Theory of Conformally Berwald Finsler Spaces and Its Applications to \((\alpha, \beta)\)-Metrics

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Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

The theory of conformal changes of Finsler metrics has been studied by M. Hashiguchi [2] in 1976 and some of the Japanese school have directed their efforts to find conformally invariant curvature tensors similar to the Weyl conformal curvature tensor of a Riemannian space and to establish the condition for a Finsler space to be conformally flat. Finally, about five years ago, S.Kikuchi [6] succeeded in finding a conformally invariant Finsler connection and giving the conformally flat condition.

We have, however, a strange and objectionable in Kikuchi’s theory. His conformally invariant connection can be only defined on an essential assumption. Whether this assumption holds or not in a Finsler space under consideration poses newly a difficult problem. Since we have not a conformally invariant connection in the Riemannian case, the assumption is, of course, not satisfied by any Riemannian space.

About ten years ago, Y.Ichijyo and M.Hashiguchi [5] defined a conformally invariant Finsler connection in a Finsler space with \((\alpha, \beta)\)-metric, where \(\alpha = (a_{ij}(x)y^iy^j)^{1/2}\) is a Riemannian metric and \(\beta = b_i(x)y^i\) is a one-form in \(y^i\), on the assumption \(b^2 = a^{ij}b_ib_j \neq 0\). They gave the condition for a Randers space with the metric \(\alpha + \beta\) to be conformally flat based on their connection. M.Matsumoto [11] showed that their theory can be applied to a Kropina space with the metric \(\alpha^2/\beta\).

The main purpose of the present paper is to consider Kikuchi’s conformally invariant Finsler connections of Finsler spaces with \((\alpha, \beta)\)-metric. Since our main interest is Kikuchi’s assumption, it is sufficient to stop our studies halfway to Finsler spaces conformal to locally Minkowski spaces. Thus we shall propose a new notion of conformally Berwald Finsler space.

Mathematics Subject Classification: 53C05, 53C60

Key words: Berwald connection, \((\alpha, \beta)\)-metric, Kropina space
1 Conformally Berwald connections

In this section, we give a conformally Berwald connection which is induced from a scalar field S with a regularity condition. 

Let us consider a Finsler space \( F^n = F^n(M^n, L) \) with the Berwald connection \( B\Gamma = (G^i_j, G^i^j, k, 0) \) and a conformal change \( L \to L = e^{c(x)}L \). The quantities of the conformally changed space \( F^n \) will be denoted by putting a bar.

We have first the conformally invariant tensors \( B_{ij} \) and \( B^{ij} \):

\[
B_{ij} = \left( \frac{2}{L^2} \right) (g_{ij} - 2l_{ij}), \quad B^{ij} = \left( \frac{L^2}{2} \right) (g^{ij} - 2l^{ij}).
\]

The matrix \( (B^{ij}) \) is the inverse of \( (B_{ij}) \) \[2\]. In the following we denote by subscripts of \( B^{ij} \) the partial differentiations of \( B^{ij} \) by \( y^h \): \( B^{ij}_{h...k} = \partial ... \partial h B^{ij} \).

Putting \( F = \frac{L^2}{2} \) and \( 2G^i = g^{ij}((\hat{\partial}_i \partial F)y^r - \partial_i F), \) we have \( G^i_j = \hat{\partial}_i G^j \) and \( G^i^j = \hat{\partial}_h G^i_j \). If we put \( c_i = \partial_i c, \) on account of the paper \[2\] we get \( \hat{G}^h = G^h - B^{hr}c_r \) and

\[
(1.1) \quad \hat{G}^h_i = G^h_i - B^{hr}c_r, \quad \hat{G}^h_i^j = G^h_i^j - B^{hr}c_r.
\]

Then we obtain the relations between the \( hu \)-curvature tensors

\[
(1.2) \quad \hat{G}^h_{i, j} = G^h_{i, j} - B^{hr}c_r.
\]

Assume that we have a conformally invariant scalar field \( S(x, y) \) which is \((r)p\)-homogeneous in \((y')\). Denoting by \((;)\) the \( h \)-covariant differentiation in \( B\Gamma \), \( (1.1) \) yields

\[
\hat{S}_i = \partial_i S - (\hat{\partial}_i S)G^r_i = \partial_i S - (\hat{\partial}_i S)(G^r_i - B^{rs}c_s),
\]

and hence

\[
(1.3) \quad \hat{S}_i - S_i = W^r_i c_r, \quad W^r_i := (\hat{\partial}_r S)B^{rj}.\]

