad [X_2, X_3] = 2A_1,
[X_2, X_4] &= -Y_1, \quad [X_2, X_5] = Y_6, \quad [X_2, X_6] = -Y_5,
[X_3, X_4] &= -Y_2, \quad [X_3, X_5] = Y_5, \quad [X_3, X_6] = Y_6,
\end{align*}
\]

We put \( p = RX_1 + RX_2 + RX_3 +RX_4 + RX_5 + RX_6 \). Then we have a Cartan decomposition of \( g \)
\[
g = k + p.
\]

For any element \( X \) of \( g \), we denote the \( k \) (resp. \( p \))-component of \( X \) by \( X_k \) (resp. \( X_p \)).

We shall identify \( p \) with the tangent space \( T_o(H_3) \) of \( H_3 \) at the origin \( o \). For a constant \( c(<0) \), we give on \( H_3 \), regarded as a symmetric space, a Riemannian metric \( \langle , \rangle \) in such a way that
\[
\langle \sqrt{-c} X_i, \sqrt{-c} X_j \rangle = \delta_{ij} \quad (1 \leq i, j \leq 6)
\]
at \( o \). Such an \( H_3 \) is the hyperbolic complex space form of constant holomorphic sectional curvature \( 4c \) of complex dimension 3, which is denoted by \( H_3(C) \). Then \( G \) acts on \( H_3(C) \) as a group of isometries. The complex structure \( J \) on \( H_3(C) \) is given by (cf. Helgason [2], p. 393)
\[
J(X_1) = X_4, \quad J(X_2) = X_3 \quad \text{and} \quad J(X_3) = X_6,
\]
where we have \( J = \text{ad} \left( -\frac{1}{4}(A_1 + A_2 + A_3) \right) \).

For any element \( \sigma \) of \( G \) and for any \( 4 \times 4 \) matrix \( X \) over \( C \), we put
\[
\text{Ad}(\sigma) X = \sigma X \sigma^{-1}.
\]

Then \( \text{Ad}(\sigma) \ (\sigma \in G) \) is an isomorphism of \( G \) as well as an inner automorphism of \( g \). The exponential map \( \exp \) of \( g \) into \( G \) is given by
\[
\exp X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \quad \text{for} \quad X \in g.
\]

Then the followings are well-known, or can be easily checked:
\[
\begin{align*}
(1.3) \quad & \text{Ad}(\sigma) p \subset p \quad \text{for} \quad \sigma \in K, \\
(1.4) \quad & \frac{d}{dt} \bigg|_0 \text{Ad}(\exp tX) Y = [X, Y] \quad \text{for} \quad X, Y \in g.
\end{align*}
\]
\[
\sigma (\exp X) \sigma^{-1} = \exp(\sigma X \sigma^{-1}) \quad \text{for} \quad \sigma \in G, \; X \in \mathfrak{g},
\]

\[
\exp(sA_1 + tA_2 + uA_3) = e^{i(s+t+u)}E_{11} + e^{-it}E_{22} + e^{-iu}E_{33} + e^{-iu}E_{44}
\]
for \( s, \; t, \; u \in \mathbb{R} , \)

(1.7) The group \( Ad(K) \) acts on any hypersphere of \( \mathfrak{p} \) centered at the origin transitively.

**Remark 1.1.** The statement (1.7) is a property of a symmetric space of rank 1.

For two subalgebras \( \mathfrak{l}_1 \) and \( \mathfrak{l}_2 \) of \( \mathfrak{g} \), if there is an element \( \sigma \) of \( K \) such that

\[ Ad(\sigma) \; \mathfrak{l}_1 = \mathfrak{l}_2 , \]

then the corresponding two orbits \( L_1(o) \) and \( L_2(o) \) are congruent in \( H_3(\mathbb{C}) \) since \( \sigma(L_1(o)) = L_2(o) \) by (1.5).

