On Normal Sections of Stiefel Submanifold

K. Arslan and C. Özbür

Abstract

In this study we consider Stiefel submanifold $V_{n,k}$ which is a projective $m$-space $P^m (n = mk)$ isometrically imbedded in $\mathbb{R}^{m+n(m+1)}$ by its first standard imbedding. We also consider the normal sections of $V_{n,k}$ and show that $V_{n,k}$ has $P_2 - PNS$ property.

Mathematics Subject Classification: 53C40, 53C42
Key words: Normal section, Stiefel manifold

1 Introduction

Let $M$ be an $n$-dimensional submanifold in $(m + d)$- dimensional Euclidean space $\mathbb{R}^{n+d}$. Let $\nabla$, $\nabla^k$, and $\nabla^l$ denote the covariant derivatives in $T(M)$, $N(M)$ and $\mathbb{R}^{n+d}$ respectively. Thus $\nabla_X$ is just the directional derivative in the direction $X$ in $\mathbb{R}^{n+d}$. Then for tangent vector fields $X$, $Y$ and $Z$ and normal vector field $v$ over $M$ we have $\nabla_X Y = \nabla_X Y + h(X,Y)$ and $\nabla_X v = -A_v X + \nabla_X v$, where $h$ is the second fundamental form and $A_v$ is the shape operator of $M$ [6]. For tangent vector fields $X, Y, Z$ over $M$ we define $\nabla h$ as usual by

$$\nabla_X h(Y, Z) = (\nabla_X h)(Y, Z) + h(\nabla_X Y, Z) + h(Y, \nabla_X Z).$$

Let $M$ be a smooth $n$-dimensional submanifold in $(n + d)$-dimensional Euclidean space $\mathbb{R}^{n+d}$. For a point $x$ in $M$ and a non-zero tangent vector $X \in T_x M$, we define the $(d+1)$-dimensional affine subspace $E(x, X)$ of $\mathbb{R}^{n+d}$ by $E(x, X) = x + \text{span} \{X, T_x M\}$. In a neighborhood of $x$ the intersection $M \cap E(x, X)$ is a regular curve $\gamma : (-\varepsilon, \varepsilon) \to M$. We suppose the parameter $t \in (-\varepsilon, \varepsilon)$ is a multiple of the arc-length such that $\gamma(0) = x$ and $\gamma'(0) = X$. Each choice of $X \in T(M)$ yields a different curve which is called the normal section of $M$ at $x$ in the direction of $X$, where $X \in T_x(M)$ [8]. For such a normal section we can write

$$\gamma(t) = x + \lambda(t)X + N(t)$$

where $N(t) \in T_x M$ and $\lambda(t) \in \mathbb{R}$.

©Balkan Society of Geometers, Geometry Balkan Press
**Definition 1.1.** The submanifold $M$ is said to have pointwise $k$-planar normal sections ($P^k - PNS$) if for each normal section $\gamma$, the higher order derivatives $\gamma'(0), \gamma''(0), \ldots, \gamma^{(k+1)}(0)$ are linearly dependent as vectors in $\mathbb{R}^{n+d}$.

If $k = 1$ then $M$ is totally geodesic. Taking $k = 2$, we note that submanifolds with pointwise 2-planar normal sections have been classified. They have parallel second fundamental form (i.e. $\nabla h = 0$) or hypersurfaces [1], see also [10] and [2].

**Theorem 1.1.** [1] Let $M$ be a submanifold of $\mathbb{R}^{n+d}$ then $M$ has pointwise 2-planar normal sections ($P^2 - PNS$) if and only if

$$\|h(X,X)\|^2 (\nabla_X h)(X,X) = \langle(\nabla_X h)(X,X), h(X,X) \rangle h(X,X).$$

2 Stiefel Manifolds $V_{n,k}$

For each $n, k$ ($k \leq n$) the Stiefel manifold $V_{n,k}$ has as its points all orthonormal frames $x = (e_1, \ldots, e_k)$ of $k$ vectors in Euclidean $n$-space (i.e. ordered sequences of $k$ orthonormal vectors in $\mathbb{R}^n$). Any orthogonal matrix $A$ of degree $n$ sends any such orthonormal frame $x$ to another, namely $Ax = ( Ae_1, \ldots, Ae_k)$, this defines an action of $O(n)$ on $V_{n,k}$ which is transitive (see [9]).