Along the lines of S.Kikuchi \[6\] and F.Ikeda \[4\] we shall suppose that

\[
(1.4) \quad \det(W^{;j}_i) \neq 0, \quad \text{and let } (V^{;j}_i) \text{ be the inverse matrix of } (W^{;j}_i).
\]

\( V^{;j}_i(x, y) \) are \((-r)p\)-homogeneous in \((y')\) and \( (1.3) \) can be written in the form

\[
(1.5) \quad \hat{V}_j - V_j = c_j, \quad V_j := S^r V^r_j.
\]

\( V_j(x, y) \) are \((0)p\)-homogeneous in \((y')\). Since \( c_j \) are functions of position, we must have

\[
(1.6) \quad \hat{\partial}_r (\hat{V}_j - V_j) = 0.
\]

Substituting from \( (1.5) \) in \( (1.1) \), we get the invariant quantities

\[
(1.7) \quad ^c G^h_i = G^h_i + B^{hr}V_r, \quad ^c G^h_i^j = G^h_i^j + B^{hr}V_r.
\]

Consequently we obtain the conformally invariant Finsler connection \( ^c B\Gamma = (^c G^h_i, ^c G^h_i^j, 0) \). This is called the conformal Berwald connection with respect to \( S \). On the other hand, \( (1.2) \) yields a conformally invariant tensor.
\[ (1.8) \quad c_{i}^{h \ ijk} = c_{i}^{\ h \ jk} + B^{hr \ ijk} V_{r}. \]

It is remarked that the \( hV \)-curvature tensor \((\partial_{k}(c_{i}^{h \ ijk}))\) of \( cB\Gamma \) is conformally invariant, but it is different from \( c_{i}^{h \ jk} \):

\[ (1.9) \quad \partial_{k}(c_{i}^{h \ jk}) = c_{i}^{h \ jk} + B^{hr \ ijk} \partial_{k} V_{r}. \]

According to the Berwald expression (Theorem 3.4) of a Finsler connection given by T. Aikou and M. Hashiguchi \[1\], the set \((L_{i}, D^{i \ k}, T^{i \ jk}, P^{i \ jk}, C^{i \ jk})\) of the essential tensor fields of \( cB\Gamma \) are

\[ (1.10) \quad L_{k} = -l_{r} B^{rs \ k} V_{s}, \quad D^{i \ k} = 0, \quad T^{i \ jk} = 0, \quad P^{i \ jk} = B^{hr \ ij} \partial_{r} V_{k}, \quad C^{i \ jk} = 0. \]

The conditions (1) and (2) mentioned in their theorem are satisfied because \( V_i(x, y) \) are \( (0)p \)-homogeneous in \( y^i \).

A Finsler space \( F^n \) is called a Berwald space if \( G^{i \ jk} \) are functions of position alone, or \( G^{i \ jk} = 0 \). In the Cartan connection \( C\Gamma = (G^{i \ jk}, F^{i \ jk}, C^{i \ jk}) \), \( F^n \) is a Berwald space if and only if \( F^{i \ jk} \) are functions of position alone, or \( C^{i \ jk} = 0 \) in terms of the \( h \)-covariant differentiation in \( C\Gamma \).

**Definition.** A Finsler space \( F^n = (M^n, L) \) is called *conformally Berwald*, if there exists a conformal change \( L \rightarrow \bar{L} = e^{c(x)} L \) such that the changed space \( \bar{F}^n = (M^n, \bar{L}) \) is a Berwald space.

We deal with a conformally invariant scalar \( S \) which satisfies \( S_{i} = 0 \) for a Berwald space. Such an invariant \( S \) is called of parallel type \[4\], (Theorem 2.1). The supposition \( \det(W_{ji}) \neq 0 \) with respect to \( S \) of parallel type is called the *Kikuchi condition*. Then we get

\[ (1.11) \quad c_{j} = \bar{V}_{j} - V_{j}, \quad V_{j} = S_{,r} V^{r \ j}, \]
on the Kikuchi condition.

