On Conformally Flat Pseudosymmetric Spaces

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Dedicated to Prof. Dr. Constantin Udriste
on the occasion of his sixtieth birthday

Abstract

In a recent paper [1] M. C. Chaki introduced and studied a type of non-flat Riemannian space \((M^n, g) (n \geq 2)\) whose curvature tensor \(R_{ijkl}^h\) satisfies the condition

\[
R_{ijkl}^h = 2\lambda_i R_{ijkl}^h + \lambda^h R_{ijkl}^h + \lambda_j R_{ijkl}^h + \lambda_k R_{ijkl}^h
\]

where \(\lambda_i\) is a non-zero vector and comma denotes covariant differentiation with respect to the metric \(g_{ij}\). Such a space was called by him a pseudo symmetric space, the vector \(\lambda_i\) was called its associated vector and an \(n\)-dimensional space of this kind has been denoted \((PS)_n\). Tarafder [2] proved that a conformally flat \((PS)_n (n \geq 3)\) with non-zero constant scalar curvature is a subprojective space in the sense of Kagou [3], if the associated vector is gradient. In the present paper we obtain the above result without assuming any restriction on the scalar curvature. Among others it is shown that a conformally flat \((PS)_n\) can be expressed as a warped product \(I \times e^t M^*\) where \(M^*\) is an Einstein space and such space is a space of quasi-constant curvature [4].

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1 Conformally flat \((PS)_n (n \geq 3)\)

It is known [1] that a conformally flat \((PS)_n (n \geq 3)\) can not be of zero scalar curvature and also it is known [2] that in a conformally flat \((PS)_n\)

\[
R_{ij} = \frac{R - t}{n - 1} g_{ij} + \frac{nt - R}{(n - 1)\lambda_i\lambda_j} \lambda_i \lambda_j
\]

where \(R\) denotes the scalar curvature and \(t\) is a scalar.

The above expression can be written as

\[
R_{ij} = \alpha g_{ij} + \beta u_i u_j
\]

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where \( \alpha = \frac{R - t}{n - 1}, \beta = \frac{nt - R}{n - 1} \) are two scalars and \( v_i = \frac{\lambda_i}{\sqrt{\lambda_i \lambda}} \) is a unit vector. On the other hand, a conformally flat space is conformally symmetric, that is, \( C_{ijkl}^h = 0 \). The above equation is equivalent to

\[
R_{jl,k} - R_{j,k,l} = \frac{1}{2(n-1)} (g_{jl} R_{,k} - g_{jk} R_{,l}).
\]

The relation (1.1) implies

\[
R_{ij,k} = \alpha_k g_{ij} + \beta_k v_i v_j + \beta(v_j v_i, k + v_i v_j, k), \text{ where } \alpha, k = \alpha_k \text{ and } k = \beta_k.
\]

Substituting (1.3) into (1.2) we obtain

\[
\alpha_k g_{jl} + \beta_k v_j v_l + \beta(v_j v_l, k + v_l v_j, k) - \alpha_l g_{jl} - \beta_l v_j v_l - \beta(v_l v_j, k + v_j v_l, k)
\]

\[
= \frac{1}{2(n-1)} (g_{jl} R_{,k} - g_{jk} R_{,l})
\]

where \( R_k = R_{,k} \).

Since \( v^i v_i = 1 \) and \( (v_{i,k}) v^i = 0 \), so by transvecting with \( g^j \), (1.4) reduces to

\[
(n-1) \alpha_k + \beta_k - (\beta_a v^a) v_k - \beta(v_k v^a, a + v^a v_k, a) = \frac{1}{2} R_k.
\]

Transvecting (1.4) with \( v^j \) we obtain

\[
(n-1) \alpha_k + \beta_k - (\beta_a v^a) v_k - \beta(v_k v^a) v_{k,a} = \frac{1}{2(n-1)} (v_l R_k - v_k R_l).
\]

Transvecting again with \( v^j \) we have

\[
(n-1) \alpha_k + \beta_k - (\beta_a v^a) v_k - \beta(v_k v^a) v_{k,a} = \frac{1}{2(n-1)} \{ R_k - (v^a R_a) v_k \}
\]

Substituting this into (1.5) we find

\[
(n-2) \alpha_k + \beta v_k v^a, a + (\alpha_a v^a) v_k - \frac{1}{2(n-1)} v_k (v^a R_a) = \frac{1}{2} \frac{n-2}{n-1} R_k
\]

Transvecting (1.7) with \( v^k \), we get

\[
(n-1)(\alpha_a v^a) - \beta v^a, a = \frac{1}{2} (R_a v^a).
\]

Thus (1.7) reduces to

\[
R_k = \lambda v_k + 2(n-1)(\alpha_k - \mu v_k)
\]

where \( \lambda = R_a v^a \) and \( \mu = \alpha_a v^a \).

