Semi-Symmetric Conformal Metrical N-Linear Connections in the Bundle of Accelerations

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Abstract

In the present paper we determine all semi-symmetric conformal metrical N-linear connections, which preserve the nonlinear connection N, in the bundle of accelerations. We study the group of transformations of these connections and its invariants.

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Key words: osculator bundle, curvature, torsion, semi-symmetric conformal metrical N-linear connection.

1 Introduction

The differential geometry of higher order Lagrange spaces was introduced and studied by R. Miron and Gh. Atanasiu in [8] — [13].

The applications of the Lagrange geometry of order k in Physics and Mechanics are quite numerous and important, [8].

The study of higher order Lagrange spaces is grounded on the k-osculator bundle notion. The bundle of accelerations corresponds in this study to k = 2, [1],[10].

In the present paper we define the notion of semi-symmetric conformal metrical N-linear connection on E = Osc²M and we determine the set of all semi-symmetric conformal metrical N-linear connections, which preserve the nonlinear connection N, on E. (§2) The group of their transformations preserving a nonlinear connection N, gives us important invariants (§3).

As to the terminology and notations we use those from [14], which are essentially based on M. Matsumoto’s book [6].

2 Notion of semi-symmetric conformal metrical N-linear connection in the bundle of accelerations

Let M be a real n-dimensional C∞-differentiable manifold and (Osc²M, π, M) its 2-osculator bundle, or the bundle of accelerations.


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The local coordinates on $E = \mathcal{O} \mathcal{S}^2 M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$.

If $N$ is a nonlinear connection on $E$, with the coefficients $N_{(1)}^{i} {}_{j}^k$, $N_{(2)}^{i} {}_{j}^k$, then let $\mathcal{D}(N) = (L^i_j, C^{(1)l}_j, C^{(2)l}_j)$ be an $N$-linear connection on $E$.

We consider a metrical $d$-structure on $E$, defined by a $d$-tensor field of the type $(0, 2)_m$, marked as $g_{ij}(x^i, y^{(1)i}, y^{(2)i})$. This $d$-tensor field is symmetric and nondegenerate.

To the metrical $d$-structure $g_{ij}$ on $E$, we associate Obata’s operators

\begin{equation}
\Omega_{ij}^{\mu} = \frac{1}{2}(\delta_{\mu}^{i} \delta_{\mu}^{j} - g_{ij} g^{\mu \nu}), \quad \Omega_{ij}^{* \mu} = \frac{1}{2}(\delta_{\mu}^{i} \delta_{\mu}^{j} + g_{ij} g^{\mu \nu}),
\end{equation}

where $(g_{ij})$ is the inverse matrix of $(g_{ij})$. They have the same properties as the ones associated with a Finsler space [14].

Let $\mathcal{S}_2(E)$ be the set of all symmetric $d$-tensor fields of the type $(0, 2)$ on $E$. It is easy to show that, the relation

\begin{equation}
a_{ij} \sim b_{ij} \Leftrightarrow \exists \rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E) \mid a_{ij} = e^{2\rho} b_{ij}; \quad a_{ij}, b_{ij} \in \mathcal{S}_2(E)
\end{equation}

is an equivalence relation on $\mathcal{S}_2(E)$.

**Definition 2.1.** The equivalence class $\hat{g}$ of $\mathcal{S}_2(E)/\sim$, to which the metrical $d$-structure $g_{ij}$ belongs, is called conformal metrical $d$-structure on $E$.

**Definition 2.2.** An $N$-linear connection $\mathcal{D}(N) = (L^i_j, C^{(1)l}_j, C^{(2)l}_j)$ on $E$ is said to be compatible with the conformal metrical $d$-structure $\hat{g}$, or a conformal metrical $N$-linear connection on $E$, if

\begin{equation}
g_{ij}^{(a)} = 2\omega_k g_{ij}, \quad g_{ij}^{(a)} \mid_k = 2\lambda_{(a)k} g_{ij}, \quad (\alpha = 1, 2),
\end{equation}

where $\omega_k = \omega_k(x, y^{(1)}, y^{(2)})$ and $\lambda_{(a)k} = \lambda_{(a)k}(x, y^{(1)}, y^{(2)})$, $(\alpha = 1, 2)$ are covariant $d$-vector fields and $\mid_k$, $(\alpha = 1, 2)$ denote the $h$- and $\nu_{(a)}$-covariant derivatives, $(\alpha = 1, 2)$ with respect to $\mathcal{D}(N)$.