Now we consider a Finsler space \( F^n \) having \( S \) satisfying the Kikuchi condition and suppose that \( F^n \) is conformal to a Berwald space \( \bar{F}^n \). Then \( \bar{S}_{i} = 0 \) and \( \bar{V}_{j} = 0 \), and hence (1.11) is reduced to

\[ (1.12) \quad c_{j} = -V_{j}, \]

which implies that \( V_{j} = V_{j}(x) \) is a gradient vector:

\[ (1.13) \quad (a) \quad \partial_{j} V_{i} = 0, \quad (b) \quad V_{i \ j} - V_{j \ i} = 0. \]

Next, since \( \bar{F}^n \) is a Berwald space, we have \( \bar{G}^{i \ jk} = 0 \) and hence (1.8) implies \( c_{i}^{\ h \ jk} = 0 \). Consequently we have

\[ (1.14) \quad c_{i}^{\ h \ jk} = 0. \]

Therefore (1.13a), (1.14) and (1.9) lead to the fact that the \( hV \)-curvature tensor of \( cB\Gamma \) vanishes.

Conversely, we consider a Finsler space \( F^n \) having \( S \) satisfying the Kikuchi condition such that \( V_{j} \) with respect to \( S \) satisfies (1.13) and \( cB\Gamma \) has the vanishing \( hV \)-curvature tensor. (1.9) with (1.13a) show \( c_{i}^{\ h \ jk} = 0 \). (1.13) gives the function
\[ c(x) \text{ satisfying (1.12) and hence we have the conformal change } L \rightarrow L = e^{c(x)}L. \]

Then \( cG^i_{\ jk} = 0 \) and (1.5) with (1.12) give \( \bar{V}_i = 0 \). Then (1.8) leads to \( \bar{G}^i_{\ jk} = 0 \) and thus the changed space \( \bar{F}^n \) is a Berwald space.

We denote by \( c\nabla \) the \( h \)-covariant differentiation in \( c\mathcal{B}\Gamma \), the \( v \)-covariant one in \( c\mathcal{B}\Gamma \) is \( \dot{\partial} \). Then (1.13) are written in terms of \( c\mathcal{B}\Gamma \) as follows:

\[(1.13') \quad (a) \quad \dot{\partial}_j V_i = 0, \quad (b) \quad c\nabla_j V_i - c\nabla_i V_j = 0.\]

Therefore we have

**Theorem 1.** Let \( F^n \) be a Finsler space having an \( S \) satisfying the Kikuchi condition. \( F^n \) is a conformally Berwald space, if and only if its conformal Berwald connection with respect to \( S \) has the vanishing \( hv \)-curvature tensor and satisfies (1.13').

## 2 Kikuchi’s assumption for \((\alpha, \beta)\)-metrics

To do justice Kikuchi’s assumption (1.4), that is the condition \( \det(W^i_j) \neq 0 \), we shall be concerned with Finsler space with \((\alpha, \beta)\)-metrics.

Let us consider a Finsler space \( F^n = (M^n, L(\alpha, \beta)) \) with \((\alpha, \beta)\)-metric where \( \alpha^2 = a_{ij}(x)y^iy^j \) is a Riemannian fundamental form and \( \beta = b_i(x)y^i \) is a 1-form in \( y^i \).

We put \( \alpha_{i...j} = \dot{\partial}_i...\dot{\partial}_j \alpha \) and have

\[(2.1) \quad \alpha\alpha_i = Y_i, \quad Y_i := a_{ir}y^r, \]

\[(2.2) \quad \alpha\alpha_{ij} = a_{ij} - Y_iY_j/\alpha^2 := k_{ij}, \]

where \( k_{ij} \) is the angular metric tensor of the Riemannian space \( \mathbb{R}^n = (M^n, \alpha) \) associated with \( F^n \). Next we have

\[(2.3) \quad \alpha\alpha_{ijk} = -(k_{ij}Y_k + (ijk))/\alpha^2, \]

where \(+ijk\) denotes cyclic permutations with respect to indices and their sum.

Further we put \( F = L^2/2 \) and the derivatives of \( F \) with respect to \((y^i, \alpha, \beta)\) are denoted by the subscripts \((i, 1, 2)\). Then \( F_i = F_{1}\alpha_i + F_{2}v_i \) and

\[(2.3') \quad F_{ij} = F_{1}\alpha_{ij} + F_{11}\alpha_i\alpha_j + F_{12}(\alpha_{b_j} + \alpha_jb_i) + F_{22}b_ib_j. \]

Since \( F_{ij} \) is the fundamental tensor \( g_{ij} \) of \( F^n \), we have from (2.1) and \( F_1 = F_{11}\alpha + F_{12}\beta \),

\[(2.3')' \quad g_{ij} = (F_{1}/\alpha)a_{ij} + F_{22}b_ib_j + (F_{12}/\alpha)(b_iY_j + b_jY_i) - (\beta F_{12}/\alpha^3)Y_iY_j. \]