Each Stiefel manifold $V_{n,k}$ can be realized as a non-singular surface in the Euclidean space $\mathbb{R}^{nk}$ in the following way. Fix an orthonormal basis for $\mathbb{R}^n$ (e.g. the standard basis), and introduce the following notation for the components with respect to this basis of any $k$-frame $(e_1, \ldots, e_k)$ (i.e. point of $V_{n,k}$):

$$e_i = (x_{i1}, \ldots, x_{in}), \quad i = 1, \ldots, k.$$

The $nk$ quantities $x_{ij}$, $i = 1, \ldots, k$; $j = 1, \ldots, n$, (in lexicographic order, say) are now to be regarded as the co-ordinates of a point in $nk$-dimensional Euclidean space $\mathbb{R}^{nk}$, related by the following $k(k+1)/2$ equations:

$$\langle e_i, e_j \rangle = \delta_{ij} \iff \sum_{s=1}^{n} x_{is} x_{js} = \delta_{ij}, \quad i, j = 1, \ldots, k, i \leq j$$

We now investigate the isotropy group of this homogeneous space. Take any orthonormal $k$-frame $e_1, \ldots, e_k$ and enlarge it to an orthonormal basis $e_1, \ldots, e_n$ for the whole of Euclidean $n$-space. Any orthogonal transformation fixing the vectors $e_1, \ldots, e_k$ must (relative to the above basis for $\mathbb{R}^n$) have the form

$$
\begin{bmatrix}
1 & 0 \\
0 & \ddots \\
0 & 1 \\
0 & A
\end{bmatrix}
$$

whence the isotropy group is isomorphic to $O(n-k)$, and $V_{n,k}$ can be identified with $O(n)/O(n-k)$. In fact $V_{n,k} \cong O(n)/O(n-k)$.

The Stiefel manifolds $V_{n,k}$ for $k < n$ are also homogeneous spaces for the group $SO(n)$ (see [9]). From this point of view the isotropy group is clearly (isomorphic to) $SO(n-k)$, and therefore also
\[ V_{n,k} \cong SO(n)/SO(n-k). \]

In particular, we have

\[ V_{n,n} \cong O(n) , \quad V_{n,n-1} \cong SO(n) , \quad V_{n,1} \cong S^{n-1}. \]

In [11] Jimenez showed that: i) \( V_{n,k} \) belongs to the class of reduced Riemannian \( \sum \)-spaces, ii) if \( n = 2k > 30 \) and \( k = n - 2 \), \( V_{n,k} \) belongs to the class of all pointwise Riemannian \( k \)-symmetric spaces \( (k \geq 2) \) and furthermore, it is not diffeomorphic to the underlying manifold of any symmetric in the class of all regular Riemannian \( k \)-symmetric spaces \( (k \geq 2) \), and iii) if \( n \) and \( k \) are appropriately chosen \( V_{n,k} \) is not diffeomorphic to the underlying manifold of a space in class of all pointwise Riemannian \( r \)-symmetric spaces.

## 3 The First Standard Imbedding Of The Real Projective Space \( \mathbb{P}^m \)

In this section we will consider the Stiefel submanifold as the first standard imbedding of the real projective space \( \mathbb{P}^m \) \( (m = nk) \) in \( \mathbb{R}^{m+1} \mathbb{S}^{m(m+1)} \). The notation here is essentially the same as in [7].

Let \( M(m+1; \mathbb{R}) \) be the space of \((m+1) \times (m+1)\) matrices over \( \mathbb{R} \). It is considered as an \((m+1)^2\)-dimensional Euclidean space with the inner product

\[ \langle A, B \rangle = \frac{1}{2} \text{trace} \; AB^T, \]

where \( B^T \) is the transpose of the matrix \( B \).

Someone consider \( \mathbb{P}^m \) as the quotient space of the hypersphere

\[ \mathbb{S}^m = \{ \zeta \in \mathbb{R}^{m+1} : \zeta^T \zeta = I_k \}; \quad m = nk, \]

obtained by identifying \( \zeta \) with \( \zeta \lambda \), where \( \zeta \) is a column vector and \( \lambda \in \mathbb{R} \) such that \( |\lambda| = 1 \).