**Definition 2.3.** An $N$-linear connection $\mathcal{D}(N) = (L^i_j, C^{(1)l}_j, C^{(2)l}_j)$ on $E$, is called semi-symmetric if the torsion $d$-tensor fields $T^{(a)}_0 {}_j^i k$, $S^{(a)}_0 {}_j^i k$, $(\alpha = 1, 2)$ have the form

\begin{equation}
T^{(a)}_0 {}_j^i k = \frac{1}{n - 1}(T^{(a)0}_0 \delta^i_k - T^{(a)0}_k \delta^i_j),
S^{(a)}_0 {}_j^i k = \frac{1}{n - 1}(S^{(a)0}_0 \delta^i_k - S^{(a)0}_k \delta^i_j),
\end{equation}

where $T^{(a)0}_0 = T^{(a)0}_0$, $S^{(a)0}_0 = S^{(a)0}_0$, $(\alpha = 1, 2)$.

**Definition 2.4.** An $N$-linear connection $\mathcal{D}(N) = (L^i_j, C^{(1)l}_j, C^{(2)l}_j)$ on $E$ is called a semi-symmetric conformal metrical $N$-linear connection if the relations (2.3) and (2.4) are verified.

If $\sigma_j = \frac{1}{n - 1} T^{(a)0}_0$, $\tau^{(a)}_0 = \frac{1}{n - 1} S^{(a)0}_0$, $(\alpha = 1, 2)$ and if we apply the Theorem 5.4.3., [8], we obtain:

**Theorem 2.1** The set of all semi-symmetric conformal metrical $N$-linear connections $\mathcal{D}(N) = (L^i_j, C^{(1)l}_j, C^{(2)l}_j)$, which preserve the nonlinear connection $N$ on $E$ is given by
(2.5) \[ L^i_{jk} = L^i_{jk} + 2\Omega^i_{jk}\sigma_r, C^i_{(\alpha)jk} = C^i_{(\alpha)jk} + 2\Omega^i_{jk}\gamma^\alpha_r, \] (\(\alpha = 1, 2\)),

where \(\delta \Gamma(N) = (L^i_{jk}, C^i_{(1)jk}, C^i_{(2)jk})\) is an arbitrary conformal metrical \(N\)-linear connection on \(E\), whose \(d\)-tensor fields \(T^\alpha_0\), \(S^\alpha_0\), (\(\alpha = 1, 2\)) are vanish.

3. Group of transformations of semi-symmetric conformal metrical \(N\)-linear connections on \(E = Osc^2 M\), which preserve the nonlinear connection \(N\).

Let us consider the transformations \(\delta \Gamma(N) \rightarrow \delta \Gamma(N)\) of semi-symmetric conformal metrical \(N\)-linear connections on \(E\), which preserve the nonlinear connection \(N\).

**Theorem 3.1** Two semi-symmetric conformal metrical \(N\)-linear connections on \(E\), \(\delta \Gamma(N) = (L^i_{jk}, C^i_{(1)jk}, C^i_{(2)jk})\), \(\delta \Gamma(N) = (\tilde{L}^i_{jk}, \tilde{C}^i_{(1)jk}, \tilde{C}^i_{(2)jk})\), are related as follows:

\[(3.1) \quad L^i_{jk} = L^i_{jk} - \delta_j^i \omega_k + 2\Omega_j^i \theta_r, \quad \tilde{C}^i_{(\alpha)jk} = C^i_{(\alpha)jk} - \delta_j^i \lambda^\alpha_k + 2\Omega_j^i \gamma^\alpha_r, \] (\(\alpha = 1, 2\)),

where we have \(\theta_r = \sigma_r - \omega_r, \gamma^\alpha_r = \tau^\alpha_r - \lambda^\alpha_r, (\alpha = 1, 2)\).