We have shown \([8], [12]\).

\[ \det(g_{ij}) = (F_1/\alpha)^{n-2}T \det(a_{ij}), \]

\[ T := DB + 2FF_1/\alpha^3, \]

\[ D := F_{11}F_{22} - F_{12}^2, \]

\[ B := b^2 - (\beta/\alpha)^2. \]
We suppose $F_1 \neq 0$, of course. As a consequence $F^n$ has the irregular metric $(\det(g_{ij}) = 0)$, if and only if $T = 0$ [7]. In the following we are concerned only with $F^n$ having regular metric.

Then the inverse matrix $(g^{ij})$ of $(g_{ij})$ may be put

$$
(2.5) \quad g^{ij} = (\alpha/F_1)a^{ij} - s_0B^iB^j - s_{-1}(B^iy^j + B^jy^i) - s_{-2}y^iy^j,
$$

where $B^i = a^i_{12}b_j$. The condition $g^{ik}g_{kj} = \delta^i_j$ for $g^{ij}$ leads to

$$
(2.6) \quad T_{s_0} = \alpha D/F_1, \quad T_{s_{-1}} = 2FF_12/\alpha^2F_1,
$$

$$
T_{s_{-2}} = -(F_{12}/\alpha^2F_1)(BF_2 + 2F\beta/\alpha^2).
$$

From (2.3) and $F_{ijk} = 2C_{ijk}$ we have

$$
2C_{ijk} = C_1 + C_2 + C_3 + C_4 + C_5,
$$

$$
C_1 = F_1(\alpha a_{ijk} + (ijk)) + F_{11}(\alpha a_{ij} + (i)),
$$

$$
C_2 = (F_{111}\alpha a_k + F_{112}b_k)a_{ij}, \quad C_3 = (F_{112}\alpha a_k + F_{122}b_k)b_{ij},
$$

$$
C_4 = (F_{122}\alpha a_k + F_{222}b_k)a_{ij}.
$$

From (2.1) and (2.2), and putting

$$
(2.7) \quad p_i := b_i - (\beta/\alpha^2)Y_i,
$$

we have $C_1 = (F_{12}/\alpha)(k_{ij}p_k + (ijk))$. Next, from $F_{ab1}\alpha + F_{aba}\beta = 0$, for $a, b = 1, 2$, we have $F_{122} = -(\beta/\alpha)F_{222}, F_{112} = (\beta/\alpha)^2F_{222}$ and $F_{111} = -(\beta/\alpha)^3F_{222}$. Then

$$
C_2 = (\beta^2/\alpha^4)F_{222}Y_iY_jp_k, \quad C_3 = -(\beta/\alpha^2)F_{222}b_iY_jp_k,
$$

$$
C_2 = C_3 = -(\beta/\alpha^2)F_{222}p_iY_jp_k,
$$

$$
C_4 = -(\beta/\alpha^2)F_{222}b_iY_jp_k, \quad C_5 = F_{222}b_iY_jp_k,
$$

$$
C_4 + C_5 = F_{222}p_iY_jp_k,
$$

$$
C_2 + C_3 + C_4 + C_5 = F_{222}p_iY_jp_k.
$$

Consequently we have [12], [11]

$$
(2.8) \quad C_{ijk} = (F_{12}/2\alpha)(k_{ij}p_k + (ijk)) + (F_{222}/2)p_ip_jp_k.
$$

Now as a conformally invariant scalar $S$ in the section 1, we shall study two cases, that is, $A^2 = g^{ij}(LC_i)(LC_j)$ in the case of a Berwald $F^n$ in this section and $\beta/\alpha$ in the next section.

In the following of this section we study the condition (1.4) i.e. Kikuchi’s assumption for $(\alpha, \beta)$–metric.