Substituting this into (1.6), we obtain

\[
(\beta_k v_l - \beta_l v_k) + \beta(v_l, k - v_k, l) = 0.
\]
Now if $v_i$ is gradient, that is, $v_{i,k} - v_{k,i} = 0$, then

\[(1.10) \quad \beta_k v_i - \beta_i v_k = 0. \quad \text{That is,} \]

\[(1.10a) \quad \beta_k = a v_k \quad \text{where} \quad a \quad \text{is a scalar.} \]

Now by (1.8), (1.9) and (1.10) the equation (1.4) reduces to

$$\beta(v_{i,j,k} - v_{k,i,j}) = \frac{1}{2(n-1)} \phi(v_k g_{j,i} - v_i g_{j,k})$$

where $\phi = \lambda - 2(n-1)\mu$.

Transvecting the above equation with $t'$ and using $v_{j,i} = v_{i,j}$, we get

\[(1.11) \quad v_{j,k} = \frac{1}{2(n-1)} \frac{\phi}{\beta} (v_k v_j - g_{j,k}). \]

Let us consider the scalar function

$$f = \frac{1}{2(n-1)} \frac{\phi}{\beta} \neq 0.$$ 

We have

$$f_k = -\frac{1}{2(n-1)} \frac{\phi}{\beta^2} \beta_k + \frac{1}{2(n-1)} \frac{\phi_k}{\beta}$$

where $f_k = f_k$ and $\phi_k = \phi_k$.

Again (1.8) implies

$$R_{k,j} = \phi_j v_k + \phi_k v_j + 2(n-1)(\alpha_{k,j} - \alpha_{j,k})$$

from which we get $\phi_j v_k = \phi_k v_j$, that is, $\phi_k = A v_k$ where $A$ is a scalar function.

Thus form (1.10a) and (1.12) $f_k = B v_k$ where

$$B = \frac{1}{2(n-1)\beta} \left( -\frac{\phi_k}{\beta} + A \right)$$

Using (1.12), it is easy to show that $\omega_i = \frac{1}{2(n-1)\beta} \phi v_i$ is a gradient vector field.

In fact, $\omega_{i,j} = v_i f_j + f v_{i,j} = \beta v_i v_j + f v_{i,j} = \omega_{j,i}$. Thus (1.11) can be written as follows:

$$v_{j,k} = -f g_{j,k} + \omega_k v_j$$

where $\omega_k$ is gradient.

Hence $v_i$ is a concircular vector field. Since $f \neq 0$, $v_i$ is a proper concircular vector.

Hence $\lambda_i$ is a proper concircular vector field.

It is known [3] that if a conformally flat space admits a proper concircular vector field, then the space is a subprojective space in the sense of Kagan. Thus we can state

**Theorem 1.** If the associated vector of a conformally flat $(PS)_n$ is gradient, then the space is a subprojective space.

In [6] K. Yano proved that in order that a Riemannian space admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form
\[ ds^2 = (dx^1)^2 + e^{q(x^1)}dx^\alpha dx^\beta \]

where \( g^\alpha_\beta = g^\alpha_\beta(x^\nu) \) are the functions of \( x^\nu \) only (\( \alpha, \beta, \gamma = 2, 3, \ldots, n \)) and \( q = q(x^1) \) is a constant function of \( x^1 \) only. Since conformally flat \((PS)_n\) admits proper conicircular vector field \( v_i \) the space under consideration is the warped product \( 1 \times e^q M^* \) where \((M^*, g^*)\) is an \((n - 1)\)-dimensional Riemannian space. Gebarowski [6] proved that the warped product \( 1 \times e^q M^* \) satisfies (1.2) iff \( M^* \) is an Einstein space. Thus we state the following theorem:

**Theorem 2.** A conformally flat \((PS)_n\) is the warped product \( 1 \times e^q M^* \) where \( M^* \) is an Einstein space.

A conformally flat Riemannian space is said to be of quasi-constant curvature [4] if the curvature tensor \( R_{hijk} \) is given by

\[ R_{hijk} = a(g_{hj}g_{ik} - g_{hk}g_{ij}) + b(g_{hj}\theta_i\theta_k - g_{hk}\theta_i\theta_j - g_{ij}\theta_k\theta_k + g_{ik}\theta_k\theta_j) \]

where \( a \) and \( b \) are differentiable functions and \( \theta_i \) is a unit vector. Since our space is conformally flat, the curvature tensor is given by

\[ R_{hijk} = \frac{1}{n - 2} \left( R_{hj}g_{ik} - R_{hj}g_{ik} + R_{ij}g_{hk} - R_{ij}g_{hk} \right) - \frac{R}{(n - 1)(n - 2)}(g_{ij}g_{hk} - g_{ik}g_{hj}) \]

Now on account of (1.1) the above equation reduces to (1.11), where

\[ \theta_i = \nu_i, \quad a = \frac{R}{(n + 1)(n - 2)} - \frac{2\alpha}{n - 2} \quad \text{and} \quad b = -\frac{\beta}{n - 2} \]

Hence we obtain

**Theorem 3.** A conformally flat \((PS)_n\) is a space of quasi-constant curvature.

**References**


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