**Theorem 3.2** The set \(\mathcal{C}_N^\alpha\) of all the transformations given by (3.1) is a transformation group of the set of all conformal metrical \(N\)-linear connections on \(E\), which preserve the nonlinear connection \(N\), together with the mapping product.

The transformation \(t : \delta \Gamma \rightarrow \delta \Gamma\) given by (3.1) is express by the product of the following two transformations:

\[(3.2) \quad \tilde{L}^i_{jk} = L^i_{jk} - \delta_j^i \omega_k, \quad \tilde{C}^i_{(\alpha)jk} = C^i_{(\alpha)jk} - \delta_j^i \lambda^\alpha_k, (\alpha = 1, 2),\]

\[(3.3) \quad \tilde{L}^i_{jk} = L^i_{jk} + 2\Omega^i_{jk}\theta_r, \quad \tilde{C}^i_{(\alpha)jk} = C^i_{(\alpha)jk} + 2\Omega^i_{jk}\gamma^\alpha_r, (\alpha = 1, 2).\]

**Theorem 3.3** The group \(\mathcal{C}_N^\alpha\) is the direct product of the group \(\mathcal{C}_N^\beta\) (of all transformations (3.2)) and the group \(\mathcal{C}_N^\gamma\) (of all transformations (3.3)).

One can notice that the invariants of the group \(\mathcal{C}_N^\alpha\) will be invariants of each of these subgroups, and reciprocally.

In our previous paper [16], starting from the tensor fields \(K^i_{hjk}, P^i_{(\alpha)hjk}, (\alpha = 1, 2), S_{(22)}^i_{hjk}\), where
we obtained the following important invariants of the group of transformations of semi-symmetric metrical \( N \)-linear connections, which preserve the nonlinear connection \( N \), on \( E \):

\[
\begin{align*}
H^{i}_{h\ jk} &= K^{i}_{h\ jk} + \frac{2}{n-2}A_{hjk}\{\Omega^{j}_{hjk}(K_{rh} - \frac{K_{gr}}{2(n-1)})\}, \\
N^{i}_{(\alpha)h\ jk} &= \mathcal{P}^{(\alpha)h\ jk} + \frac{2}{n-2}A_{hjk}\{\Omega^{j}_{hjk}(\mathcal{P}^{(\alpha)rk} - \frac{\mathcal{P}^{(\alpha)gr}}{2(n-1)})\}, \quad (\alpha = 1, 2), \\
M^{i}_{(22)h\ jk} &= S^{i}_{(22)h\ jk} + \frac{2}{n-2}A_{hjk}\{\Omega^{j}_{hjk}(S^{r}_{(22)rk} - \frac{S^{r}_{(22)gr}}{2(n-1)})\},
\end{align*}
\]

where

\[
K_{h\ jk} = K^{i}_{h\ jki}, \quad \mathcal{P}^{(\alpha)h\ jk} = \mathcal{P}^{(\alpha)h\ jki}, \quad S^{i}_{(22)h\ jk} = S^{i}_{(22)h\ jki},
\]

\[
K = g^{hj}K_{h\ jk}, \quad \mathcal{P}^{(\alpha)} = g^{hj}\mathcal{P}^{(\alpha)h\ jk}, \quad S^{i}_{(22)} = g^{hj}S^{i}_{(22)h\ jk}, \quad (\alpha = 1, 2).
\]

If we replace \( K^{i}_{h\ jk}, \mathcal{P}^{(\alpha)h\ jk}, \quad (\alpha = 1, 2), \quad S^{i}_{(22)h\ jk} \) by the tensor fields \( K^{*\ i}_{h\ jk}, \mathcal{P}^{*\ (\alpha)h\ jk}, \quad (\alpha = 1, 2), \quad S^{*\ i}_{(22)h\ jk} \) defined by:

\[
\begin{align*}
K^{*\ i}_{h\ jk} &= K^{i}_{h\ jk} - \frac{1}{n}\delta^{i}_{hK}^{s\ jk}, \\
\mathcal{P}^{*\ (\alpha)h\ jk} &= \mathcal{P}^{(\alpha)h\ jk} - \frac{1}{n}\delta^{i}_{h\mathcal{P}^{(\alpha)s\ jk}}, \quad (\alpha = 1, 2), \\
S^{*\ i}_{(22)h\ jk} &= S^{i}_{(22)h\ jk} - \frac{1}{n}\delta^{i}_{h\mathcal{S}^{s}_{(22)s\ jk}}
\end{align*}
\]

we can obtain the invariants of the group of transformations of semi-symmetric conformal metrical \( N \)-linear connections on \( E \), which preserve the nonlinear connection \( N \):