Let us find $A^2 = g^{ij}(LC_i)(LC_j)$ for $F^n$ with $(\alpha, \beta)$–metric. From (2.5) and (2.8) we have

$$
C_1 = C_{ijk}g^{ik} = C_{ijk}(\alpha a^{jk}/F_1 - s_0B^jB^k).
$$

Also we have
\[ p_k a^{jk} = B^j - (\beta/\alpha^2)y^j, \quad kij p_k a^{jk} = p_i, \quad p_j p_k a^{jk} = B, \]

\[ C_{ijk} = ((n + 1)F_{12}/2\alpha + F_{222}\beta/2)p_i, \]

\[ k_{ij}B_j = p_i, \quad p_k B^k = B, \quad k_{jk}B^j B^k = B, \]

\[ C_{ijk}B^j B^k = (3F_{12}B/2\alpha + F_{222}B^2/2)p_i. \]

Thus we get

\[
C_i = E p_i, \\
E = (F_{12}/\alpha)((n + 1)\alpha/2F_1 - s_0 B) + (F_{222}B/2)(\alpha/F_1 - s_0 B).
\]

Consequently we have

\[
A^2 = 4F^2 E^2 B/\alpha^2 T.
\]

For the later use we are concerned with two examples where we put \( t = \beta/\alpha \).

**Ex. 1.** Randers metric \( L = \alpha + \beta \),

\[
T = (1 + t)^3, \quad \beta E = (n + 1)t/2(1 + t), \quad A^2 = (n + 1)^2 B/4(1 + t), \quad B = b^2 - t^2.
\]

**Ex. 2** Kropina metric \( L = \alpha^2/\beta \),

\[
T = 2b^2/t^6, \quad \beta E = -n + t^2/b^2, \quad A^2 = (B/2b^2)(t^2/b^2 - n - 2)^2.
\]

It is remarked that \( \dot{p}_t = p_i/\alpha, \quad \partial_t A^2 = (\partial A^2/\partial t)p_i/\alpha \) for the above two examples.

We now approach to our problem by another way from the homogeneity of \( F(\alpha, \beta) \).

If we put

\[ F(\alpha, \beta) = \alpha^2 f(t), \quad f(t) = F(1, t), \quad t = \beta/\alpha, \]

then we have the following:

\[
(F_1, F_2) = \alpha(\phi(t), f'(t)), \quad \phi(t) = 2f - tf',
\]

\[
(F_{11}, F_{12}, F_{22}) = (\phi - tf', \phi', f''),
\]

\[
(F_{111}, F_{112}, F_{122}, F_{222}) = (f'''/\alpha)(-t^3, t^2, -t, 1).
\]

\[
D = \delta(t) := 2f f'' - (f')^2, \quad B = B(b, t) := b^2 - t^2,
\]

\[
T = \delta(t)B(b, t) + 2f(t)\phi(t) = T(b, t),
\]

\[
T_{s_0} = \delta(t)/\phi(t), \quad E = \Psi(b, t)/\alpha,
\]

\[
\Psi(b, t) = [(n + 1)\alpha\phi' + B[(n - 1)\alpha\phi'/2\phi'' + f''')]T, \quad A^2 = 4(f\Psi)^2 B/T := \Pi(b, t).
\]

Consequently we have

\[ \dot{\partial}_t A^2 = \Pi p_i/\alpha. \]
Putting \( W^j_i = (\partial_r A^2) B^r j_i \), \( W_0^j_i = (\partial_r t) B^r j_i \), we obtain
\[
W^j_i = \Pi_r W_0^j_i.
\]
As an example of this process we show a case of Kropina metric \( L = \alpha^2 / \beta, \)
\[
f(t) = 1/2t^2, \quad \phi(t) = 2/t^2, \quad \delta(t) = 2/t^6, \quad T = 2b^2/t^6,
\]
\[
\Psi = (t^2 - (n + 2)b^2)/b^2 t, \quad A^2 = (B/2b^2)(t^2/b^2 - (n + 2))^2.
\]
Thus we have

**Theorem 2.** If a non-Riemannian Finsler space \( F^n \) with \( L(\alpha, \beta) \) has the non-zero \( \Pi_t \), then \( S = \alpha^2 \) satisfies the Kikuchi’s condition and Theorem 1 can be applicable to \( F^n \).