Define a mapping \( \bar{\varphi} : \mathbb{S}^m \to H(m + 1; \mathbb{R}) = \{ A \in M(m + 1; \mathbb{R}) : A^T = A \} \) as follows

\[ \bar{\varphi}(\zeta) = \zeta \zeta^T = \begin{pmatrix} |\zeta_0|^2 & \zeta_0 \zeta_1 & \cdots & \zeta_0 \zeta_n \\ \zeta_1 \zeta_0 & |\zeta_1|^2 & \cdots & \zeta_1 \zeta_n \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_n \zeta_0 & \zeta_n \zeta_1 & \cdots & |\zeta_n|^2 \end{pmatrix} \]

for \( \zeta = (\zeta_i) \in \mathbb{S}^m \subset \mathbb{R}^{m+1}, 0 \leq i \leq m \). Then it is easy to verify that \( \bar{\varphi} \) induces a mapping \( \varphi \) of \( \mathbb{P}^m \) into \( H(m + 1; \mathbb{R}) \):

\[ \varphi(\pi(\zeta)) = \bar{\varphi}(\zeta) = \zeta \zeta^T. \]

where \( \pi : \mathbb{S}^m \to \mathbb{P}^m \) is a Riemannian submersion [7]. We simply denote \( \varphi(\pi(\zeta)) \) by \( \varphi(\zeta) \).

From (1) the image of \( \mathbb{P}^m \) under \( \varphi \) is given by

\[ \varphi(\mathbb{P}^m) = \{ A \in H(m + 1; \mathbb{R}) : A^2 = A \text{ and trace } A = 1 \}, \quad m = nk. \]

Let \( A = \zeta \zeta^T \) be a point in \( \varphi(\mathbb{P}^m) \). Consider the curve
\( A(t) = \zeta T; \; \zeta \in \mathbb{R}^{m+1} \)

in \( \varphi(P^m) \) with \( A(0) = A \) and \( A'(0) = X \in T_A(P^m) \). From \( A^2(t) = A(t) \) one gets \( XA^T + AX^T = X \). So we have

\[
T_A(P^m) = \{ X \in H(m + 1; \mathbb{R}) : XA^T + AX^T = X \}.
\]

**Proposition 3.1** [7]. Let \( Y \) be a vector field tangent to \( P^m \) and \( X \in T_A(P^n) \). Consider a curve \( A(t) \) in \( \varphi(P^m) \) so that \( A(0) = A \) and \( A'(0) = X \). Denote by \( Y(t) \) the restriction of \( Y \) to \( A(t) \). Then

\[
A(t)Y(t) + Y(t)A(t) = Y(t).
\]

**Corollary 3.2.** Let \( A(t) = \zeta(t)\zeta'(t)^T; \; \zeta(t) \in \mathbb{R}^{m+1} \) be a curve in \( \varphi(P^m) \) then

\[
A(0) = X = \xi a^T + a\xi^T
\]

where

\[
\zeta(0) = \xi; \zeta'(0) = a.
\]

**Proof.** Let \( A(t) = \zeta \zeta^T \) be a curve in \( \varphi(P^m) \) with \( A(0) = A \) and \( A'(0) = X \in T_A(P^m) \). Differentiating (5) we get

\[
A'(t) = \zeta \zeta^T + \zeta \xi^T + \zeta^T a
\]

At point \( t = 0 \) since \( \zeta(0) = \xi; \zeta'(0) = a \) we get the result.

A vector \( \nu \) in \( H(m + 1; \mathbb{R}) \) is normal to \( P^m \) at \( A \) if and only if \( < X, \nu >= 0 \) for all \( X \) in \( T_A(P^m) \). Thus, \( \nu \) is in \( T_A^-(P^m) \) if and only if \( trace(X\nu) = 0 \) for all \( X \) in \( T_A(P^m) \). Therefore by (5) we obtain

\[
T_A^-(P^m) = \{ \nu \in H(m + 1; \mathbb{R}) : A\nu = \nu A \}.
\]

**Theorem 3.3** [7]. The isometric imbedding \( \varphi : S^n \rightarrow H(m + 1; \mathbb{R}) \), determined by (3), is the first standard imbedding of \( P^n \) into \( H(n + 1; \mathbb{R}) \) and \( P^n \) lies in a hypersphere \( S(r) \) of \( H(m + 1; \mathbb{R}) \) centered at \( \frac{1}{m+1}I \) and with radius \( r = \sqrt{\frac{m}{2}(n + 1)} \).

**Definition 3.1.** A real projective \( m \)-space \( P^n \) (\( m = nk \)) isometrically imbedded in \( \mathbb{R}^{m+\frac{m(m+1)}{2}} \) by its first standard imbedding \( \varphi \) determined by (3) is called the Stiefel submanifold \( V_{n,k} \).