Put \( Z_1 = X_1 + Y_5, \; Z_2 = X_2 + Y_5, \; Z_3 = X_5 - A_3, \; Z_4 = X_6 \) and \( Z_5 = X_3 - Y_6 \). Then it follows from (1.1) that

\[
\begin{align*}
[Z_1, \; Z_2] &= [Z_1, \; Z_3] = [Z_1, \; Z_5] = [Z_2, \; Z_3] = [Z_2, \; Z_5] = 0, \\
\end{align*}
\]

(1.8) If we define a subspace \( \mathfrak{l} \) of \( \mathfrak{g} \) by

\[(1.9) \quad \mathfrak{l} = \mathbb{R}Z_1 + \mathbb{R}Z_2 + \mathbb{R}Z_3 + \mathbb{R}Z_4 + \mathbb{R}Z_5 , \]

then we see from (1.8) that \( \mathfrak{l} \) is a solvable Lie subalgebra of \( \mathfrak{g} \).

Now we can state our example as follows.

**Theorem 1.1.** Let \( L \) be the connected Lie subgroup of \( G \) whose associated Lie algebra is \( \mathfrak{l} \) given in (1.9), and denote by \( \sigma_t \) the 1-parameter subgroup \( \exp tX_4 \) of \( G \). Then, for any \( t \in \mathbb{R} \), the orbit \( L(\sigma_t(o)) \) of the point \( \sigma_t(o) \) under \( L \) is an extrinsically homogeneous real hypersurface in \( H_3(\mathbb{C}) \) whose structure vector is not principal.

In order to prove Theorem 1.1, we must make some preparations. Let \( \nabla \) be the Riemannian connection of \( H_3(\mathbb{C}) \) with respect to the Riemannian metric \( \langle \; , \; \rangle \). Each element \( X \) of \( \mathfrak{g} \) induces a differentiable vector field \( X^* \) on \( H_3(\mathbb{C}) \) such as

\[ X_p^* = \frac{d}{dt} \bigg|_{t=0} \; (\exp tX)(p), \quad p \in H_3(\mathbb{C}). \]

Then the following results are well-known in the theory of a symmetric spaces

\[ (1.10) \quad \text{For any} \; X \in \mathfrak{g} , \; X^* \text{ is a Killing vector field on } H_3(\mathbb{C}) , \]

\[ (1.11) \quad [X^*, \; Y^*] = -[X, \; Y]^* \quad \text{for any} \; X, \; Y \in \mathfrak{g} , \]

\[ (1.12) \quad \nabla_{X^*} Y^* = 0 \quad \text{for any} \; X, \; Y \in \mathfrak{p} . \]
It is clear that, for any $X \in \mathfrak{g}$, we have $X_*^o = X_p \in \mathfrak{p} \equiv T_o(H_3(C))$. In particular, we see that

$$X_*^o = \begin{cases} 
0 & \text{if } X \in \mathfrak{k} \\
X & \text{if } X \in \mathfrak{p}.
\end{cases}$$

To find out the shape operator of an orbit under $G$, we shall prove

**Proposition 1.2.** Let $\mathfrak{m}$ be any Lie subalgebra of $\mathfrak{g}$, and $M$ be the corresponding analytic Lie subgroup of $G$ such that $\dim M(o) \leq 5$. Let $\nu \in \mathfrak{p}$ be a normal vector of the orbit $M(o)$ at $o$. Then the shape operator $T_\nu$ of $M(o)$ in the direction $\nu$ is given by $T_\nu(X_p) = [X_k, \nu]_M$ for $X \in \mathfrak{m}$, where $[X_k, \nu]_M$ indicates the $T_\nu(M)$-component of $[X_k, \nu]$.

**Proof.** First we assert that

$$(1.13) \quad \nabla_{X_*^o} Y = -[X, Y] \quad \text{for any } X \in \mathfrak{p} \text{ and } Y \in \mathfrak{k}.$$ 

In fact, we have

$$\nabla_{X_*^o} Y = \nabla Y * + [X_*^o, Y] = \nabla Y * - [X, Y]^*$$

by (1.11). Evaluating this equation at $o$, we obtain (1.13).