### 3 Another assumption for \( (\alpha, \beta) \)-metrics

As conformally changed \( \tilde{L}(\tilde{\alpha}, \tilde{\beta}) = e^{c(x)} L(\alpha, \beta) = L(e^{c(x)} \alpha, e^{c(x)} \beta) \) by (1) \( p \)-homogeneity of \( L, \beta/\alpha \) is conformally invariant [5]. Let us take \( S = \beta/\alpha \) and put \( W_{ij} = g_{ir}(\partial_r S) B^{rj} \). Then we have \( \partial_s S = p_s/\alpha \) and
\[
W_{ij} = g_{ir}(p_s/\alpha)(y_j g^{rs} - \delta^s_j y^r - \delta^r_j y^s - L^2 C^{rs} j_i) = (p_i y_j - p_j y_i - 2F p^r C_{rij})/\alpha.
\]

We put \( P^r = a^r p_i \), and have from (2.5), (2.7) and (2.6)
\[
p^r = g^{rs} p_i = (\alpha/F_1 - s_0 B)p^r - (s_0 \beta/\alpha^2 + s_1)B y^r
\]
\[
= (2F/\alpha^2 T) P^r - (B/\alpha^2 T)(\alpha \beta D + 2FF_1) y^r.
\]

Since [7] shows \( \alpha \beta D + 2FF_1 = F_1 F_2 \), we find
\[
(3.1) \quad P^r = (2F P^r - BF_2 y^r)/\alpha^2 T.
\]

Thus (2.8) together with \( P^i p_i = B \) and \( P^i k_{ij} = p_j \) leads to
\[
(3.2) \quad P^r C_{rij} = (F/\alpha^3 T)(F_1 B k_{ij} + (2F_1 + \alpha F_222 B)p_i p_j).
\]

From (2.3′) we have
\[
(3.3) \quad y_i = F_2 p_i + (2F/\alpha^2) Y_i.
\]

Consequently we obtain
\[
W_{ij} = Q a_{ij} + Q_0 p_i p_j + Q_{-1}(p_i Y_j - p_j Y_i) + Q_{-2} Y_i Y_j,
\]
\[
Q = -2F^2 F_12 B/\alpha^4 T,
\]
\[
Q_0 = -(2F^2/\alpha^4 T)(2F_12 + \alpha BF_222),
\]
\[
Q_{-1} = 2F/\alpha^3,
\]
\[
Q_{-2} = -Q/\alpha^2.
\]
$Q = 0$, if and only if $F_{12} = 0$, that is, $F$ is of the form $c_1 \alpha^2 + c_2 \beta^2$ with constant $c$'s. Thus, suppose that $F^n$ is not Riemannian, then $Q \neq 0$.

Next, we put $V^{jk} = V^j, g^{rk}$. Then

\[(3.5) \quad W^i_j V^{jk} = \delta^i_k.\]

Let us put

\[(3.6) \quad V^{jk} = a^{jk}/Q + R_0 P^j P^k + R_{-1}(P^j y^k - P^k y^j) + R_{-2} g^j y^k.\]

Then (3.5) yields as coefficients of the following terms,

\[
p_i P^k : (Q + Q_0 B) R_0 - \alpha^2 Q_{-1} R_{-1} = -Q_0/Q,
\]

\[
y_i P^k : -Q_{-1} B R_0 = Q_{-1}/Q,
\]

\[
p_i y^k : (Q + Q_0 B) R_{-1} + \alpha^2 Q_{-1} R_{-2} = -Q_{-1}/Q,
\]

\[
y_i y^k : -1/Q B R_{-1} = 1/\alpha^2.
\]

Therefore we obtain

\[(3.7) \quad R_0 = -1/BQ, \quad R_{-1} = -1/\alpha^2 B Q_{-1},\]

\[
R_{-2} = -1/\alpha^2 Q + (Q + Q_0 B)/\alpha^2(Q_{-1})^2 B.
\]

In an interesting paper [3] concerned with Finsler spaces equipped with a linear connection, M.Kashiguchi and Y.Ichijyo showed that if $b_i$ of a Finsler space $F^n$ with $(\alpha, \beta)$-metric is parallel with respect to the Levi-Civita connection $\gamma = (\gamma^i_jk(x))$ of the associated Riemannian space, then $F^i_jk$ of the Cartan connection $CT = (F^i_jk, G^i_j, C^i_jk)$ coincide with $\gamma^i_jk(x)$ and hence $F^n$ is a Berwald space. This is also shown directly from the equation which gives the difference $B^i_jk = G^i_jk - \gamma^i_jk [9]$.

\[L_\alpha B^k_j i y^i y^k = \alpha L_\beta (b_{ij} - B^k_j b_k)y^i.\]

If $b_{ij} = 0$, then the uniqueness of the theorem leads to $B^k_j i = 0$ immediately. The converse is not true; $b_{ij} = 0$ is not necessary for $F^n$ to be a Berwald space. For instance, as has been shown in [9], a Randers space with $L = \alpha + \beta$ is a Berwald space, if and only if $b_{ij} = 0$, while a Kropina space with $L = \alpha^2/\beta$ is a Berwald space, if and only if there exists a vector field $f_i(x)$ satisfying $b_{ij} = (f^i b_r) a_{ij} + b_i f_j - b_j f_i$.