## 4 Normal Sections Of \( V_{n,k} \)

An element \( A \in V_{n,k} \) can be written as \( A = \zeta \zeta^T \) where \( \zeta \zeta^T \zeta = I_{k \times k} \) and \( \zeta = \xi m + \eta k \), \( \xi^T \eta = 0, m \in M_{k \times k}(R) \). Since \( \zeta \) is not unique we can replace \( \zeta \) by \( \zeta u \) where \( u \in O(k) \). Put \( m = \xi^T \zeta \) then we can find \( u \in O(k) \) so that \( mu \) is lower triangular and the diagonal element of \( mu \) are \( \geq 0 \) by using the Gram-Schmidt orthonormalization process. Further \( mu \) is uniquely defined by \( m \) and \( \xi^T (\zeta - \zeta m) = \xi^T \zeta - m = 0 \). Thus we can write \( A = \zeta \zeta^T \) where \( \zeta = \xi m + \eta k \) where \( m \) is lower triangular \( k \times k \) with diagonal term \( \geq 0 \) and \( \xi^T \eta = 0, m \in M_{k \times k}(R) \) where \( m \) and \( \eta \) are unique. So

\[
A = \xi mm^T \xi^T + \xi m \eta^T + \eta m^T \xi^T + \eta^T.\]

Interchanging \( A \) with the curve \( A(t) \) in \( \varphi(P^m) \) we get
(6) \[ A(t) = \zeta(t)\zeta(t)^T = \xi m(t)m(t)^T \xi^T + \xi \eta(t)^T + \eta(t)m(t)^T \xi^T + \eta(t)\eta(t)^T. \]

**Lemma 4.1.** Let \( A(t) \) be a normal section of \( V_{n,k} \) at point \( A(0) \) in the direction of \( A'(0) = X \). Then

\[ \lambda(t)X = \xi m(t)\eta(t)^T + \eta(t)m(t)^T \xi^T \]

\[ N(t) = \xi m(t)m(t)^T \xi^T + \eta(t)\eta(t)^T \]

where \( a\xi^T + \xi a^T = X \), \( \xi^T \xi = I_k \) and \( a^T \xi + \xi^T a = 0 \), \( a = \eta(0) \in M_{n \times k} \).

**Proof.** Since \( \xi^T \eta(t) = 0 \), then \( m(t)\eta(t)^T \xi + \xi^T \eta(t)m(t)^T = 0 \). Therefore \( \xi m(t)\eta(t)^T + \eta(t)m(t)^T \xi^T \in T_x(V_{n,k}) \). Also \( \eta(t)^T \eta(t) \in N_x(V_{n,k}) \), since \( \text{trace}(a^T\eta(t)\eta(t)^T\xi) = 0 \) (whatever \( a \in M_{n \times k} \)). Also

\[
a^T (\xi m(t)m(t)^T \xi^T) a = \text{trace}(a^T \xi m(t)m(t)^T \xi^T a) = \frac{1}{2} \text{trace}(a^T \xi + \xi^T a)m(t)m(t)^T = 0
\]

if \( a^T \xi + \xi^T a = 0 \). Therefore \( \xi m(t)m(t)^T \xi^T \in N_x(V_{n,k}) \). Hence comparing (1) with (6) we get the result.

**Lemma 4.2.** Let \( A(t) = \zeta(t)\zeta(t)^T \) be a normal section of \( V_{n,k} \) at point \( A(0) \) in the direction of \( A'(0) = X \). Then

\[ m(t)^Tm(t) + \eta(t)^T\eta(t) = I_k. \]

**Proof.** Since \( \zeta(t) = \xi m(t) + \eta(t) \),

\[ \zeta(t)^T\zeta(t) = I_k, \]

and

\[ \xi^T \eta(t) = 0, \eta \in M_{n \times k} \]

we get the result.

**Theorem 4.3.** Stiefel submanifold \( V_{n,k} \) (considered as a projective \( m \)-space \( P^n \) \( (n = mk) \) isometrically imbedded in \( R^{m(m+1)/2} \) by its first standard imbedding defined by (3)) has \( P^2 - PNS \) property.