Next, by use of the result in R. Takagi and T. Takahashi ([4], 471p) and the equations (1.12) and (1.13), we get

$$T_\nu(X_p) = -\nabla_\nu X_*^o = -\nabla_{\nu_*^o}(X_k + X_p) = [X_k, \nu]_M,$$

and this completes the proof. \qed

**Remark 1.2.** As seen from the above proof, Proposition 1.2 holds for any symmetric space and the isometry group of it.

**Proof of Theorem 1.1.** It is clear by definition that the orbit $L(\sigma_t(o))$ is an extrinsically homogeneous real hypersurface in $H_3(C)$.

Since the orbit $L(\sigma_t(o))$ is congruent to the orbit $(Ad(\sigma_t^{-1})L)(o)$ under $Ad(\sigma_t^{-1})L$ in $H_3(C)$, we shall investigate the shape operator and the structure vector on the latter. For simplicity, we put $c_t = \cosh t$ and $s_t = \sinh t$. Then we see that

$$\sigma_t = c_t E_{11} + c_t E_{22} + E_{33} + E_{44} + s_t E_{12} + s_t E_{21}.$$

By a simple calculation, we have

$$\begin{align*}
Ad(\sigma_t)Z_1 &= -2s tc_t A_2 + c_t Y_3 + (c_t^2 + s_t^2) X_1 + s_t X_5, \\
Ad(\sigma_t)Z_2 &= s_t Y_1 + Y_5 + c_t X_2, \\
Ad(\sigma_t)Z_3 &= -s_t^2 A_2 - A_3 + s_t Y_3 + s_t c_t X_1 + c_t X_5, \\
Ad(\sigma_t)Z_4 &= s_t Y_4 + c_t X_6, \\
Ad(\sigma_t)Z_5 &= s_t Y_2 - Y_6 + c_t X_3.
\end{align*}$$

(1.14)

Since $\nu = X_3$ is the normal vector of $(Ad(\sigma_t^{-1})L)(o)$, it follows from (1.1), Proposition 1.2 and (1.14) that

$$\begin{align*}
T_\nu((c_t^2 + s_t^2) X_1 + s_t X_5) &= -4s_t c_t X_1 - c_t X_5, \\
T_\nu(s_t c_t X_1 + c_t X_5) &= -(2s_t^2 + 1) X_1 - s_t X_3, \\
T_\nu(c_t X_2) &= -s_t X_2, \\
T_\nu(c_t X_3) &= -s_t X_3, \\
T_\nu(c_t X_6) &= -s_t X_6.
\end{align*}$$
Therefore, with respect to a basis \{X_1, X_2, X_3\} for \(T \circ (\text{Ad}(\sigma_i^{-1})L)(\alpha)\), the shape operator \(T := T_{\sqrt{-c} X_3}\) of \((\text{Ad}(\sigma_i^{-1})L)(\alpha)\) is expressed by

\[
\begin{align*}
T(X_1) &= \sqrt{-c} (\tanh^3 t - 3 \tanh t)X_1 - \sqrt{-c} \sech^3 t X_5, \\
T(X_2) &= -\sqrt{-c} \tanh t X_2, \\
(1.15) \quad T(X_3) &= -\sqrt{-c} \tanh t X_3, \\
T(X_5) &= -\sqrt{-c} \sech^3 t X_1 - \sqrt{-c} \tanh^3 t X_5, \\
T(X_6) &= -\sqrt{-c} \tanh t X_6.
\end{align*}
\]

Since the structure vector of \((\text{Ad}(\sigma_i^{-1})L)(\alpha)\) is \(X_1\) by (1.2) and \(\sech t \neq 0\) for any \(t \in \mathbb{R}\), (1.15) shows that \(X_1\) is not principal. This completes the proof. \(\square\)

**Remark 1.3.** By the above computation we see that the vector \(X_2, X_3\) and \(X_6\) are always principal, and for any \(t \in \mathbb{R}\) the orbit \(L(\sigma_i(\alpha))\) has the following three distinct principal curvatures \(-\sqrt{-c} \tanh t\) (of multiplicity 3),

\[-\sqrt{-c} \left(\frac{3}{2} \tanh t \pm \sqrt{1 - \frac{3}{4} \tanh^2 t}\right)\].

In particular, the orbit \(L(\alpha)\) is minimal.

**References**


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