**Definition.** Let a Finsler space $F^n$ with $L(\alpha, \beta)$-metric $(\alpha, \beta)$ be a Berwald space. If $b_i$ is necessarily parallel in the associated Riemannian space, then $F^n$ is called a **parallel Berwald space** and $L(\alpha, \beta)$ is of **parallel type**.

We give here some example of parallel Berwald spaces.

**Ex. 1** [13]

\[L = (\alpha^s + \ldots + c_k \alpha^{s-k} \beta^k + \ldots + \beta^s)^r,\]

where $rs = 1$ and const. $c$'s, is of parallel type.

**Ex. 2** [9], [10].

\[L = c_1 \alpha + c_2 \beta + \alpha^2/\alpha, \quad c_2 \neq 0,\]

\[L = c_1 \alpha + c_2 \beta + \alpha^2/\beta, \quad c_1 \neq 0,\]

where const. $c$'s, are of parallel type.
We consider a Finsler space $F^n$ with $L(\alpha, \beta)$ of parallel type and conformal to a Berwald space $\bar{F}^n$. Then $S = \beta / \alpha$ is conformally invariant and $\bar{S}_{ij} = 0$ in the Levi-Civita connection $\bar{\gamma}$ of $\bar{F}^n$. Therefore we have

**Theorem 3.** Let $F^n = (M^n, L(\alpha, \beta))$ be a Finsler space with $(\alpha, \beta)$-metric of parallel type. $F^n$ is a conformally Berwald space, if and only if the conformal Berwald connection with respect to $\beta / \alpha$ has the vanishing hv-curvature tensor and satisfies $(1.12')$.

\section{Conformally Berwald Randers spaces and Kropina spaces}

The last section is devoted to the conditions for Finsler spaces of Randers type and Kropina type to be conformally Berwald. We shall use the symbols

$$r_{ij} = (b_{i;j} + b_{j;i})/2, \quad s_{ij} = (b_{i;j} - b_{j;i})/2, \quad s_j = b^i s_{ij},$$

where the covariant differentiation ($;)$ is the one with respect to the associated Riemannian space with the metric $\alpha$. By a conformal change $L \to \bar{L} = e^c(x) L$ various quantities are changed as follows:

$$\bar{a}_{ij} = e^{2c} a_{ij}, \quad \bar{b}_i = e^c b_i.$$

Putting $c_i = \partial_i c$ and $c^i = a^r c_r$, the Christoffel symbols $\gamma^i_j k$ constructed from $a_{ij}$ are changed to

$$\bar{\gamma}^i_j k = \gamma^i_j k + \delta^i_j c_k + \delta^i_k c_j - c^l a_{jk},$$

and hence we obtain

$$\bar{b}_{i;j} = e^c (b_{i;j} - c_i b_j + b^r c_r a_{ij}).$$

First we are concerned with a Randers space with the metric $L = \alpha + \beta$. It is a Berwald space, if and only if $b_{i;j} = 0 \ [9]$. Consequently the space is conformally Berwald, if and only if there exists a gradient $c_i(x)$ satisfying

\begin{equation}
(4.1) \quad b_{i;j} - c_i b_j + b^r c_r a_{ij} = 0.
\end{equation}

From (4.1) we get

$$b^i b_{i;j} = b^2 c_i - b^r c_r b_i, \quad a^{ij} b_{i;j} = -(n - 1) b^r c_r.$$

Consequently we have

\begin{equation}
(4.2) \quad c_i = (b^r b_{i;r} - a^r s b_{r;j} b_i / (n - 1)) / b^2.
\end{equation}

Since $c_i$ is a gradient vector, we have

\begin{equation}
(4.3) \quad c_{i;j} - c_{j;i} = 0.
\end{equation}

(4.1) can be written as

$$r_{ij} = (c_i b_j + c_j b_i)/2 - b^r c_r a_{ij}, \quad s_{ij} = (c_i b_j - c_j b_i)/2.$$
These give respectively

\[ a^r r_s = -(n-1)b^r c_r, \quad s_j = (b^r c_r b_j - b^2 c_j)/2. \]