**Theorem 3.4** Let

\[
K^{*\ h\ j} = K^{*\ i}_{h\ jki}, \quad \mathcal{P}^{*\ (\alpha)h\ j} = \mathcal{P}^{*\ (\alpha)i}_{h\ jki}, \quad (\alpha = 1, 2), \quad S^{*\ i}_{(22)h\ j} = S^{*\ i}_{(22)h\ jki}, \quad K^{*} = g^{hj}K^{*\ h\ j}, \quad \mathcal{P}^{*\ (\alpha)} = g^{hj}\mathcal{P}^{*\ (\alpha)h\ j}, \quad (\alpha = 1, 2), \quad S^{*\ i}_{(22)} = g^{hj}S^{*\ i}_{(22)h\ j},
\]

For \( n > 2 \) the following tensor fields
\[
\begin{aligned}
H^*_{h \, jk} &= K^*_{h \, jk} + \frac{2}{n-2} A_{jk} \{ \Omega^r_j^r(K^*_{rj} - \frac{K^*_{rj} g_{rj}}{2(n-1)}) \}, \\
N^*_{(\alpha)h \, jk} &= P^*_{(\alpha)h \, jk} + \frac{2}{n-2} A_{jk} \{ \Omega^r_j^r(P^*_{(\alpha)rj} - \frac{P^*_{(\alpha)rj} g_{rj}}{2(n-1)}) \}, (\alpha = 1, 2), \\
M^*_{(22)h \, jk} &= S^*_{(22)h \, jk} + \frac{2}{n-2} A_{jk} \{ \Omega^r_j^r(S^*_{(22)rj} - \frac{S^*_{(22)rj} g_{rj}}{2(n-1)}) \},
\end{aligned}
\]

determined by of semi-symmetric conformal metrical N-linear connections, which preserve the nonlinear connection N, on E, are invariants of the group \( \mathcal{C}^*_N \): 

**Proof.** We can easily show that \( H^*_{h \, jk}, N^*_{(\alpha)h \, jk}, (\alpha = 1, 2), M^*_{(22)h \, jk} \) are invariants of \( C^u_N \). Owing to Theorem 3.3, it suffices to prove the theorem for \( C^u_N \).

From Theorem 2.3 of [16], the tensor fields \( K^*_{h \, jk}, P^*_{(\alpha)h \, jk}, (\alpha = 1, 2), S^*_{(22)h \, jk} \) of a semi-symmetric metrical N-linear connection \( DF(N) \), are transformed on the basis of the relations (3.3) of \( DF(N) \) to \( DF(N) \) as follows:

\[
\begin{aligned}
\bar{K}^*_{h \, jk} &= K^*_{h \, jk} + 2A_{jk} \{ \Omega^r_j^r \sigma_{rj} \}, \\
\bar{P}^*_{(\alpha)h \, jk} &= P^*_{(\alpha)h \, jk} + 2A_{jk} \{ \Omega^r_j^r \rho_{(\alpha)rj} \}, (\alpha = 1, 2), \\
\bar{S}^*_{(22)h \, jk} &= S^*_{(22)h \, jk} + 2A_{jk} \{ \Omega^r_j^r \tau_{(2)rj} \},
\end{aligned}
\]

where \( \sigma_{rj}, \rho_{(\alpha)rj}, \tau_{(2)rj}, (\alpha = 1, 2) \) are some \( \alpha \)-tensor fields determined from \( DF(N) \). Since \( \Omega^r_j^r = 0 \), the tensor fields \( K^*_{h \, jk}, P^*_{(\alpha)h \, jk}, (\alpha = 1, 2), S^*_{(22)h \, jk} \) obey the same transformation laws as (3.8), Hence, (3.7) follows from the well-known elimination method used in Theorem 2.4 of [16].

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**References**


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