**Proof.** The normal section \( A(t) \) of \( V_{n,k} \) at point \( A(0) = \xi \xi^T \) in the direction of \( A'(0) = X \) is given by \( A(t) = \zeta(t)\zeta(t)^T \), where \( \zeta(t) = \xi m(t) + \eta(t) \), \( \eta, m \in M_{n \times k} \).

\[ m(t) = \xi T \zeta(t) \]

and

\[ \eta(t)m(t)^T = \lambda(t)a. \]

At point \( t = 0 \) we get

\[
\begin{align*}
\zeta(0) &= \xi \\
m(0) &= I_k \\
\eta(0) &= 0 \\
A(0) &= \xi \xi^T
\end{align*}
\]
at least for sufficiently small \( A(t) \). Thus we now differentiate (9)-(13) and so determine \( m', \eta', \lambda', m'', \eta'', \lambda'' \). Then using \( N(t) = \xi m(t)m(t)^T \xi + \eta(t)\eta(t)^T \) we compute \( N', N'' \), etc.

Differentiating (9), (12) and (11) (surprising the dependence of \( \zeta \) on \( t \) to simplify the notation) we get

\[
A'(t) = \zeta'(t)\zeta(t)^T + \zeta(t)\zeta'(t)^T = \lambda'(t)X + N'(t),
\]

\[
\zeta'(t)^T\zeta(t) + \zeta(t)^T\zeta'(t) = 0,
\]

\[
m'(t) = \xi^T\zeta'(t)
\]

Putting \( t = 0 \) we get

\[
A'(0) = X = \xi a^T + a\xi^T.
\]

\[
\zeta(0) = \xi, \zeta'(0) = \eta' = a,
\]

\[
m'(0) = \xi^T a = 0.
\]

Differentiating (13) two times with respect to \( t \) we also get

\[
\eta''(t)m(t)^T + 2\eta'(t)m'(t)^T + \eta(t)m''(t)^T = \lambda''(t)a.
\]

At \( t = 0 \) by the use of (14) and (20) the above equation gives

\[
\eta''(0) = 0.
\]

Differentiating (9) and (8) we get

\[
0 = \lambda''(t) + 2m'(t)^Tm(t) + m'(t)m''(t) + \eta''(t)^T\eta(t) + 2\eta'(t)^T\eta'(t) + \eta(t)\eta''(t)^T
\]

and

\[
N''(t) = \xi m''(t)m(t)^T \xi + 2\xi m'(t)m'(t)^T \xi + \xi m(t)m''(t)^T \xi + \eta''(t)\eta(t)^T + 2\eta'(t)^T\eta'(t)^T + \eta(t)\eta''(t)^T.
\]

At \( t = 0 \) by (14), (20) and (21) we have

\[
m''(0)^T + m''(0) = -2a^T a
\]

and

\[
N''(0) = h(X, X) = -2\xi a^T a \xi + 2a^T a.
\]

Differentiating (22) and (23) with respect to \( t \) we get
\begin{equation}
0 = m'''(t)^T m(t) + 3m''(t)^T m'(t) + 3m'(t)^T m''(t) + m^T(t)m'''(t) + 
+ \eta'''(t)^T \eta(t) + 3\eta''(t)^T \eta'(t) + 3\eta'(t)^T \eta''(t) + \eta^T(t)\eta'''(t)
\end{equation}

and
\begin{equation}
N'''(t) = \xi m'''(t)m(t)^T \xi^T + 3\xi m''(t)m'(t)^T \xi^T + 3\xi m'(t)m''(t)^T \xi^T + 
+ \xi m(t)m'''(t)^T \xi^T + \eta'''(t)^T \eta(t)^T + 3\eta''(t)^T \eta'(t)^T + 
+ 3\eta'(t)^T \eta''(t)^T + \eta(t)^T \eta'''(t)^T.
\end{equation}

At \( t = 0 \), substituting (14), (20) and (21) into (25) and (26) one can get respectively
\[ m'''(0)^T + m'''(0) = 0, \]
and
\[ N'''(0) = (\nabla_X h)(X,X) = \xi m'''(0)^T \xi^T + \xi m'''(0)^T \xi^T = 0. \]

This means that \( V_{n,k} \) has parallel second fundamental form. Therefore by Theorem 1.1 \( M \) has \( P2-PNS \) property.

**Remark.** For \( k = 1 \) the Stiefel submanifold \( V_{n,1} \) becomes a Veronese submanifold. In [3] we have shown that \( V_{n,1} \) has \( P2-PNS \) property.

**Acknowledgements.** This research is supported by Uludağ University research fund.

**References**


Uluðag University
Faculty of Art and Sciences
Department of Mathematics
Campus of Göürükle, 16059
Bursa-TURKEY

e-mail: arslan@uludag.edu.tr

e-mail: cozgur@balikesir.edu.tr