Hence we have

\[ r_{ij} = (r^s / (n-1))(a_{ij} - b_i b_j / b^2) - (b_i s_j + b_j s_i) / b^2 \]

(4.4)

\[ s_{ij} = (b_i s_j - b_j s_i) / b^2. \]

(4.5)

Now (4.2) can be written as

\[ c_i = (b^r r_s - s_i - a^r r_s b_i / (n-1)) / b^2, \]

and (4.4) gives \( b^r r_s = -s_i \). Therefore we have

\[ r_{ij} = (r^s / (n-1))(a_{ij} - b_i b_j / b^2) - (b_i s_j + b_j s_i) / b^2 \]

(4.6)

\[ c_i = -(2s_i + r^s b_i / (n-1)) / b^2. \]

Therefore we have

**Theorem 4.** A Randers space is conformally Berwald, if and only if (4.4) and (4.5) hold and \( c_i \) given by (4.6) is gradient, that is, satisfies (4.3).

Let \( F^n = (M^n, L = \alpha^2 / \beta^2) \) be a Kropina space and \( \tilde{F}^n = (M^n, \tilde{L}) \) a conformally changed space with \( \tilde{L} = e^{c(x)} L \). The latter is a Berwald space [9], if and only if there exists \( f_i \) satisfying

\[ \tilde{b}_{ij} = e^c (b_{ij} - c_i b_j + b^r c_r a_{ij} + b_i f_j - b_j f_i). \]

From \( \tilde{b}_{ij} = e^c (b_{ij} - c_i b_j + b^r c_r a_{ij}) \) and \( \tilde{b}^i = e^{-c} b^i \) the above is written as

(4.7)

\[ b_{ij} - c_i b_j + b^r c_r a_{ij} = b^r f_r a_{ij} + b_i f_j - b_j f_i, \]

which is equivalent to

\[ r_{ij} - (b_i c_j + b_j c_i) / 2 + b^r c_r a_{ij} = b^r f_r a_{ij}. \]

(4.8)

\[ s_{ij} = (b_i c_j + b_j c_i) / 2 = b_i f_j - b_j f_i. \]

(4.9)

Multiplying \( b^i \) to (4.9) yields

\[ s_j = b^2 (f_j - c_j / 2) - b_i (f_i - c_i / 2) b^i. \]

(4.10)

Consequently, eliminating \( f_i \) from (4.9) we obtain

\[ s_{ij} = (b_i s_j - b_j s_i) / b^2. \]

(4.11)

Next we deal with (4.8). Put

\[ u = b^r (c_r - f_r), \quad b^r r_{ij} = b r_{ij} \text{ and } b^r r_j = br. \]

(4.12)

\[ b r_{ij} = (b^r c_r b_j + b^2 c_j) / 2 + ub_j = 0. \]

(4.13)
Multiplying $b^i$ and from $b^2 \neq 0$ for the Kropina space, we obtain $r + u = b^i c_i$. Then (4.13) gives
\begin{equation}
(4.14) \quad \frac{c_j}{b^2} = \frac{2 br_j + (u - r) b_j}{b^2}.
\end{equation}

As a consequence (4.8) may be written in the form
\begin{equation}
(4.15) \quad r_{ij} = \frac{b_i r_j + b_j r_i}{b} + (u - r) b_i b_j / b^2 - u a_{ij}.
\end{equation}

(4.14) gives $b^i c_j = u + r$, and hence (4.12) yields $b^r f_r = r$. Consequently (4.10) yields
\begin{equation}
(4.16) \quad f_i = s_i / b^2 + r_i / b.
\end{equation}

Conversely, we consider a Kropina space $F^n$ such that (4.15) and (4.11) are satisfied and $c_j$ of (4.14) is gradient $(c_j = \partial_j c(x))$. We make the conformally changed $\bar{F}^n$ from $F^n$ by the conformal change $L \rightarrow \bar{L} = e^{c(x)} L$. Then (4.15), (4.11) and (4.14) lead to
\begin{align*}
b_{ij} - c_i b_j + b^r c_r a_{ij} &= r_{ij} + s_{ij} - c_i b_j + b^r c_r a_{ij} = \\
&= ra_{ij} + b_i (r_j / b + s_j / b^2) - b_j (r_i / b + s_i / b^2).
\end{align*}

Thus, (4.16) immediately leads to (4.7).

**Theorem 5.** A Kropina space is conformally Berwald, if and only if (4.15) and (4.11) hold and $c_j$ of (4.14) is gradient.

**References**


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